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A COMPACT FINITE DIFFERENCE SCHEME FOR SPACE-TIME FRACTIONAL DIFFUSION EQUATIONS WITH TIME DISTRIBUTED-ORDER DERIVATIVE

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In this paper, we present a compact finite difference scheme for the Dirichlet problem of a class of space-time fractional diffusion equations with time distributed-order derivative, where the time fractional derivative is defined in the sense of Caputo derivative, and the space fractional derivative is defined by the Riesz derivative. The term involving time distributed-order derivative is discretized by use of the compound Simpson formula, and the Caputo fractional derivative is approximated by the Lagrange interpolation formula, while the Riesz space fractional derivative is approximated by the compact fractional center difference formula. The proposed difference scheme is proved to be uniquely solvable, unconditionally stable and convergent with accuracy of fourth order in both space and time directions. Numerical experiments for supporting the theoretical analysis are given.

Keywords: space-time fractional diffusion equation; time distributed-order derivative; compact difference scheme; fourth order accuracy; unconditionally stable

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1. Introduction

Fractional derivatives have been proved to be very useful in describing the memory and hereditary properties of materials and processes, and fractional differential equations are widely used various domains including physics, biology, engineering, signal processing, systems identification, control theory, finance, fractional dynamics and so on [1-5]. One of the most important applications for fractional differential equations is to model the process of subdiffusion and superdiffusion of particles in physics, where the fractional diffusion equations are extensively used [6-8].

Due to the complexity of fractional calculus, it is difficult to obtain exact solutions for fractional differential equations. So it becomes important to develop effective numerical methods for seeking numerical solutions for fractional differential equations. Among the numerical methods existing in the literature, the finite difference method is the most popular one, which has been used by many authors to construct efficient difference schemes for a variety of fractional differential equations. Also there have been many effective finite difference schemes for solving fractional diffusion equations. For example, in [9-12], finite difference schemes were established for time fractional subdiffusion equations and diffusion-wave equations, where the time fractional derivative is defined in the sense of the Caputo derivative, and was approximated mainly by use of the Lagrange interpolation formulas (L1 or L2). In [13-18], the authors developed various finite difference schemes for space

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fractional diffusion equations, while in [19, 20], difference schemes for space-time fractional equations were investigated. Recently, there have also been some papers investigating difference methods for distributed-order fractional differential equations. In [21], Ye et al. derived a second order compact difference scheme to approximate a distributed-order time-fractional diffusion-wave equation, while in [22], Morgado and Rebelo presented an implicit scheme for the distributed order time-fractional reaction-diffusion equation with a nonlinear source term. In [23, 24], Gao and Sun developed alternating direction implicit difference schemes for the two-dimensional diffusion and wave equations with fractional derivatives in time. In [25], Ye et al. researched a class of space-time fractional diffusion equations with time distributed-order, and proposed a second order difference scheme to approximate both time and space fractional derivatives.

In this paper, we investigate difference schemes for space-time fractional diffusion equations with time distributed-order derivative, and the following problem will be considered:

$$\begin{cases} \mathcal{D}_t^{\omega(\alpha)}u(x,t) = p(x)\dfrac{\partial^\beta u(x,t)}{\partial|x|^\beta} + f(x,t),\ x\in[a,b],\ t\in[0,T], \\ u(x,0)=\varphi(x),\ x\in[a,b], \\ u(a,t)=u(b,t)=0, \end{cases} \tag{1}$$

where $\alpha\in(0,1)$, $\beta\in(1,2)$, u is smooth enough, p is continuous with $p(x)\geq L>0$ for $x\in(a,b)$, $\dfrac{\partial^\beta u(x,t)}{\partial|x|^\beta}$ denotes the Riesz fractional derivative, $\mathcal{D}_t^{\omega(\alpha)}u(x,t)$ denotes the time-fractional derivative of distributed order defined by

$$\begin{cases} \mathcal{D}_t^{\omega(\alpha)}u(x,t) = \int_0^1 \omega(\alpha)\,{}_0^C D_t^\alpha u(x,t)d\alpha, \\ {}_0^C D_t^\alpha u(x,t) = \dfrac{1}{\Gamma(1-\alpha)}\int_0^t \dfrac{u_t'(x,s)}{(t-s)^\alpha}ds, \\ \omega(\alpha)\geq 0,\ \int_0^1\omega(\alpha)d\alpha = K>0, \end{cases} \tag{2}$$

where ω is smooth enough.

The fractional diffusion equation with time distributed-order is useful for modeling a mixture of delay sources [26]. The Riesz fractional derivative can be used for describing anomalous diffusion [27]. In general, the most popular methods available for approximating the Riesz fractional derivative in the case $0<\alpha<2$, $\alpha\neq 1$, are the Grunwald-Letnikov and the fractional center difference approximation methods. In [28], Shen et al. established implicit and explicit finite difference methods with Grünwald-Letnikov derivative approximation to a linear Riesz fractional diffusion equation, and proved that the explicit method is conditionally stable, while the implicit method is unconditionally stable. In [29], the authors investigated a discrete random walk model based on an explicit finite-difference approximation for the Riesz fractional advection-dispersion equation, and presented explicit and implicit difference schemes using Grünwald-Letnikov derivative approximation. In [30], Yang et al. presented the standard and shifted Grünwald-Letnikov derivative approximations, the method of lines, the matrix transform method, the Lagrange approximation method and a spectral representation method for a Riesz fractional advection-dispersion equation on a finite domain, while in [31], Zhang et al. established an implicit finite difference method for a non-linear Riesz fractional diffusion equation with Grünwald-Letnikov derivative approximation. In [32], Çelik et al. used the fractional centered difference to approximate the Riesz fractional derivative, and established a second order accuracy Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative.

We organize the rest of this paper as follows. In Section 2, we present some notations and preliminaries. Then in Section 3, we develop a compact finite difference scheme for the problem (1). First, the spatial fractional derivative will be approximated by a compact scheme of fourth order accuracy. Second, the integral term in the time distributed-order derivative denoted by the first equation of (2) will be approximated by the compound Simpson formula, which is also of fourth order accuracy, and after this approximation, the time distributed-order derivative can be decomposed to multi-term time fractional derivatives. Third, each time fractional derivative will be approximated by the Lagrange interpolation formula. In Section 4, Theoretical analysis including unique solvability, stability and convergence for the present finite difference scheme are fulfilled. In Section 5, numerical examples are given for testing the present finite difference scheme. Finally, in Section 6, some concluding comments are proposed.

2. **Preliminaries**

Let $J,\ M,\ N$ be positive integers, and $\Delta\alpha = \frac{1}{2J},\ h = \frac{b-a}{M},\ \tau = \frac{T}{N}$. Define $\alpha_l = l\Delta\alpha (0 \le l \le 2J),\ x_i = a + i * h(0 \le i \le M),\ t_n = n\tau(0 \le n \le N),\ \Omega_h = \{x_i | 0 \le i \le M\},\ \Omega_\tau = \{t_n | 0 \le n \le N\},\ (i,n) = (x_i, t_n)$, and then the domain $[a,b] \times [0,T]$ is covered by $\Omega_h \times \Omega_\tau$. By $U_i^n = u(x_i, t_n)$ and u_i^n we denote the exact solution and numerical solution at the point (i,n) respectively. Denote $U^n = (U_1^n,\ U_2^n, ...,\ U_{M-1}^n)^T,\ u^n = (u_1^n,\ u_2^n, ...,\ u_{M-1}^n)^T$.

Define the grid functions spaces $V_h = \{u | u = (u_1,\ u_2, ... u_{M-1})^T\}$ and $V_h^0 = \{u | u \in V_h,\ u_0 = u_M = 0$. For $u,\ v \in V_h^0$, define inner product as $(u,v) = h \sum\limits_{i=1}^{M-1} u_i v_i$, and corresponding discrete L_2 norm by $\|u\| = \sqrt{(u,u)} = (\sum\limits_{i=1}^{M-1} h|u_i|^2)^{\frac{1}{2}}$.

For further use, denote

$$\delta_x u_{i-\frac{1}{2}}^n = \frac{u_i^n - u_{i-1}^n}{h},\ \delta_x^2 u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}.$$

Definition 1. For $n-1 \le \beta < n,\ n \in \mathbb{N}$, the left-side Riemann-Liouville derivative, the right-side Riemann-Liouville derivative, and the Riesz derivative of order β for the function $u(x,t)$ are defined by

$$\begin{cases} {}_{-\infty}D_x^\beta u(x,t) = \frac{d^n}{dx^n}(\frac{1}{\Gamma(n-\beta)} \int_{-\infty}^x (x-\sigma)^{n-1-\beta} u(\sigma,t) d\sigma), \\ {}_xD_\infty^\beta u(x,t) = (-1)^n \frac{d^n}{dx^n}(\frac{1}{\Gamma(n-\beta)} \int_x^\infty (\sigma - x)^{n-1-\beta} u(\sigma,t) d\sigma), \\ \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} = -\frac{1}{2\cos(\beta\pi/2)}({}_{-\infty}D_x^\beta u(x,t)) +_x D_\infty^\beta u(x,t)) \end{cases} \tag{3}$$

respectively, where in the definition of the Riesz derivative it satisfies that $\beta \ne 2k+1,\ k = 0, 1, ...$

Remark 1. Considering the homogeneous boundary value conditions in the problem (1), we can extend the spatial definition of the function u to the whole $\mathbb{R}$, and then it holds that

$$\begin{cases} {}_{-\infty}D_x^\beta u(x,t) =_a D_x^\beta u(x,t)), \\ {}_xD_\infty^\beta u(x,t) =_x D_b^\beta u(x,t), \\ \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} = -\frac{1}{2\cos(\beta\pi/2)}({}_aD_x^\beta u(x,t)) +_x D_b^\beta u(x,t)). \end{cases} \tag{4}$$

Lemma 1 (The compound Simpson formula). Suppose $f(\alpha) \in C^4[0,1]$. Then it holds that

$$\int_0^1 f(\alpha) d\alpha = \Delta\alpha \sum_{l=0}^{2J} d_l f(\alpha_l) - \frac{(\Delta\alpha)^4}{180} f^{(4)}(\eta),\ \eta \in (0,1), \tag{5}$$

where

$$d_l = \begin{cases} \frac{1}{3}, & l = 0, 2J, \\ \frac{2}{3}, & k = 2, 4, ..., 2J-4, 2J-2, \\ \frac{4}{3}, & k = 1, 3, ..., 2J-3, 2J-1. \end{cases}$$

Lemma 2 [9, Lemma 2.1](The Lagrange interpolation formula). Suppose $0 < \alpha < 1$, and $u(t) \in C^2[0, t_n]$. Then it holds that

$$|{}_0^C D_t^{\alpha} u(t) - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[a_0^{(\alpha)} u(t_n) - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})u(t_k) - a_{n-1}^{(\alpha)} u(t_0)]|$$
$$\leq \frac{1}{\Gamma(2-\alpha)}[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1 + 2^{-\alpha})] \max_{t_0 \leq t \leq t_n} |u''(t)| \tau^{2-\alpha}, \tag{6}$$

where $a_k^{(\alpha)} = (k+1)^{1-\alpha} - k^{1-\alpha},\ k \geq 0$, and satisfies $(1-\alpha)(k+1)^{-\alpha} < a_k^{(\alpha)} < (1-\alpha)k^{-\alpha}$.

Remark 2. Note that Lemma 2 also holds for $\alpha = 0$ under the same conditions and for $\alpha = 1$ if the coefficients are modified by $a_0^1 = 1,\ a_k^1 = 0,\ k = 1, 2, ...n-1$.

Lemma 3 [33, Theorem 2.4]. Let $1 < \beta < 2$, $f \in C^7(\mathbb{R})$ and all its derivatives up to order five belonging to $L_1(\mathbb{R})$. Define the fractional center difference

$$\Delta_h^{\beta} f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\beta)} f(x - kh),$$

where $g_k^{(\beta)} = \dfrac{(-1)^k \Gamma(\beta+1)}{\Gamma(\frac{\beta}{2} - k + 1)\Gamma(\frac{\beta}{2} + k + 1)}$. Then

$$-\frac{1}{h^{\beta}} \Delta_h^{\beta} f(x) = \frac{\beta}{24} \frac{\partial^{\beta} f(x-h)}{\partial |x|^{\beta}} + (1 - \frac{\beta}{12}) \frac{\partial^{\beta} f(x)}{\partial |x|^{\beta}} + \frac{\beta}{24} \frac{\partial^{\beta} f(x+h)}{\partial |x|^{\beta}} + O(h^4). \tag{7}$$

Furthermore, the coefficients $g_k^{(\beta)}$ satisfy the following properties:

$$\begin{cases} g_0^{(\beta)} = \dfrac{\Gamma(\beta+1)}{\Gamma(\frac{\beta}{2}+1)^2} > 0,\ g_k^{(\beta)} = \left(1 - \dfrac{1+\beta}{k + \frac{\beta}{2}}\right) g_{k-1}^{(\beta)},\ k = 1, 2, ..., \\ g_k^{(\beta)} = g_{-k}^{(\beta)},\ k = 1, 2, ...,\ \sum_{k=-\infty}^{\infty} g_k^{(\beta)} = 0. \end{cases}$$

3. **Derivation of the compact finite difference scheme**

Let the operator $\mathcal{A}$ be defined as $\mathcal{A}u_i = \frac{\beta}{24}u_{i-1} + (1 - \frac{\beta}{12})u_i + \frac{\beta}{24}u_{i+1}$. The first equation of (1) can be rewritten as

$$\frac{1}{p(x)} \mathcal{D}_t^{\omega(\alpha)} u(x,t) = \frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + \frac{f(x,t)}{p(x)}. \tag{8}$$

Applying the operator $\mathcal{A}$ to (8) at the point (i, n) and by use of Lemma 3 one can deduce that

$$\mathcal{A}[\frac{1}{p_i} \mathcal{D}_t^{\omega(\alpha)} U_i^n] = -\frac{1}{h^{\beta}} \Delta_h^{\beta} U_i^n + \mathcal{A}(\frac{f_i^n}{p_i}) = -\frac{1}{h^{\beta}} \sum_{k=-M+i}^{i} g_k^{(\beta)} U_{i-k}^n + \mathcal{A}(\frac{f_i^n}{p_i}). \tag{9}$$

On the other hand, by (2) and Lemma 1 one has

$$\mathcal{D}_t^{\omega(\alpha)} U_i^n = \omega(\alpha) {}_0^C D_t^{\alpha} U_i^n = \Delta\alpha \sum_{l=0}^{2J} d_l \omega(\alpha_l) {}_0^C D_t^{\alpha_l} U_i^n + O(\Delta\alpha)^4. \tag{10}$$

Furthermore, from Lemma 2 and Remark 2 one can obtain that

$${}_0^C D_t^{\alpha_l} U_i^n = \frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[a_0^{(\alpha_l)} U_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)} - a_{n-k}^{(\alpha_l)})U_i^k - a_{n-1}^{(\alpha_l)} U_i^0] + O(\tau^{2-\alpha_l}). \tag{11}$$

Combining (9)-(11) we have

$$\mathcal{A}\{\frac{\Delta\alpha}{p_i}\sum_{l=0}^{2J} d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[a_0^{(\alpha_l)}U_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)} - a_{n-k}^{(\alpha_l)})U_i^k - a_{n-1}^{(\alpha_l)}U_i^0]\}$$
$$= -\frac{1}{h^\beta}\sum_{k=-M+i}^{i} g_k^{(\beta)}U_{i-k}^n + \mathcal{A}(\frac{f_i^n}{p_i}) + O(\tau + h^4 + (\Delta\alpha)^4). \quad (12)$$

Then the finite difference scheme for the problem (1) can be formulated as follows:

$$\begin{cases} \mathcal{A}\{\frac{\Delta\alpha}{p_i}\sum_{l=0}^{2J} d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[a_0^{(\alpha_l)}u_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)} - a_{n-k}^{(\alpha_l)})u_i^k - a_{n-1}^{(\alpha_l)}u_i^0]\} \\ \qquad = -\frac{1}{h^\beta}\sum_{k=-M+i}^{i} g_k^{(\beta)}u_{i-k}^n + \mathcal{A}(\frac{f_i^n}{p_i}), 1 \le i \le M-1,\ 1 \le n \le N, \\ u_i^0 = \varphi(x_i),\ 1 \le i \le M-1, \\ u_0^n = u_M^n = 0. \end{cases} \quad (13)$$

4. **Unique solvability, stability and convergence analysis**

In this section, we prove the unique solvability, stability and convergence of the finite difference scheme (13).

Setting $\widetilde{u}_i^n = \frac{u_i^n}{p_i}$, $\widetilde{f}_i^n = \frac{f_i^n}{p_i}$, the first equation of (13) can be rewritten

$$\mathcal{A}\{\Delta\alpha\sum_{l=0}^{2J} d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[a_0^{(\alpha_l)}\widetilde{u}_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)} - a_{n-k}^{(\alpha_l)})\widetilde{u}_i^k - a_{n-1}^{(\alpha_l)}\widetilde{u}_i^0]\}$$
$$= -\frac{1}{h^\beta}\sum_{k=-M+i}^{i} g_k^{(\beta)}p_{i-k}\widetilde{u}_{i-k}^n + \mathcal{A}\widetilde{f}_i^n. \quad (14)$$

By the definition of the operator $\mathcal{A}$ we have $\mathcal{A}u_i = (1 + \frac{\beta}{24}h^2\delta_x^2)u_i$, and for $\forall u \in V_h^0$,

$$(\mathcal{A}u, u) = (u, u) + \frac{\beta}{24}h^2(\delta_x^2 u, u) = \|u\|^2 - \frac{\beta}{24}h^2(\delta_x u, \delta_x u)$$
$$= \|u\|^2 - \frac{\beta}{24}h^2\|\delta_x u\|^2.$$

On the other hand, by [34, Lemma 2.1.1] we have $\|\delta_x u\| \le \frac{2}{h}\|u\|$. So one can obtain that $(\mathcal{A}u, u) \ge \|u\|^2 - \frac{\beta}{6}\|u\|^2$. Since $\beta \in (1, 2)$, then $\frac{2}{3}\|u\|^2 \le (\mathcal{A}u, u) \le \|u\|^2$. Furthermore, for $\forall u,\ v \in V_h^0$, we can define one discrete inner product as $(u, v)_{\mathcal{A}} = (\mathcal{A}u, v) = h\sum_{i=1}^{M-1}(\mathcal{A}u_i)v_i$, while define the discrete norm as $\|u\|_{\mathcal{A}} = \sqrt{(\mathcal{A}u, u)}$. As one can see from above, $\frac{2}{3}\|u\|^2 \le \|u\|_{\mathcal{A}}^2 \le \|u\|^2$.

Lemma 4. For $\beta \in (1, 2)$, $p(x) \ge L > 0$, $v \in V_h^0$, it holds that

$$-\frac{h}{h^\beta}\sum_{i=1}^{M-1}[\sum_{k=-M+i}^{i} g_k^{(\beta)}p_{i-k}v_{i-k}v_i] \le -c_*^\beta L[2(b-a)]^{-\beta}h\sum_{i=1}^{M-1}v_i^2,$$

where $c_*^\beta = \frac{2}{\beta}r_\beta$, and $r_\beta = e^{-2}[\frac{(4-\beta)(2-\beta)\beta}{(6+\beta)(4+\beta)(2+\beta)}][\frac{\Gamma(\beta+1)}{\Gamma^2(\beta/2+1)}](3+\frac{\beta}{2})^{\beta+1}$.

The proof of Lemma 4 is similar to [34, Lemma 5.1.2].

Theorem 1. The finite difference scheme (13) is uniquely solvable.

Proof. We need to prove that there is only zero solution for the corresponding homogeneous difference equation of (13), which is formulated as follows due to (14)

$$\mathcal{A}\{\Delta\alpha\sum_{l=0}^{2J}d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}a_0^{(\alpha_l)}\widetilde{u}_i^n\}=-\frac{1}{h^\beta}\sum_{k=-M+i}^{i}g_k^{(\beta)}p_{i-k}\widetilde{u}_{i-k}^n. \quad (15)$$

Taking the inner product of (15) with $\widetilde{u}^n$, considering $a_0^{(\alpha_l)}=1$, from Lemma 4 one can deduce that

$$[\Delta\alpha\sum_{l=0}^{2J}d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}]\|\widetilde{u}^n\|_{\mathcal{A}}^2\le 0.$$

Since $p(x)$ is continuous on the compact interval $[a,b]$, there exists a positive constant κ such that $p(x)\le\kappa$ for $x\in(a,b)$, and furthermore, there exists one integer j satisfying $\|u^n\|_\infty=|u_j^n|=|p_j\frac{u_j^n}{p_j}|\le\kappa|\frac{u_j^n}{p_j}|\le\kappa\|\widetilde{u}^n\|\le\sqrt{\frac{3}{2}}\kappa\|\widetilde{u}^n\|_{\mathcal{A}}\le 0$, which implies (15) has only zero solution. Furthermore, the finite difference scheme (13) has unique solution. The proof is complete.

Theorem 2. The finite difference scheme (13) is unconditionally stable with respect to the initial datum and the right source term f.

Proof. Set $\mu=\Delta\alpha\sum_{l=0}^{2J}d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}$. Then according to (2) and Lemma 1 one can deduce that there exists a positive constant K_μ such that $\mu\ge K_\mu$.

Taking the inner product of the first equation of (14) with $\widetilde{u}^n$, considering $p(x)\ge L>0$, by use of Lemma 4 one can deduce that

$$\begin{aligned}
&\Delta\alpha\sum_{l=0}^{2J}d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[a_0^{(\alpha_l)}(\mathcal{A}\widetilde{u}^n,\widetilde{u}^n)-\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)}-a_{n-k}^{(\alpha_l)})(\mathcal{A}\widetilde{u}^k,\widetilde{u}^n)\\
&-a_{n-1}^{(\alpha_l)}(\mathcal{A}\widetilde{u}^0,\widetilde{u}^n)]=-\frac{1}{h^\beta}\sum_{i=1}^{M-1}\sum_{k=-M+i}^{i}g_k^{(\beta)}p_{i-k}\widetilde{u}_{i-k}^n\widetilde{u}_i^n+(\mathcal{A}\widetilde{f}^n,\widetilde{u}^n)\\
&\le -c_*^\beta L[2(b-a)]^{-\beta}h\sum_{i=1}^{M-1}\widetilde{u}_i^2+(\mathcal{A}\widetilde{f}^n,\widetilde{u}^n)\\
&=-c_*^\beta L[2(b-a)]^{-\beta}\|\widetilde{u}^n\|^2+(\mathcal{A}\widetilde{f}^n,\widetilde{u}^n)\\
&\le -c_*^\beta L[2(b-a)]^{-\beta}\|\widetilde{u}^n\|^2+\{c_*^\beta L[2(b-a)]^{-\beta}\|\widetilde{u}^n\|^2+\frac{1}{4c_*^\beta L[2(b-a)]^{-\beta}}\|\mathcal{A}\widetilde{f}^n\|^2\}\\
&=\frac{1}{4c_*^\beta L[2(b-a)]^{-\beta}}\|\mathcal{A}\widetilde{f}^n\|^2,
\end{aligned}$$

that is,

$$\begin{aligned}
&\mu\|\widetilde{u}^n\|_{\mathcal{A}}^2\le\Delta\alpha\sum_{l=0}^{2J}d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}\\
&[\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)}-a_{n-k}^{(\alpha_l)})(\widetilde{u}^k,\widetilde{u}^n)_{\mathcal{A}}+a_{n-1}^{(\alpha_l)}(\widetilde{u}^0,\widetilde{u}^n)_{\mathcal{A}}]+K_1\|\mathcal{A}\widetilde{f}^n\|^2, \quad (16)
\end{aligned}$$

where $K_1=\frac{1}{4c_*^\beta L[2(b-a)]^{-\beta}}$.

Now we prove the following inequality by use of the mathematical induction method

$$\|\widetilde{u}^n\|_{\mathcal{A}}^2\le\|\widetilde{u}^0\|_{\mathcal{A}}^2+\frac{2K_1}{\mu}\max_{1\le s\le n}\|\mathcal{A}\widetilde{f}^s\|^2. \quad (17)$$

If $n=1$, from (16) one has

$$\mu\|\widetilde{u}^1\|_{\mathcal{A}}^2\le\mu\|\widetilde{u}^0\|_{\mathcal{A}}^2+K_1\|\mathcal{A}\widetilde{f}^1\|^2,$$

which implies (17) holds.

Suppose (17) holds for $1,2,...,n-1$, then for the level n, from (16) one can deduce that

$$\mu\|\widetilde{u}^n\|_{\mathcal{A}}^2\le\Delta\alpha\sum_{l=0}^{2J}d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)}-a_{n-k}^{(\alpha_l)})(\frac{\|\widetilde{u}^n\|_{\mathcal{A}}^2+\|\widetilde{u}^k\|_{\mathcal{A}}^2}{2})$$

$$+a_{n-1}^{(\alpha_l)}(\frac{\|\widetilde{u}^n\|_{\mathcal{A}}^2+\|\widetilde{u}^0\|_{\mathcal{A}}^2}{2})] + K_1\|\mathcal{A}\widetilde{f}^n\|^2,$$

which is followed by

$$\mu\|\widetilde{u}^n\|_{\mathcal{A}}^2 \le \Delta\alpha\sum_{l=0}^{2J} d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)}-a_{n-k}^{(\alpha_l)})\|\widetilde{u}^k\|_{\mathcal{A}}^2$$
$$+a_{n-1}^{(\alpha_l)}\|\widetilde{u}^0\|_{\mathcal{A}}^2] + 2K_1\|\mathcal{A}\widetilde{f}^n\|^2.$$

Then

$$\mu\|\widetilde{u}^n\|_{\mathcal{A}}^2 \le \Delta\alpha\sum_{l=0}^{2J} d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)}-a_{n-k}^{(\alpha_l)})(\|\widetilde{u}^0\|_{\mathcal{A}}^2+\frac{2K_1}{\mu}\max_{1\le s\le k}\|\mathcal{A}\widetilde{f}^s\|^2)$$
$$+a_{n-1}^{(\alpha_l)}\|\widetilde{u}^0\|_{\mathcal{A}}^2] + 2K_1\|\mathcal{A}\widetilde{f}^n\|^2,$$

and furthermore,

$$\mu\|\widetilde{u}^n\|_{\mathcal{A}}^2 \le \Delta\alpha\sum_{l=0}^{2J} d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)}-a_{n-k}^{(\alpha_l)})+a_{n-1}^{(\alpha_l)}]\|\widetilde{u}^0\|_{\mathcal{A}}^2$$
$$+\Delta\alpha\sum_{l=0}^{2J} d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)}-a_{n-k}^{(\alpha_l)})+a_{n-1}^{(\alpha_l)}]\frac{2K_1}{\mu}\max_{1\le s\le k}\|\mathcal{A}\widetilde{f}^s\|^2+2K_1\|\mathcal{A}\widetilde{f}^n\|^2$$
$$=\mu\|\widetilde{u}^0\|_{\mathcal{A}}^2+2K_1\max_{1\le s\le n-1}\|\mathcal{A}\widetilde{f}^s\|^2+2K_1\|\mathcal{A}\widetilde{f}^n\|^2.$$

Moreover, we have

$$\mu\|\widetilde{u}^n\|_{\mathcal{A}}^2 \le \mu\|\widetilde{u}^0\|_{\mathcal{A}}^2+2K_1\max_{1\le s\le n}\|\mathcal{A}\widetilde{f}^s\|^2.$$

So (17) also holds for $\forall n\ge 1$ according to the mathematical induction method.

From (17) we have

$$\frac{2}{3}\|\widetilde{u}^n\|^2 \le \|\widetilde{u}^0\|^2+\frac{2K_1}{\mu}\max_{1\le s\le n}\|\mathcal{A}\widetilde{f}^s\|^2 \le \frac{1}{L}\|u^0\|^2+\frac{2K_1}{L\mu}\max_{1\le s\le n}\|\mathcal{A}f^s\|^2. \quad (18)$$

Furthermore, there exist $K_2>0$ and an integer $j\in[1,M-1]$ such that $|p(x)|\le K_2$, and

$$\|u^n\|_\infty = |u_j^n| = |p_j\widetilde{u}_j^n| \le K_2|\widetilde{u}_j^n| \le K_2\|\widetilde{u}^n\|. \quad (19)$$

Combining (18) and (19) we have

$$\|u^n\|_\infty^2 \le \frac{3K_2^2}{2L}\|u^0\|^2+\frac{3K_1K_2^2}{L\mu}\max_{1\le s\le n}\|\mathcal{A}f^s\|^2$$
$$\le \frac{3K_2^2(b-a)}{2L}\|u^0\|_\infty^2+\frac{3K_1K_2^2(b-a)}{L\mu}\max_{1\le s\le n}\|\mathcal{A}f^s\|_\infty^2$$
$$\le \frac{3K_2^2(b-a)}{2L}\|u^0\|_\infty^2+\frac{3K_1K_2^2(b-a)}{LK_\mu}\max_{1\le s\le n}\|\mathcal{A}f^s\|_\infty^2. \quad (20)$$

From (20) one can see that the solution u^n of the finite difference scheme (13) depends continuously on the initial datum u^0 and on the term f on the right, which shows that the difference scheme (13) is unconditionally stable. The proof is complete.

For the convergence of the finite difference scheme (13), we have the following theorem.

Theorem 3. The finite difference scheme (13) is convergent with the accuracy $O(\tau + h^4 + (\Delta\alpha)^4)$.

Proof. Let $\epsilon^n = U^n - u^n,\ n=0,1,...,N$ denote the errors between the exact solutions and the numerical solutions, and $\epsilon^n = (\epsilon_1^n,\ \epsilon_2^n,\ \epsilon_2^n, ..., \epsilon_{M-1}^n)^T$, $\widetilde{\epsilon}_i^n = \frac{\epsilon_i^n}{p_i}$. Then from (12)-(14) one can obtain that

$$\begin{cases} \mathcal{A}\{\Delta\alpha\sum_{l=0}^{2J} d_l\omega(\alpha_l)\frac{\tau^{-\alpha_l}}{\Gamma(2-\alpha_l)}[a_0^{(\alpha_l)}\widetilde{\epsilon}_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha_l)}-a_{n-k}^{(\alpha_l)})\widetilde{\epsilon}_i^k-a_{n-1}^{(\alpha_l)}\widetilde{\epsilon}_i^0]\} \\ = -\frac{1}{h^\beta}\sum_{k=-M+i}^{i} g_k^{(\beta)}p_{i-k}\widetilde{\epsilon}_{i-k}^n + R^n(\tau,h,\Delta\alpha),\ 1\le n\le N,\ 1\le i\le M-1, \\ \epsilon_i^0 = 0,\ 1\le i\le M-1, \end{cases} \quad (21)$$

where $R^n(\tau, h, \Delta\alpha) = O(\tau + h^4 + (\Delta\alpha)^4)$.

Similar to the proof process of Theorem 2 one can deduce that

$$\frac{2}{3}\|\widetilde{\epsilon}^n\|^2 \leq \|\widetilde{\epsilon}^0\|^2 + \frac{2K_1}{\mu}\max_{1\leq s\leq n}\|R^s(\tau, h, \Delta\alpha)\|^2 = \frac{2K_1}{\mu}\max_{1\leq s\leq n}\|R^s(\tau, h, \Delta\alpha)\|^2$$

and

$$\|\epsilon^n\|_\infty^2 \leq \frac{3K_1K_2^2(b-a)}{K_\mu}\max_{1\leq s\leq n}\|R^s(\tau, h, \Delta\alpha)\|_\infty^2, \tag{22}$$

where $K_1,\ K_2,\ K_\mu$ are defined as in Theorem 2. Then there exist positive constants $C_1,\ C_2,\ C_3$ such that

$$\|\epsilon^n\|_\infty \leq C_1\tau + C_2h^4 + C_3(\Delta\alpha)^4.$$

The proof is complete.

5. Numerical experiments

In this section, we carry out numerical experiments for testing the efficiency of the finite difference scheme (13).

Example 1. Consider the problem (1) with the following conditions

$$\begin{cases} a = 0,\ b = 1, \\ \omega(\alpha) = \Gamma(3-\alpha), \\ p(x) = x^2 + 1, \\ \varphi(x) = x^2(1-x)^2, \\ f(x,t) = \dfrac{2t(t-1)x^2(1-x)^2}{\ln(t)} + \dfrac{(x^2+1)}{2\cos(\beta\pi/2)}\sum\limits_{n=2}^{4}[\dfrac{c_n n! x^{-\alpha+n}}{\Gamma(1-\alpha+n)} + \dfrac{c_n n!(1-x)^{-\alpha+n}}{\Gamma(1-\alpha+n)}], \end{cases}$$

where $c_2 = c_4 = 1,\ c_3 = -2$. The exact solution is $u(x,t) = (t^2+1)x^2(1-x)^2$ for the problem above.

Let $\|e^n\| = \|U^n - u^n\|$ denote the absolute error in L_2 norm between the exact solution and the numerical solution. In Table 1, the numerical error results at different time steps are shown.

Table 1: The absolute errors at $\tau = 0.01$, $h = \frac{1}{10}$, $\Delta\alpha = 0.05$

	$\beta = 1.3$	$\beta = 1.5$	$\beta = 1.7$	$\beta = 1.8$
time steps	$\|e\|_\infty$	$\|e\|_\infty$	$\|e\|_\infty$	$\|e\|_\infty$
10	1.5210×10^{-4}	2.4070×10^{-4}	3.7246×10^{-4}	4.5876×10^{-4}
20	1.7650×10^{-4}	2.7418×10^{-4}	4.1557×10^{-4}	5.0644×10^{-4}
30	1.9259×10^{-4}	2.9658×10^{-4}	4.4586×10^{-4}	5.4137×10^{-4}
40	2.0823×10^{-4}	3.1944×10^{-4}	4.7861×10^{-4}	5.8032×10^{-4}
50	2.2562×10^{-4}	3.4567×10^{-4}	5.1728×10^{-4}	6.2687×10^{-4}
60	2.4564×10^{-4}	3.7635×10^{-4}	5.6308×10^{-4}	6.8230×10^{-4}
70	2.6871×10^{-4}	4.1198×10^{-4}	6.1659×10^{-4}	7.4721×10^{-4}
80	2.9506×10^{-4}	4.5285×10^{-4}	6.7811×10^{-4}	8.2191×10^{-4}

In Figs. 1-2, comparisons between the exact solutions and the numerical solutions are demonstrated.

It follows from Table 1 that the absolute errors are stable with the increment of the computation time steps, which coincide with the theoretical analysis results in Theorem 2. From Figs. 1-2 one can see that the numerical solutions can approximate the exact solutions in a prefect manner.

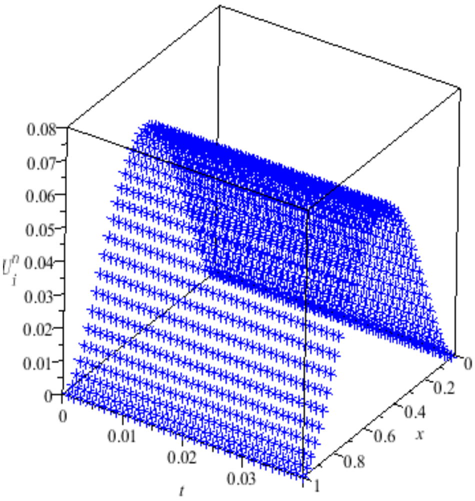

Fig 1. The exact solutions with h=1/41;τ=0.001;Δα=0.1;β=1.5

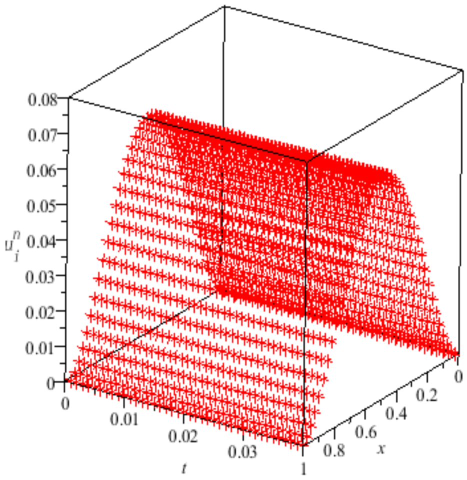

Fig 2. The numerical solutions with h=1/41;τ=0.001;Δα=0.1;β=1.5

Example 2. Consider the problem (1) with the following conditions

$$\begin{cases} a = 0, \ b = 1, \\ \omega(\alpha) = \Gamma(4-\alpha), \\ p(x) = \sin(x) + 1, \\ \varphi(x) = x^3(1-x)^3, \\ f(x,t) = \dfrac{6t^2(t-1)x^3(1-x)^3}{\ln(t)} + \dfrac{(\sin(x)+1)}{2\cos(\beta\pi/2)} \sum_{n=3}^{6} [\dfrac{c_n n! x^{-\alpha+n}}{\Gamma(1-\alpha+n)} + \dfrac{c_n n!(1-x)^{-\alpha+n}}{\Gamma(1-\alpha+n)}], \end{cases}$$

where $c_3 = 1, \ c_4 = -3, \ c_5 = 3, \ c_6 = -1$.

The exact solution is $u(x,t) = (t^3+2)x^3(1-x)^3$.

In Table 2, the numerical error results at different time steps are shown, while in Figs. 3-4, the exact solutions and the numerical solutions are demonstrated respectively under certain selected parameters.

Table 2: The absolute errors at $\tau = 0.01$, $h = \frac{1}{10}$, $\Delta\alpha = 0.05$

	$\beta = 1.3$	$\beta = 1.5$	$\beta = 1.7$	$\beta = 1.8$
time steps	$\|e\|_\infty$	$\|e\|_\infty$	$\|e\|_\infty$	$\|e\|_\infty$
10	$5.\times10^{-4}$	1.9228×10^{-4}	3.7246×10^{-4}	4.5876×10^{-4}
20	1.2317×10^{-4}	1.9371×10^{-4}	2.5216×10^{-4}	3.2158×10^{-4}
30	1.3152×10^{-4}	1.9362×10^{-4}	2.7956×10^{-4}	3.4235×10^{-4}
40	1.3789×10^{-4}	1.9440×10^{-4}	2.8691×10^{-4}	3.7008×10^{-4}
50	1.5634×10^{-4}	1.9706×10^{-4}	3.2354×10^{-4}	4.2164×10^{-4}
60	1.6107×10^{-4}	2.0238×10^{-4}	3.5211×10^{-4}	4.6018×10^{-4}
70	1.6986×10^{-4}	2.1101×10^{-4}	4.1026×10^{-4}	5.1127×10^{-4}
80	1.8206×10^{-4}	2.2363×10^{-4}	4.3257×10^{-4}	5.8765×10^{-4}

From Table 2 one can see that the numerical results are stable, and Figs. 3-4 show that the numerical solutions can approximate the exact solutions satisfactorily.

Remark 3(Comparison with other method). In [25], the author considered the following space-time fractional diffusion equations with time distributed-order derivative

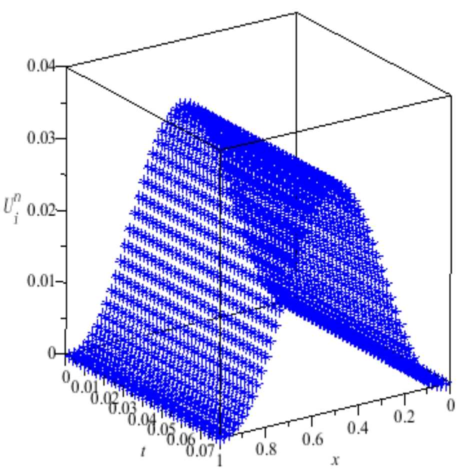

Fig 3. The exact solutions with h=1/51;τ=0.002;Δα=0.05;β=1.5

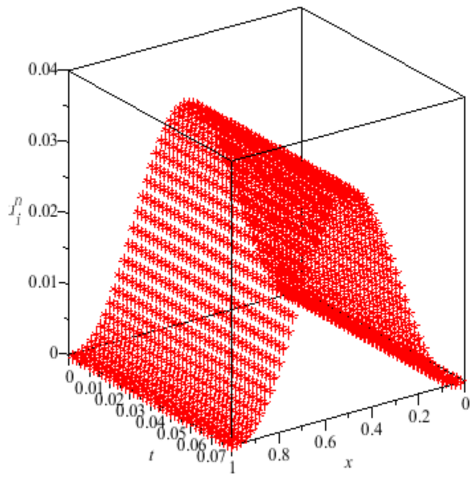

Fig 4. The numerical solutions with h=1/51;τ=0.002;Δα=0.05;β=1.5

$$\mathcal{D}_t^{\omega(\alpha)} u(x,t) = K_\beta \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t)$$

with the same initial and boundary value conditions as in (1), where $K_\beta > 0$ is a constant. A finite difference scheme was proposed with the error is in fact $O(\tau + h^2 + (\Delta\alpha)^2)$ in [25]. From the results in Tables 1-2 one can see that the orders of magnitude of the errors are 10^{-4}, that is, h^{-4}, while the orders of magnitude of the errors in Ref. [25] is h^{-2}. So our method is evidently of higher precision than that in Ref. [25].

6. Conclusions

We have developed a compact difference scheme with accuracy $O(\tau+h^4+(\Delta\alpha)^4)$ for a class of space-time fractional diffusion equations with time distributed-order derivative, and proved the uniquely solvability, unconditionally stable and convergence for it. Numerical experiments for testing the theoretical analysis results were carried out, and the numerical results show that they are in good agreement with the theoretical analysis.

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