

GEOMETRIC PROGRAMMING APPROACHES OF RELIABILITY ALLOCATION

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One of the important problems in the reliability design of a system is to allocate the reliability values to diverse constitutive units of the system. Every system has a reliability goal that needs to be achieved. Reliability allocations are used to set the goals for various subsystem or functional blocks such that the overall system level reliability can be achieved in an effective way. Our model discussed the posynomial cost function, taking into account all its properties regarding multivariate monotony and convexity (either Euclidean or with respect to a connection). Such a cost and the reliability constraint, associated to reliability polynomial, lead us to geometric programming method.

Keywords: geometric programming, reliability, allocation.

MSC2010: 49K45, 62N05, 60K30

1. Introduction

Reliability Allocation addresses the setting of reliability goals for individual components in order to meet a specified reliability goal and achieve a proper balance between the component goals. A proper balance usually refers to approximately equal relative time of development, difficulty, risk, or to reducing overall development costs. The model assigns reliability to a component according to the cost of increasing its reliability. In a complex system, it is necessary to translate overall system characteristics, including reliability, into detailed specifications, for the numerous units that make up the system. The process of assigning reliability requirements to individual units to attain the desired system reliability is known as reliability allocation. The allocation of system reliability involves solving basic program. Any design encapsulate the assignment of the values of many associated decision variables, for example, with thicknesses, lengths, widths, and proportions of various materials. The initial values allocated to the system itself should either be the specified values for the distinct reliability metrics of the system, or a set of reliability values which are slightly more difficult to obtain than the specified values. This is due to two factors: firstly, the system's reliability equation is required as an input by the model; secondly, the model also requires cost as a function of the component's reliability as an input. This cost function will function as the requital of increasing the a reliability of a component. The total system cost, which is the objective function to be reduced, is assumed as the sum of the costs of each individual component. Such types of problems can also be found in mechanical, electrical or computer hardware and software systems, and they have been examined in a number of

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studies. The model allocates reliability to a component according to the cost of increasing its reliability.

This paper examines possible approaches to allocate the reliability values such that the total cost is minimized. It is organized as follows. Section 2 gives new properties of monomials regarding the monotony and convexity. Section 3 describes geometric programming models for series systems, parallel systems, general systems. Section 4 underlines the possibility of using generalized geometric programming to solve reliability allocation problems. Finally, Section 5 provides some conclusions and a discussion of potential future research for a geometric programming reliability problem. Our ideas sprang from a desire to diversify following topics: systems dependability assessment [1], geometric programming [2], [7], [10]-[12], [15], geometric modeling in probability [3], Euclidean convexity [4], Riemannian convexity [13], reliability optimization [5], [8], [9], [14], multivariate function monotony [6].

2. Cost function as posynomial

A *geometric program* (GP) is a type of mathematical optimization problem characterized by objective and constraint functions that have a special form (monomial, posynomial, signomial).

2.1. Monomial and posynomial functions

Let x_1, \dots, x_m denote m real positive variables, and $x = (x_1, \dots, x_m)$ the associated vector. A real valued function of the form

$$f(x) = cx_1^{a_1} \cdots x_m^{a_m}$$

where $c > 0$ and $a_i \in \mathbb{R}$, is called a *monomial function*. We refer to the constant c as the coefficient of the monomial, and we refer to the constants a_1, \dots, a_m as the exponents of the monomial. Monomials are closed under product, division, positive scaling, power, inverse.

Also, any monomial of type $\varphi(x) = x_1^{a_1} x_2^{a_2}$ is: (i) *monotonically increasing in each variable*, if $a_1 > 0, a_2 > 0$; (ii) *bimonotonically increasing*, i.e.,

$$(x_1, x_2) \leq (y_1, y_2) \Rightarrow \varphi(x_1, y_2) + \varphi(y_1, x_2) \leq \varphi(x_1, x_2) + \varphi(y_1, y_2),$$

iff $a_1 a_2 > 0$; (iii) *monotonically of Lebesgue type*, i.e., for each domain $\mathbb{D} \in \mathbb{R}_+^2$, the function attains the extremum values on boundary $\partial\mathbb{D}$.

Moreover, the monomial $\varphi(x)$ is *Euclidean convex* if and only if

$$a_1 < 0, a_2 < 0, \text{ or } a_1 + a_2 < 1 \text{ and } a_1 a_2 < 0.$$

Theorem 2.1. *The general monomial $\psi(x) = x_1^{a_1} \cdots x_m^{a_m}$ is: (i) monotonically increasing in each variable, if $a_1 > 0, \dots, a_m > 0$; (ii) multi-monotonically increasing, i.e.,*

$$(x_1, \dots, x_m) \leq (y_1, \dots, y_m) \Rightarrow \sum_{A \subset \mathbb{N}_m} (-1)^{\text{card } A} \prod_{i \in A} x_i^{a_i} \prod_{j \in \mathbb{N}_m \setminus A} y_j^{a_j} \geq 0$$

iff $a_1 \cdots a_m > 0$; (iii) *monotonically of Lebesgue type*, i.e., for each domain $\mathbb{D} \in \mathbb{R}_+^m$, the function attains the extremum values on boundary $\partial\mathbb{D}$.

Proof. (ii) We use the identity

$$\sum_{A \subset \mathbb{N}_m} (-1)^{\text{card } A} \prod_{i \in A} x_i^{a_i} \prod_{j \in \mathbb{N}_m \setminus A} y_j^{a_j} = \prod_{i=1}^m (y_i^{a_i} - x_i^{a_i}).$$

If $y_i \geq x_i$, then $\text{sgn}(y_i^{a_i} - x_i^{a_i}) = \text{sgn } a_i$. □

For a general monomial $\psi(x) = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, to formulate necessary and sufficient conditions for convexity is a difficult task. A simpler sufficient condition is obtained using the function $g(x) = \ln \psi(x)$, $g(x) = a_1 \ln x_1 + \dots + a_m \ln x_m$. The function $\psi(x) = e^{g(x)}$ is Euclidean convex for $a_i < 0, i = 1, \dots, m$, since $g(x)$ is Euclidean convex in this case.

Theorem 2.2. *There exist an infinity of linear symmetric connections $\Gamma_{jk}^i(x)$ on \mathbb{R}_+^m such that the monomial $\psi(x)$ to be convex.*

Proof. We start by computing $dg(x) = \frac{a_1}{x_1} dx_1 + \dots + \frac{a_m}{x_m} dx_m$ and $d^2g(x) = -\frac{a_1}{x_1^2} dx_1^2 - \dots - \frac{a_m}{x_m^2} dx_m^2$. Since

$$Hess_{ii}g(x) = -\frac{a_i}{x_i^2} - \Gamma_{ii}^h(x) \frac{a_h}{x_h}, \quad Hess_{jk}g(x) = -\Gamma_{jk}^h(x) \frac{a_h}{x_h},$$

there exist an infinity of connections $\Gamma_{jk}^i(x)$ satisfying the algebraic system

$$-\frac{a_i}{x_i^2} - \Gamma_{ii}^h(x) \frac{a_h}{x_h} = 0, \quad \Gamma_{jk}^h(x) \frac{a_h}{x_h} = 0.$$

The function $g(x)$ is linear affine with respect to a connection and hence $\psi(x) = e^{g(x)}$ is convex with respect to that connection. Indeed,

$$Hess_{jk}\psi = e^g \left(\frac{\partial^2 g}{\partial x_j \partial x_k} + \frac{\partial g}{\partial x_j} \frac{\partial g}{\partial x_k} - \Gamma_{jk}^h \frac{\partial g}{\partial x_h} \right) = e^g \frac{a_j}{x_j} \frac{a_k}{x_k},$$

which is positive semidefinite. \square

The linear connection in the previous Theorem can be selected to be Riemannian. As for example, the monomial Riemannian metric

$$g_{11} = x_1^{\alpha_1}, \dots, g_{mm} = x_m^{\alpha_m}$$

determine the monomial Christoffel symbols Γ_{jk}^i with non-zero components $\Gamma_{ii}^i = \frac{1}{2} \frac{\alpha_i}{x_i}$ and the geodesics

$$\ddot{x}_i(t) + \frac{1}{2} \frac{\alpha_i}{x_i} (\dot{x}_i)^2(t) = 0, \quad i = 1, \dots, m,$$

i.e.,

$$x_i(t) = \begin{cases} (k_i t + C_i)^{\frac{2}{\alpha_i + 2}} & \text{if } \alpha_i \neq -2 \\ C_i e^{k_i t} & \text{if } \alpha_i = -2. \end{cases}$$

Then, for the connection in Theorem 2.2, we need to select the parameter $\alpha = (\alpha_1, \dots, \alpha_m) = (-2, \dots, -2)$.

Remark 2.1. *Moving to a non-trivial Riemannian metric or moving to a non-trivial linear connection on \mathbb{R}_+^m , we preserve, create or destroy convexity of functions. These changes are based on the idea of convexity in relation to geodesics or auto-parallel curves.*

A natural and valid objection might be raised as to why actually started to speak about the convexity in general sense, when there is a simpler Euclidean convexity. We can allege at defendendum before the Court of Conscience that we have got chummy with differential geometry in so far as we can understand its reasoning with equal case as we understand Euclidean context. Though, we should mention that does not matter how we obtain the convexity since once it is created we have all ingredients for convex programming theory (see [14]).

If the monotony and convexity are objectives in a problem, then the exponents a_i are used as decision parameters.

A function of the form

$$f(x) = \sum_{k=1}^K c_k x_1^{a_1 k} \cdots x_m^{a_m k},$$

where $c_k > 0$, is called a *posynomial function*. Being a finite linear combination of monomials, with positive coefficients, its properties regarding monotony and convexity are coming from those of constitutive monomials. Also, posynomials are closed under sum, product, positive scaling, division by monomial, positive integer power.

2.2. Cost function as posynomial

In reliability problems, we accept that partial costs are not offset each other. Also, having in mind the properties of monomials, the cost function can be defined as a posynomial

$$C(R_1, \dots, R_m) = \sum_{i \in [k]} a_i \prod_{j=1}^m R_j^{a_{ij}}, \quad 0 < R_j \leq 1, a_i > 0.$$

When an exponent a_{ij} is negative and the corresponding $R_j \rightarrow 0$, the cost explodes, i.e., $C \rightarrow \infty$. Specification of coefficients a_i and exponents a_{ij} depend on the system design.

3. Geometric programming models

We assume that a system has been designed at a higher level as an assembly of appropriated connected subsystems. In general the functionality of each subsystems can be unique, however there can be several choices for many of the subsystems providing the same functionality, but differently reliability levels.

3.1. Series system

Notations: $0 < R_j \leq 1$ is the reliability of component j ; the posynomial

$$C(R_1, \dots, R_m) = \sum_{i \in [k]} a_i \prod_{j=1}^m R_j^{a_{ij}}$$

is the total system cost, where $a_i > 0$ are cost coefficients, R_S is the system reliability, R_G is reliability goal.

Consider a series system consisting of m components, with the reliability $R_S = R_1 \cdots R_m$. The objective is to allocate reliability to all or some of the components of the system, in order to meet that goal with a minimum cost. The problem is generally a non-linear programming problem:

$$\begin{aligned} \min_R C(R_1, \dots, R_m) &= \sum_{i \in [k]} a_i \prod_{j=1}^m R_j^{a_{ij}}, \quad a_i > 0 \\ \text{s.t.} \quad R_S &= R_1 \cdots R_m \geq R_G \\ 0 < R_j &\leq 1, \quad j = 1, \dots, m. \end{aligned}$$

This program is designed to achieve a minimum total system cost.

We reformulate the previous program in standard geometric programming form

$$\begin{aligned} \min_R C(R_1, \dots, R_m) &= \sum_{i \in [k]} a_i \prod_{j=1}^m R_j^{a_{ij}} \\ \text{s.t.} \quad R_G R_1^{-1} R_2^{-1} \cdots R_m^{-1} &\leq 1, \quad 0 < R_j \leq 1, \quad j = 1, 2, \dots, m. \end{aligned}$$

Using the matrix of the exponents

$$A = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n-11} & -1 \\ a_{12} & a_{22} & \dots & a_{n-12} & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{1m} & a_{2m} & \dots & a_{n-1m} & -1 \end{pmatrix},$$

the associated *dual program* can be written

$$\max_{\alpha} P(\alpha) = \left\{ \left(\frac{a_1}{\alpha_1} \right)^{\alpha_1} \left(\frac{a_2}{\alpha_2} \right)^{\alpha_2} \dots \left(\frac{R_G}{\alpha_n} \right)^{\alpha_n} \right\} \alpha_n^{\alpha_n}$$

s. t.

$$\begin{aligned} \alpha_1 + \alpha_2 + \dots + \alpha_n &= 1 \text{ (normality)} \\ A\alpha &= 0, \alpha = (\alpha_1, \dots, \alpha_n)^T \text{ (orthogonality)}. \end{aligned}$$

Further, the solution of the dual program will give the solution of the primal program.

3.2. Solving dual program

Ingredients: n = number of terms; m = numbers of variables.

Algorithm: (1) find degree of difficulty = $n - m - 1$; (2) find $(\alpha_1, \alpha_2, \dots, \alpha_n)$; (3) evaluate $P^*(\alpha)$; (4) find $u_i = \alpha_i P^*$, $i = 1, 2, \dots, n$; (5) calculate $(R_1^*, R_2^*, \dots, R_m^*)$, using the algebraic system

$$\prod_{j=1}^m R_j^{a_{ij}} = \frac{\alpha_i P^*}{a_i}, \quad i = 1, 2, \dots, n.$$

This last nonlinear system is easily linearized by taking logarithms, and denoting $R_j^* = \exp(\omega_j)$:

$$\sum_{j=1}^m a_{ij} \omega_j = \ln \frac{u_i}{a_i}, \quad i = 1, 2, \dots, n.$$

Example 3.1. Consider a series system consisting of 3 components and the geometric programming problem

$$\begin{aligned} \min_R C &= a_1 R_2^2 R_3^3 + a_2 R_1^2 R_3^2 + a_3 R_1^3 R_2^2 \\ \text{s.t. } R_G R_1^{-1} R_2^{-1} R_3^{-1} &\leq 1, \quad 0 < R_i \leq 1. \end{aligned}$$

Solution: $m=3, n=4$; degree of difficulty $4 - 3 - 1 = 0$. Dual G.P.

$$\max_{\alpha} P = \left(\frac{a_1}{\alpha_1} \right)^{\alpha_1} \left(\frac{a_2}{\alpha_2} \right)^{\alpha_2} \left(\frac{a_3}{\alpha_3} \right)^{\alpha_3} \left(\frac{R_G}{\alpha_4} \right)^{\alpha_4} (\alpha_4)^{\alpha_4}$$

$$A = \begin{pmatrix} 0 & 2 & 3 & -1 \\ 2 & 0 & 2 & -1 \\ 3 & 2 & 0 & -1 \end{pmatrix}, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$$

$$\text{s.t. } \alpha_1 + \alpha_2 + \alpha_3 = 1 \text{ (normality)}$$

$$A\alpha = 0 \text{ (orthogonality)}.$$

We obtain $\alpha_1 = 0.4; \alpha_2 = 0.2; \alpha_3 = 0.4; \alpha_4 = 1.6$. Now calculate P^* and

$$u_1 = a_1 R_2^2 R_3^3 = 0.4 P^*, \quad u_2 = a_2 R_1^2 R_3^2 = 0.2 P^*,$$

$$u_3 = a_3 R_1^3 R_2^2 = 0.4 P^*, \quad u_4 = R_G R_1^{-1} R_2^{-1} R_3^{-1} = 1.6 P^*.$$

The optimal reliabilities are solutions of the system

$$R_2^2 R_3^3 = \frac{0.4 P^*}{a_1}, \quad R_1^2 R_3^2 = \frac{0.2 P^*}{a_2}, \quad R_1^3 R_2^2 = \frac{0.4 P^*}{a_3},$$

and furthermore

$$R_1^{-1} R_2^{-1} R_3^{-1} = \frac{1.6 P^*}{R_G}.$$

To solve the foregoing system, we use the logarithm, and $\omega_i = \ln R_i$:

$$2\omega_2 + 3\omega_3 = \ln \left(\frac{0.4 P^*}{a_1} \right) = m_1$$

$$2\omega_1 + 2\omega_3 = \ln \left(\frac{0.2 P^*}{a_2} \right) = m_2$$

$$3\omega_1 + 2\omega_2 = \ln \left(\frac{0.4 P^*}{a_3} \right) = m_3.$$

We find

$$\omega_1 = -\frac{m_1}{6} + \frac{m_2}{4} + \frac{m_3}{6}, \omega_2 = \frac{m_1}{4} - \frac{3m_2}{8} + \frac{m_3}{4}, \omega_3 = \frac{m_1}{6} + \frac{m_2}{4} - \frac{m_3}{6}$$

and hence

$$R_1^* = e^{\omega_1}, R_2^* = e^{\omega_2}, R_3^* = e^{\omega_3}.$$

The problem has solution since the exponents are negative.

3.3. Parallel system

A parallel system is a configuration such that, as long as not all of the system components fail, the entire system work and the total system reliability is higher than the reliability of any single system component.

Theorem 3.1. *A series system S is dual to a parallel system S' via the diffeomorphism $R_{S'} = 1 - R_S$, $R'_i = 1 - R_i$, only if they have the same number of components.*

Proof The multivariate reliability polynomial $R_S = \prod_{i=1}^m R_i$ is changed into $1 - R_{S'} = \prod_{i=1}^m (1 - R'_i)$.

Corollary 3.1. *The geometric program for a parallel system is equivalent to the geometric program for a series system.*

In fact, it is enough to replace R_i by $1 - R_i$.

3.4. General system

Let R_S be the reliability polynomial associated to the system S . Assume that this polynomial contains both "positively" and "negatively" terms. We group the "positive" terms in Σ_1 and the "negative" terms in Σ_2 , i.e., we write $R_S = \Sigma_1 - \Sigma_2$. A constraint of the form $R_S \geq R_G$ is transformed in two standard geometric programming constraints, using a new positive (dummy) variable T and imposing

$$\Sigma_1 \leq T, T \geq R_G + \Sigma_2.$$

Then, the initial constraint is equivalent to

$$T^{-1}\Sigma_1 \leq 1, T^{-1}R_G + T^{-1}\Sigma_2 \leq 1.$$

4. Generalized geometric programming problem

Geometric programming approaches of reliability allocation allows use the *generalized posynomials*: a generalized posynomial is a function formed using addition, multiplication, positive power, and maximum, starting from posynomials or composition of posynomials. Of course it appear a generalized geometric programming problem: minimize generalized posynomials over upper bound inequality constraints on other generalized posynomials. A generalized geometric programming problem involves at least two distinct phases: in the first, a feasible point is found (if there is one); in the second, an optimal point is found. Several extensions are readily handled. For example, if f is a posynomial and g is a monomial, then a constraint of the form $f \leq g$ can be handled by expressing it as $f/g \leq 1$ (since f/g is posynomial).

Any generalized geometric programming can be turned into equivalent standard geometric programming.

5. Conclusions

Every system has a reliability goal that needs to be achieved. The main focus of this paper is allocating components with constraints for a reliability system, discussing in detail the development of an integrated reliability model. A system reliability optimization problem through reliability allocation at the component level was conceived using geometrical connotations. We proposed a model for allocating a system reliability requirement with a given confidence level. The model formulas discussed the posynomial cost function in general and some inequalities constraints associated to the reliability polynomial. The problem was solved by applying geometric programming, and was confirmed using an example solution. The suggested model would be able to be applied to system design with a reliability goal with resource constraints for large scale reliability optimization problems. The true interpretation of the coefficients and exponents used for cost posynomial, in the context of reliability theory, arise from physical and economic interpretations of system design.

Acknowledgements. The work was partially supported by University Politehnica of Bucharest, and by the Academy of Romanian Scientists.

REFERENCES

- [1] J.-F. Auby, N. Brinzei, *Systems Dependability Assessment Modeling with Graphs and Finite State Automata*, John Wiley and Sons, Inc., 2015.
- [2] C. Beightler, D. T. Phillips, *Applied Geometric Programming*, John Wiley & Sons, Inc. 1976.
- [3] O. Calin, C. Udriște, *Geometric Modeling in Probability and Statistics*, Springer, 2014.
- [4] J.-S. Chen, C. -H. Hunang, *A note on convexity of two signomial functions*, Journal of Nonlinear and Convex Analysis, 3 (2009), 429-435.
- [5] W. Kuo, V. R. Prasad, *An annotated overview of system-reliability optimization*, IEEE Transactions on Reliability, 49 (2000), 176-187.
- [6] H. Lebesgue, *Sur le probleme de Dirichlet*, Rend. del Circolo Matematico di Palermo, 24 (1907), p. 371.
- [7] S. -T. Liu, *Geometric programming with fuzzy parameters in engineering optimization*, International Journal of Approximate Reasoning 46, 3 (2007), 484-498.
- [8] B. S. Mahapatra, G. S. Mahapatra, *Reliability and cost analysis of series system models using fuzzy parametric geometric programming*, Fuzzy Information and Engineering, 2, 4 (2010), 399-411.
- [9] A. Mettas, *Reliability allocation and optimization for complex systems*, Proceedings Annual Reliability and Maintainability Symposium, Los Angeles, CA, January (2000), 216-221.
- [10] J. Singh, V. Nookala, Z-Q. Luo, S. Sapatnekar, *Robust gate sizing by geometric programming*, In: Proceedings of the 42nd IEEE/ACM Design Automation Conference (DAC), (2005), 315-320.
- [11] J-F. Tsai, H-L. Li, N-Z. Hu, *Global optimization for signomial discrete programming problems in engineering design*, Eng. Optim. 34, 6 (2002),613-622.
- [12] C. Udriște, E. Tănăsescu, *Minima and Maxima of Real Functions of Real Variables (in Romanian)*, Technical Editorial House, Bucharest, 1980.
- [13] C. Udriște, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic Publisheres, 1994.

- [14] C. Udriște, S. A. Abed, A. S. Rasheed, *Optimal reliability allocation*, American Review of Mathematics and Statistics, 4, 2 (2016), 82-91; Communicated at International Conference on Applied Mathematics and Numerical Methods (ICAMNM), University of Craiova, April 14-16, (2016), Craiova, Romania.
- [15] J.-H. Yang, B.-Y. Cao, *Fuzzy geometric programming and its application*, Fuzzy Information and Engineering 2, 1 (2010), 101-112.