AUTOMATIC CONTINUITY OF (δ, ε)-DOUBLE DERIVATIONS ON C^*-ALGEBRAS

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Let $\mathcal{A}$ be an algebra, $\mathcal{M}$ be an $\mathcal{A}$-bimodule and $\delta : \mathcal{A} \to \mathcal{M}$, $\varepsilon : \mathcal{A} \to \mathcal{A}$ be linear mappings. We say that a linear mapping $d : \mathcal{A} \to \mathcal{M}$ is a (δ, ε)-double derivation if $d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$ holds for all $a, b \in \mathcal{A}$. In the case that $\mathcal{A} = \mathcal{M}$, by a δ-double derivation we mean a (δ, δ)-double derivation. In this article, we prove that if $\mathcal{A}$ is a C^*-algebra, $\mathcal{M}$ is a Banach $\mathcal{A}$-bimodule and the above-mentioned mappings δ, ε are continuous, then every (δ, ε)-double derivation $d : \mathcal{A} \to \mathcal{M}$ is automatically continuous.

Keywords: derivation, δ-double derivation, (δ, ε)-double derivation, automatic continuity.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be an algebra, $\mathcal{M}$ be an $\mathcal{A}$-bimodule and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be linear mappings. A linear mapping $d : \mathcal{A} \to \mathcal{M}$ is called a (σ, τ)-derivation if

$$d(ab) = d(a)\sigma(b) + \tau(a)d(b),$$

holds for all $a, b \in \mathcal{A}$. In the case that $\sigma = \tau$, the linear mapping d is called a σ-derivation. Clearly, if $\sigma = \tau = id$, the identity mapping on $\mathcal{A}$, then we reach to the usual notion of a derivation on the algebra $\mathcal{A}$ (see [3, 4, 8]). M. Mirzavaziri and E. O. Tehrani [7] introduced the concept of a (δ, ε)-double derivations which is a different notion of the current paper. In this paper, we generalized their definition as follows. Let $\delta : \mathcal{A} \to \mathcal{M}$, $\varepsilon : \mathcal{A} \to \mathcal{A}$ be linear mappings. A linear mapping $d : \mathcal{A} \to \mathcal{M}$ is called a (δ, ε)-double derivation if $d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$ holds for all $a, b \in \mathcal{A}$. In the case that $\mathcal{A} = \mathcal{M}$, by a δ-double derivation we mean a (δ, δ)-double derivation, i.e. $d(ab) = d(a)b + ad(b) + 2\delta(a)\delta(b)$ holds for all $a, b \in \mathcal{A}$.

The theory of automatic continuity of derivations has a long history. Results on automatic continuity of linear mappings defined on Banach algebras comprise a fruitful area of research developed during the last sixty years. See [2] for a comprehensive survey of results in this regard. Let us to investigate a background of our study. In 1958, I. Kaplansky [5] conjectured that every derivation on a C^*-algebra is continuous. Two years later, in 1960, S. Sakai [10] proved this conjecture. Indeed,

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he proved that every derivation on a C^*-algebra is automatically continuous and later in 1972, J. R. Ringrose [9], by getting idea from [1] and using its techniques showed that every derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is automatically continuous. Furthermore, the problem of automatic continuity has been considered for σ-derivations (see [3, 4, 8]). In 2009, M. Mirzavaziri and E. O. Tehrani [7] acquired some results about automatic continuity of δ-double derivations. For instance, they proved that if $\mathcal{A}$ is a C^*-algebra and $\delta : \mathcal{A} \to \mathcal{A}$ is a continuous linear mapping, then any $*$-δ-double derivation $d : \mathcal{A} \to \mathcal{A}$ is continuous ([7], Theorem 3.3). Moreover, they proved that if $\mathcal{A}$ is a C^*-algebra, $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ are two continuous linear mappings and $d : \mathcal{A} \to \mathcal{A}$ is a $* - (\delta, \varepsilon) - double$ derivation, then d is continuous ([7], Theorem 3.7).

In this study, we consider the same problem for (δ, ε)-double derivations from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{M}$. Indeed, by getting idea and using some techniques of [9], we prove the following main theorem.

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule. Suppose that $\delta : \mathcal{A} \to \mathcal{M}$, $\varepsilon : \mathcal{A} \to \mathcal{A}$ are continuous linear mappings and $d : \mathcal{A} \to \mathcal{M}$ is a (δ, ε)-double derivation. Then, d is continuous.

Using the above-mentioned theorem, we obtain some results concerning the automatic continuity of σ-derivations and (σ, τ)-derivations on C^*-algebras.

2. Results and Proofs

We begin with the following Lemmas which will be used extensively to prove our main theorem.

Lemma 2.1. *Let $\mathcal{I}$ be a closed bi-ideal in a C^*-algebra $\mathcal{A}$. Suppose that $a_1, a_2, a_3, \ldots \in \mathcal{I}$, $\sum_{n=1}^{\infty} \|a_n\|^2 \leq 1$. Then, there exist elements $b, c_1, c_2, \ldots$ of $\mathcal{I}$ such that $b \geq 0$, $\|c_n\| \leq 1$, and $a_n = bc_n$.*

Proof. See exercise 4.6.40 of [6]. □

Lemma 2.2. *Suppose that $\mathcal{A}$ is an infinite-dimensional C^*-algebra. Then, there is an infinite sequence $\{a_1, a_2, a_3, \ldots\}$ of non-zero elements of $\mathcal{A}^+$ such that $a_j a_k = 0$ when $j \neq k$.*

Proof. See exercise 4.6.13 of [6]. □

Lemma 2.3. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, and φ is a $*$-homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Suppose that $\{b_1, b_2, b_3, \ldots\}$ is a sequence of elements of $\mathcal{B}^+$ such that $b_j b_k = 0$ when $j \neq k$. Then, there is a sequence $\{a_1, a_2, a_3, \ldots\}$ of elements of $\mathcal{A}^+$ such that $a_j a_k = 0$ when $j \neq k$, and $\varphi(a_j) = b_j$ for each $j = 1, 2, 3, \ldots$.*

Proof. See exercise 4.6.20 of [6]. □

Definition 2.1. *Let $\mathcal{A}$ be an algebra, $\mathcal{M}$ be an $\mathcal{A}$-bimodule and $\delta : \mathcal{A} \to \mathcal{M}$, $\varepsilon : \mathcal{A} \to \mathcal{A}$ be two linear mappings. A linear mapping $d : \mathcal{A} \to \mathcal{M}$ is called a (δ, ε)-double derivation if $d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$ holds for all $a, b \in \mathcal{A}$. In* *the case that $\mathcal{A} = \mathcal{M}$, by a δ-double derivation we mean a (δ,δ)-double derivation, i.e. $d(ab) = d(a)b + ad(b) + 2\delta(a)\delta(b)$ holds for all $a, b \in \mathcal{A}$.*

Our main theorem reads as follows.

Theorem 2.1. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule. Then each (δ,ε)-double derivation $d : \mathcal{A} \to \mathcal{M}$ with continuous δ and ε is automatically continuous.*

Proof. We break up the proof into five steps.
For each $a \in \mathcal{A}$, we consider the maps $\eta_a : \mathcal{A} \to \mathcal{M}$, $\eta_a(t) = d(at)$ and $S_a : \mathcal{A} \to \mathcal{M}$, $S_a(t) = ad(t)$. Let $\mathcal{I} = \{a \in \mathcal{A} \mid \eta_a \textit{ is continuous}\}$.

Step 1: $\mathcal{I} = \{a \in \mathcal{A} \mid S_a \textit{ is continuous}\}$.
Set $\mathcal{V} = \{a \in \mathcal{A} \mid S_a \textit{ is continuous}\}$. Our task is to show that $\mathcal{V} = \mathcal{I}$. Let a be an element of $\mathcal{I}$. It is clear that the mapping $t \mapsto d(at) - d(a)t - \delta(a)\varepsilon(t) - \varepsilon(a)\delta(t) = ad(t)$ is continuous. It means that $a \in \mathcal{V}$ and thus, $\mathcal{I} \subseteq \mathcal{V}$. Now, we prove that $\mathcal{V} \subseteq \mathcal{I}$. Let a be an element of $\mathcal{V}$. From this and the continuity of δ, ε and the mapping $t \mapsto d(a)t$, we obtain that the mapping $t \mapsto d(a)t + ad(t) + \delta(a)\varepsilon(t) + \varepsilon(a)\delta(t) = d(at)$ is continuous. Hence, $a \in \mathcal{I}$ and it means that $\mathcal{V} \subseteq \mathcal{I}$. Consequently, $\mathcal{I} = \{a \in \mathcal{A} \mid S_a \textit{ is continuous}\}$.

Step 2: $\mathcal{I}$ is a closed two sided-ideal of $\mathcal{A}$.
First, we show that $\mathcal{I}$ is a two sided-ideal of $\mathcal{A}$. Note that for every element $b \in \mathcal{A}$, the linear mapping $\theta : \mathcal{A} \to \mathcal{A}$ defined by $\theta(t) = bt$ is continuous. Assume that a and b are two arbitrary elements of $\mathcal{I}$ and $\mathcal{A}$, respectively. It is evident that the mapping $\eta_a o \theta : \mathcal{A} \to \mathcal{M}$ defined by $\eta_a o \theta(t) = d(abt)$ is continuous, and so $ab \in \mathcal{I}$. It means that $\mathcal{I}$ is a right ideal of $\mathcal{A}$. Moreover, we have $d(bat) = bd(at) + d(b)at + \delta(b)\varepsilon(at) + \varepsilon(b)\delta(at)$. Note that the mappings $t \mapsto d(b)at$ and $t \mapsto bd(at)$ are continuous, and by using the assumption that δ and ε are also continuous, we conclude that the mappings $t \mapsto \varepsilon(b)\delta(at)$ and $t \mapsto \delta(b)\varepsilon(at)$ are continuous. Therefore, the mapping $t \mapsto d(bat)$ is a continuous linear mapping and consequently, $ba \in \mathcal{I}$. It means that $\mathcal{I}$ is a left ideal of $\mathcal{A}$. Therefore, $\mathcal{I}$ is a two sided-ideal of $\mathcal{A}$. In the following, we show that $\mathcal{I}$ is closed. If $a \in \overline{\mathcal{I}}$, then there exists a sequence $\{a_n\}$ in $\mathcal{I}$ such that $a_n \to a$. It is enough to show that the mapping $S_a : \mathcal{A} \to \mathcal{M}$ is continuous, i.e. $a \in \mathcal{V}$. Since $\{a_n\}$ is a sequence in $\mathcal{V}$, the linear mapping $S_{a_n} : \mathcal{A} \to \mathcal{M}$ is continuous for every $n \in \mathbb{N}$. We have $\lim_{n\to\infty} S_{a_n}(t) = S_a(t)$. By the principle of uniform boundedness, S_a is norm continuous and so $a \in \mathcal{V} = \mathcal{I}$. Therefore, $\mathcal{I}$ is a closed two sided-ideal of $\mathcal{A}$. Thereby, our assertion is proved

Step 3: $d|_{\mathcal{I}}$ is continuous.
Suppose that $d|_{\mathcal{I}}$ is an unbounded linear mapping. It means that $\|d|_{\mathcal{I}}\| = \sup\{\|d(a_n)\| : \|a_n\| \leq 1,\ a_n \in \mathcal{I}\} = \infty$. Then, we can choose a sequence $\{a_n\}$ in $\mathcal{I}$ such that $\|d(a_n)\| \to \infty$, $\sum_{n=1}^{\infty} \|a_n\|^2 \leq 1$. Now we define $b = (\sum_{n=1}^{\infty} a_n a_n^*)^{\frac{1}{4}}$, and since $\mathcal{I}$ is a closed bi-ideal of $\mathcal{A}$, b is a positive element of $\mathcal{I}$, i.e. $b \in \mathcal{I}^+$. We have $\|b\|^4 = \|b^4\| = \|\sum_{n=1}^{\infty} a_n a_n^*\| \leq \sum_{n=1}^{\infty} \|a_n a_n^*\| = \sum_{n=1}^{\infty} \|a_n\|^2 \leq 1$. So, $\|b\| \leq 1$. It

follows from Lemma 2.1 that for every $n \in \mathbb{N}$ there exists an element $c_n \in \mathcal{J}$ such that $\|c_n\| \leq 1$, $a_n = bc_n$. Note that $\|d(bc_n)\| = \|d(a_n)\| \to \infty$. We therefore have $\infty = \sup\{\|d(bc_n)\| \ : \ \|c_n\| \leq 1\} \leq \sup\{\|d(bt)\| \ : \ \|t\| \leq 1\}$, and consequently, the mapping $\eta_b : \mathcal{A} \to \mathcal{M}$ defined by $\eta_b(t) = d(bt)$ is unbounded. But this is a contradiction of the fact that $b \in \mathcal{J}$. Hence, the restriction $d|_{\mathcal{J}}$ is continuous.

Step 4: $\frac{\mathcal{A}}{\mathcal{J}}$ is finite-dimensional.
To obtain a contradiction, assume that $\frac{\mathcal{A}}{\mathcal{J}}$ is an infinite-dimensional C^*-algebra. It follows from Lemma 2.2 that there exists an infinite sequence $\{b_1, b_2, b_3, ...\}$ of non-zero, positive elements in $\frac{\mathcal{A}}{\mathcal{J}}$ such that $b_j b_k = 0$ where $j \neq k$. Since $\|b_j^2\| = \|b_j\|^2 > 0$, $b_j^2 \neq 0$. We know that the natural mapping $\pi : \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{J}}$ is a $*$-homomorphism from the C^*-algebra $\mathcal{A}$ onto the C^*-algebra $\frac{\mathcal{A}}{\mathcal{J}}$. According to Lemma 2.3, there exists a sequence $\{s_1, s_2, s_3, ...\}$ of elements of $\mathcal{A}^+$ such that $\pi(s_j) = b_j$, $s_j s_k = 0$, where $j \neq k$. If we now replace s_j by an appropriate scalar multiple, we may suppose also that $\|s_j\| \leq 1$. Since $\pi(s_j^2) = b_j^2 \neq 0$, $s_j^2 \notin \mathcal{J}$. This fact along with the definition of $\mathcal{J}$, imply that the mapping $\eta : \mathcal{A} \to \mathcal{M}$ defined by $t \mapsto d(s_j^2 t)$ is unbounded. Hence there is a sequence $\{t_j\}$ in $\mathcal{A}$ such that $\|t_j\| \leq 2^{-j}$, and $\|d(s_j^2 t_j)\| \geq j + m\|d(s_j)\| + m\|\delta(s_j)\|\|\varepsilon(c)\| + m\|\varepsilon(s_j)\|\|\delta(c)\|$ where m is the bound of the bilinear mapping $(a, x) \mapsto xa : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$. Since $\sum \|s_j t_j\| \leq \sum \|s_j\|\|t_j\| \leq \sum 2^{-j} < \infty$, the series $\sum s_j t_j$ is convergent to an element c of $\mathcal{A}$, i.e. $\sum s_j t_j = c$. We have $s_j c = s_j(\sum s_j t_j) = s_j s_1 t_1 + s_j s_2 t_2 + ... + s_j s_j t_j + ... = s_j^2 t_j$. Hence, $\|c\| = \|\sum s_j t_j\| \leq \sum \|s_j t_j\| \leq \sum 2^{-j} \leq 1$, and further

$$\begin{aligned}
\|s_j d(c)\| &= \|d(s_j c) - d(s_j)c - \delta(s_j)\varepsilon(c) - \varepsilon(s_j)\delta(c)\| \\
&\geq \|d(s_j c)\| - \|d(s_j)c\| - \|\delta(s_j)\varepsilon(c)\| - \|\varepsilon(s_j)\delta(c)\| \\
&= \|d(s_j^2 t_j)\| - \|d(s_j)c\| - \|\delta(s_j)\varepsilon(c)\| - \|\varepsilon(s_j)\delta(c)\| \\
&\geq j + m\|d(s_j)\| + m\|\delta(s_j)\|\|\varepsilon(c)\| + m\|\varepsilon(s_j)\|\|\delta(c)\| - \|d(s_j)c\| \\
&- \|\delta(s_j)\varepsilon(c)\| - \|\varepsilon(s_j)\delta(c)\| \\
&\geq j + m\|d(s_j)\| + m\|\delta(s_j)\|\|\varepsilon(c)\| + m\|\varepsilon(s_j)\|\|\delta(c)\| - m\|d(s_j)\| \\
&- m\|\delta(s_j)\|\|\varepsilon(c)\| - m\|\varepsilon(s_j)\|\|\delta(c)\| \\
&= j.
\end{aligned}$$

Since $\|s_j\| \leq 1$ and the mapping $t \mapsto td(c) : \mathcal{A} \to \mathcal{M}$ is bounded, the non-equality $\|s_j d(c)\| \geq j$ is a contradiction. This contradiction proves our claim that $\frac{\mathcal{A}}{\mathcal{J}}$ is finite-dimensional.

Step 5: d is continuous.
Since the algebra $\frac{\mathcal{A}}{\mathcal{J}}$ is finite-dimensional, we can consider the elements $a_1, a_2, ..., a_r$ of $\mathcal{A}$ such that $\pi(a_1), \pi(a_2), ..., \pi(a_r)$ forms a basis for the algebra $\frac{\mathcal{A}}{\mathcal{J}}$. Suppose that

$\tau_1, \tau_2, ..., \tau_r$ are linear functionals on $\frac{\mathcal{A}}{\mathcal{J}}$ such that

$$\tau_j(\pi(a_k)) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

As an easy exercise in functional analysis, we know that every τ_j is continuous for $1 \leq j \leq r$. Since $\{\pi(a_1), \pi(a_2), ..., \pi(a_r)\}$ is a basis for the algebra $\frac{\mathcal{A}}{\mathcal{J}}$, for every element $a \in \mathcal{A}$ we have $\pi(a) = \sum_{j=1}^{r} c_j\pi(a_j)$, where $c_j \in \mathbb{C}$. Hence, $\tau_j(\pi(a)) = c_1\tau_j(\pi(a_1)) + c_2\tau_j(\pi(a_2)) + ... + c_j\tau_j(\pi(a_j)) + ... + c_r\tau_j(\pi(a_r)) = c_j$. Having defined a continuous linear functional $\delta_j = \tau_j o\pi$, we have

$$\begin{aligned} \pi(a) &= \sum_{j=1}^{r} c_j\pi(a_j) \\ &= \sum_{j=1}^{r} \tau_j(\pi(a))\pi(a_j) \\ &= \sum_{j=1}^{r} \delta_j(a)\pi(a_j). \end{aligned}$$

Consequently, $a - \sum_{j=1}^{r} \delta_j(a)a_j \in \mathcal{J}$. Now we define $\Delta : \mathcal{A} \to \mathcal{J}$ by $\Delta(a) = a - \sum_{j=1}^{r} \delta_j(a)a_j$. Obviously, the linear mapping Δ is continuous, and so $d|_{\mathcal{J}} o\Delta : \mathcal{A} \to \mathcal{M}$ defined by $(d|_{\mathcal{J}} o\Delta)(a) = d(a - \sum_{j=1}^{r} \delta_j(a)a_j) = d(a) - \sum_{j=1}^{r} \delta_j(a)d(a_j)$ is continuous. The continuity of the mapping $d|_{\mathcal{J}} o\Delta$ along with the continuity of $\delta_1, \delta_2, ..., \delta_r$ imply that the linear mapping $a \mapsto d(a) - \sum_{j=1}^{r} \delta_j(a)d(a_j) + \sum_{j=1}^{r} \delta_j(a)d(a_j) = d(a) : \mathcal{A} \to \mathcal{M}$ is continuous and our ultimate goal is achieved. □

The following corollary is a comprehensive generalization of Theorem 3.7 of [7].

Corollary 2.1. *Let $\mathcal{A}$ be a C^*-algebra and $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ be continuous linear mappings. Then every (δ, ε)-double derivation $d : \mathcal{A} \to \mathcal{A}$ is continuous.*

Proof. This is Theorem 2.1 whenever $\mathcal{A} = \mathcal{M}$. □

The corollaries below are the generalizations of Theorem 3.8 and Theorem 4.3 of [8], respectively.

Corollary 2.2. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule, and let $\sigma : \mathcal{A} \to \mathcal{A}$ be a continuous homomorphism. Then every σ-derivation $d : \mathcal{A} \to \mathcal{M}$ is continuous.*

Proof. Since σ is a homomorphism, the module $\mathcal{M}$ equipped with the module multiplications $a \ltimes x = \sigma(a)x$ and $x \rtimes a = x\sigma(a)$ $(a \in \mathcal{A}, x \in \mathcal{M})$ is an $\mathcal{A}$-bimodule which is denoted by $\mathcal{M}^\bullet$. Since σ is bounded, we see that $\max\{\|a \ltimes x\|, \|x \rtimes a\|\} \leq \|\sigma\|\|a\|\|x\|$ for all $a \in \mathcal{A}$, $x \in \mathcal{M}$. It implies that $\mathcal{M}^\bullet$ is a Banach $\mathcal{A}$-bimodule. Note that

$$d(ab) = d(a)\sigma(b) + \sigma(a)d(b) = d(a) \rtimes b + a \ltimes d(b),$$

for all $a, b \in \mathcal{A}$. It means that $d : \mathcal{A} \to \mathcal{M}^{\bullet}$ is a derivation from the C^*-algebra $\mathcal{A}$ into the Banach $\mathcal{A}$-bimodule $\mathcal{M}^{\bullet}$. If we assume that $\delta = \varepsilon = 0$, then we have $d(ab) = d(a)\sigma(b) + \sigma(a)d(b) = d(a) \rtimes b + a \ltimes d(b) + \delta(a) \rtimes \varepsilon(b) + \varepsilon(a) \ltimes \delta(b)$. So, d is a (δ, ε)-double derivation from $\mathcal{A}$ into $\mathcal{M}^{\bullet}$. Now, Theorem 2.1 is exactly what we need to complete the proof. □

Corollary 2.3. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule, and let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be continuous homomorphisms. Then every (σ, τ)-derivation $d : \mathcal{A} \to \mathcal{M}$ is continuous.*

Proof. It is clear that $\mathcal{M}$ is a Banach $\mathcal{A}$-bimodule by the following module actions:

$$a \ltimes m = \tau(a)m, \quad m \rtimes a = m\sigma(a) \quad (a \in \mathcal{A},\ m \in \mathcal{M}).$$

We denote the above module by $\widehat{\mathcal{M}}$. Similar to the proof of Corollary 2.2, we may assume that $\delta = \varepsilon = 0$. Then we have $d(ab) = d(a)\sigma(b) + \tau(a)d(b) = d(a) \rtimes b + a \ltimes d(b) + \delta(a) \rtimes \varepsilon(b) + \varepsilon(a) \ltimes \delta(b)$. So, d is a (δ, ε)-double derivation from $\mathcal{A}$ into $\widehat{\mathcal{M}}$. Thus, by Theorem 2.1, d is continuous. □

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