A NEW CUMULATIVE DISTRIBUTION FUNCTION BASED ON m EXISTING ONES

Christophe Chesneau[1], Tassaddaq Hussain[2], Hassan S. Bakouch[3]

In this note, we present a new cumulative distribution function using sums and products of m existing cumulative distribution functions. Consequently, some new functions are discussed with focusing on one of them and providing two practical data examples.

Keywords: Cumulative distribution function transformations, Probability density function, Statistical distributions.

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1. Introduction

In literature, several transformations exist to obtain a new cumulative distribution function (cdf) using other(s) well-known cdf(s). The most famous of them is the power transformation introduced by [4]. Using a cdf $F(x)$, the considered cdf is $G(x) = (F(x))^\alpha$, $\alpha > 0$. For extensions and applications, see [5], [13] and [14], and the references therein. Another popular transformation is the quadratic rank transmutation map (QRTM) introduced by [15], where the considered cdf is $G(x) = (1+\lambda)F(x) - \lambda(F(x))^2$, $\lambda \in [-1, 1]$. Recent developments can be found in [2, 3], [7] and [8], and the references therein. Modern ideas include the DUS transformation proposed by [9]: $G(x) = \frac{1}{e-1}(e^{F(x)} - 1)$, the SS transformation introduced by [10]: $G(x) = \sin\left(\frac{\pi}{2}F(x)\right)$ and the MG transformation studied by [11]: $G(x) = e^{1-\frac{1}{F(x)}}$. Other transformations obtained by compounding can be found in [16]. An interesting approach is also given by the M transformation developed by [12], where using two cdfs $F_1(x)$ and $F_2(x)$, the considered cdf is $G(x) = \frac{F_1(x)+F_2(x)}{1+F_1(x)}$. In particular, [12] showed that the M transformation has great applications in data analysis. With specific cdfs $F_1(x)$ and $F_2(x)$, it can better fit practical data in comparison to some exploited distributions.

In this study, we propose a generalized version of the M transformation, called GM transformation. It is constructed from sums and products of m cdfs with $m \geq 1$. In comparison to the M transformation, it offers more possibility of cdf, mainly thanks to more flexibility on the denominator term. Then new distributions are derived, with the associated probability density function (pdf). Some mathematical properties of the new distributions are presented. A statistical study using two practical data sets is given: estimation of parameters of some of new distributions is performed using the method of maximum likelihood. We consider the Kolmogorov-Smirnov statistic to compare our models with some existing models. Better fits are obtained for our distributions.

The note is organized as follows. In Section 2, we present our new transformation. In Section 3, we apply it with specific well-known distributions, defining the associated pdfs

LMNO, University of Caen Normandie, France, e-mail: christophe.chesneau@unicaen.fr
[2]Department of Mathematics, Mirpur University of Science and Technology (MUST),Mirpur, AJK 10250, Pakistan
[3]Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt

and some mathematical properties of these distributions are described. Section 4 is devoted to a statistical study involving some of our new distributions as models, considering two practical data examples. A conclusion is given in Section 5.

2. GM transformation

Let $m \geq 1$ be an integer, $F_1(x), \ldots, F_m(x)$ be m cdfs of continuous distribution(s) with common support and $\delta_1, \ldots, \delta_m$ be m binary numbers, i.e. $\delta_k \in \{0,1\}$ for any $k \in \{1, \ldots, m\}$.

We introduce the following transformation of $F_1(x), \ldots, F_m(x)$:

$$G(x) = \frac{\sum_{k=1}^{m} F_k(x)}{m - 1 + \prod_{k=1}^{m} (F_k(x))^{\delta_k}}, \tag{1}$$

with the imposed value $\delta_m = 0$ in the special case where $m = 1$. The support of $G(x)$ is the common one of $F_1(x), \ldots, F_m(x)$.

The role of $\delta_1, \ldots, \delta_m$ is to activate or not the chosen cdfs in the product in the denominator. For examples, taking $m = 2$, $\delta_1 = 1$, and $\delta_2 = 1$, the function (1) becomes $G(x) = \frac{F_1(x)+F_2(x)}{1+F_1(x)F_2(x)}$. This cdf will be at the heart of Section 3. Taking $m = 3$, $\delta_1 = 1$, $\delta_2 = 1$ and $\delta_3 = 0$, the function (1) becomes $G(x) = \frac{F_1(x)+F_2(x)+F_3(x)}{2+F_1(x)F_2(x)}$; $F_3(x)$ is excluded of the denominator.

The following result motivates the interest of (1).

Theorem 2.1. *The function $G(x)$ (1) possesses the properties of a cdf.*

Proof of Theorem 2.1. For any $k \in \{1, \ldots, m\}$, let $f_k(x)$ be a pdf associated to the cdf $F_k(x)$. Recall that $F_k(x)$ is continuous with $F_k(x) \in [0,1]$, $\lim_{x \to +\infty} F_k(x) = 1$, $\lim_{x \to -\infty} F_k(x) = 0$ and $f_k(x) = F_k'(x)$ almost everywhere with $f_k(x) \geq 0$. Let us now investigate the sufficient conditions for $G(x)$ to be a cdf.

- Since $\sum_{k=1}^{m} F_k(x)$ and $m - 1 + \prod_{k=1}^{m} (F_k(x))^{\delta_k}$ are continuous functions with $m - 1 + \prod_{k=1}^{m} (F_k(x))^{\delta_k} \neq 0$, $G(x)$ is a continuous function of x.
- Let us prove that $G(x) \in [0,1]$. Owing to $\sum_{k=1}^{m} F_k(x) \geq 0$ and $m-1+\prod_{k=1}^{m} (F_k(x))^{\delta_k} > 0$, we have $G(x) \geq 0$. On the other hand, using the inequality: $\prod_{k=1}^{m} (1 - x_k) \geq 1 - \sum_{k=1}^{m} x_k$, $x_k \in [0,1]$, with $x_k = 1 - (F_k(x))^{\delta_k} \in [0,1]$ and observing that $(F_k(x))^{\delta_k} \geq F_k(x)$, we obtain

$$\begin{aligned} \prod_{k=1}^{m} (F_k(x))^{\delta_k} &\geq 1 - \sum_{k=1}^{m} (1 - (F_k(x))^{\delta_k}) = 1 - m + \sum_{k=1}^{m} (F_k(x))^{\delta_k} \\ &\geq 1 - m + \sum_{k=1}^{m} F_k(x). \end{aligned}$$

  Hence $G(x) \leq 1$.
- Let us prove that $G'(x) \geq 0$, implying that $G(x)$ is increasing. For any derivable function $u(x)$, note that $((u(x))^{\delta_k})' = \delta_k u'(x)$ since $\delta_k \in \{0,1\}$. Therefore we have

$G'(x) = \dfrac{A(x)}{B(x)}$ almost everywhere, where

$$\begin{aligned} A(x) &= \left(\sum_{k=1}^{m} f_k(x)\right)\left(m-1+\prod_{k=1}^{m}(F_k(x))^{\delta_k}\right) \\ &- \left(\sum_{k=1}^{m} F_k(x)\right)\left(\sum_{k=1}^{m}\delta_k f_k(x)\prod_{\substack{u=1\\u\neq k}}^{m}(F_u(x))^{\delta_u}\right) \end{aligned}$$

and

$$B(x) = \left(m-1+\prod_{k=1}^{m}(F_k(x))^{\delta_k}\right)^2.$$

We have $B(x) > 0$. Let us now investigate the sign of $A(x)$. The following decomposition holds: $A(x) = A_1(x) + A_2(x)$, where

$$A_1(x) = \sum_{k=1}^{m}\delta_k f_k(x)\left(m-1+\prod_{u=1}^{m}(F_u(x))^{\delta_u} - \left(\sum_{v=1}^{m}F_v(x)\right)\prod_{\substack{u=1\\u\neq k}}^{m}(F_u(x))^{\delta_u}\right)$$

and

$$A_2(x) = \sum_{k=1}^{m}(1-\delta_k)f_k(x)\left(m-1+\prod_{k=1}^{m}(F_k(x))^{\delta_k}\right).$$

Since $A_2(x) \geq 0$ as a sum of positive terms, let us focus on the sign of $A_1(x)$. Observe that, if $\delta_k = 1$, we have $F_k(x)\prod_{\substack{u=1\\u\neq k}}^{m}(F_u(x))^{\delta_u} = \prod_{u=1}^{m}(F_u(x))^{\delta_u}$. If $\delta_k = 0$, the k-th term in the sum of $A_1(x)$ is zero. Therefore we can write

$$A_1(x) = \sum_{k=1}^{m}\delta_k f_k(x)\left(m-1-\left(\sum_{\substack{v\neq k\\v=1}}^{m}F_v(x)\right)\prod_{\substack{u=1\\u\neq k}}^{m}(F_u(x))^{\delta_u}\right).$$

Since $F_v(x)\prod_{\substack{u=1\\u\neq k}}^{m}(F_u(x))^{\delta_u} \leq 1$, we have $m-1-\left(\sum_{\substack{u\neq k\\u=1}}^{m}F_u(x)\right)\prod_{\substack{u=1\\u\neq k}}^{m}(F_u(x))^{\delta_u} \geq 0$, implying that $A_1(x) \geq 0$. Therefore $A(x) \geq 0$, so $G'(x) \geq 0$.

- Let us now investigate $\lim_{x\to-\infty} G(x)$ and $\lim_{x\to+\infty} G(x)$. If $m \geq 2$, we have $m-1+\prod_{k=1}^{m}(F_k(x))^{\delta_k} \geq m-1 > 0$. Since $\lim_{x\to-\infty}\sum_{k=1}^{m}F_k(x) = 0$, we have $\lim_{x\to-\infty}G(x) = 0$. If $m = 1$, recall that we have imposed $\delta_m = 0$, so $\lim_{x\to-\infty}G(x) = \lim_{x\to-\infty}F_1(x) = 0$. On the other hand, for any $m \geq 1$, we have $\lim_{x\to+\infty}G(x) = \dfrac{m}{m-1+1} = 1$.

□

Let us now present some immediate examples existing in the literature. Taking $m = 1$ (so $\delta_1 = 0$), we obtain the simple cdf $G(x) = F_1(x)$. The choice $\delta_1 = \ldots = \delta_m = 0$ gives an uniform mixture of cdfs: $G(x) = \frac{1}{m}\sum_{k=1}^{m}F_k(x)$. Finally, for $m = 2$, $\delta_1 = 1$ and $\delta_2 = 0$, we obtain the M transformation introduced by [12]: $G(x) = \frac{F_1(x)+F_2(x)}{1+F_1(x)}$.

For this reason, we will call (1) as the GM transformation (as Generalization of the M transformation). To the best of our knowledge, it is new in literature.

New cdfs can also be derived by the GM transformation and existing transformations. Some of them using only one cdf are described below.

- For any cdf F of continuous distribution and $\delta_1, \ldots, \delta_m$ such that $\sum_{k=1}^{m} \delta_k = q$ with $q \in \{0, \ldots, m\}$, the GM transformation yields the following cdf:
$$G(x) = \frac{mF(x)}{m - 1 + (F(x))^q}.$$
- For any cdf F of continuous distribution with support equal to $\mathbb{R}$ or $[0, +\infty)$ or $(-\infty, 0)$ and any real numbers $\beta_1, \ldots, \beta_m$, where $\beta_k > 0$ for any $k \in \{1, \ldots, m\}$, the GM transformation includes the following cdf:
$$G(x) = \frac{\sum_{k=1}^{m} F(\beta_k x)}{m - 1 + \prod_{k=1}^{m} (F(\beta_k x))^{\delta_k}}.$$
- Combining the GM transformation and the power transformation introduced by [4], for any cdf F of continuous distribution and any real numbers $\alpha_1, \ldots, \alpha_m$, where $\alpha_k > 0$ for any $k \in \{1, \ldots, m\}$, we obtain the cdf:
$$G(x) = \frac{\sum_{k=1}^{m} (F(x))^{\alpha_k}}{m - 1 + \prod_{k=1}^{m} (F(x))^{\delta_k \alpha_k}}.$$
- Combining the GM transformation and the transformation using QRTM introduced by [15], for any cdf F of continuous distribution and any real numbers $\lambda_1, \ldots, \lambda_m$, where $\lambda_k \in [-1, 1]$ for any $k \in \{1, \ldots, m\}$, we obtain the cdf:
$$G(x) = \frac{\sum_{k=1}^{m} \left((1 + \lambda_k)F(x) - \lambda_k(F(x))^2\right)}{m - 1 + \prod_{k=1}^{m} ((1 + \lambda_k)F(x) - \lambda_k(F(x))^2)^{\delta_k}}.$$

Remark 2.1. *Others interesting combinations are possible according to the problem. Thanks to their adaptability, with a specific $F(x)$, these cdfs are of interest from the theoretical and applied aspects.*

3. A particular case with some related new distributions

Let us now focus our attention on a simple configuration already mentioned. If we chose $m = 2$ and $\delta_1 = \delta_2 = 1$, then the GM transformation is reduced to the following form
$$G(x) = \frac{F_1(x) + F_2(x)}{1 + F_1(x)F_2(x)}.$$
The main difference with $G(x)$ and the cdf proposed by [12] is the function F_2 in the denominator introducing more flexibility, and leading new cdf. The associated pdf is given by
$$g(x) = \frac{f_1(x)(1 - (F_2(x))^2) + f_2(x)(1 - (F_1(x))^2)}{(1 + F_1(x)F_2(x))^2}.$$

We now described some of its mathematical properties. The quantile function $Q(x)$ can be obtained via the nonlinear equation:

$$G(Q(x)) = x \quad \Leftrightarrow \quad F_1(Q(x)) + F_2(Q(x)) = x\left(1 + F_1(Q(x))F_2(Q(x))\right).$$

For x such that $F_1(x) \in (0,1)$ and $F_2(x) \in (0,1)$, using the geometric series, we can expand $G(x)$ as

$$G(x) = \sum_{k=0}^{+\infty}(-1)^k \left((F_1(x))^{k+1} (F_2(x))^k + (F_1(x))^k (F_2(x))^{k+1} \right).$$

An expansion of the pdf $g(x)$ is given by $g(x) = \sum_{k=0}^{+\infty}(-1)^k u_k(x)$, where

$$\begin{aligned} u_k(x) &= \big((k+1)\left(F_1(x)F_2(x)\right)^k [f_1(x) + f_2(x)] \\ &+ k\left(F_1(x)F_2(x)\right)^{k-1} \left(f_2(x)\left(F_1(x)\right)^2 + f_1(x)\left(F_2(x)\right)^2\right)\big). \end{aligned}$$

These expansions can be used to determine the r-th moments of a random variable X having the cdf $G(x)$, and other crucial quantities. Their expressions are however long to express in full generality.

Two special cases arising from $G(x)$ and using the usual distributions are described below.

New Two Component Weibull (NTCW) distribution.: Considering the cdf F_1 of the Weibull distribution with parameters $k_1 > 0$ and $\lambda_1 > 0$ and the cdf F_2 of the Weibull distribution with parameters $k_2 > 0$ and $\lambda_2 > 0$. Then we have $F_1(x) = \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)$, $F_2(x) = \left(1 - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}}\right)$,

$$G(x) = \frac{2 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}}}{2 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} + e^{-\left(\frac{x}{\lambda_1}\right)^{k_1} - \left(\frac{x}{\lambda_2}\right)^{k_2}}}$$

and

$$\begin{aligned} g(x) &= \frac{\frac{k_1}{\lambda_1}\left(\frac{x}{\lambda_1}\right)^{k_1-1} e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} \left(1 - \left(1 - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}}\right)^2\right)}{\left(2 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} + e^{-\left(\frac{x}{\lambda_1}\right)^{k_1} - \left(\frac{x}{\lambda_2}\right)^{k_2}}\right)^2} \\ &+ \frac{\frac{k_2}{\lambda_2}\left(\frac{x}{\lambda_2}\right)^{k_2-1} e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} \left(1 - \left(1 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}}\right)^2\right)}{\left(2 - e^{-\left(\frac{x}{\lambda_1}\right)^{k_1}} - e^{-\left(\frac{x}{\lambda_2}\right)^{k_2}} + e^{-\left(\frac{x}{\lambda_1}\right)^{k_1} - \left(\frac{x}{\lambda_2}\right)^{k_2}}\right)^2}, \qquad x > 0. \end{aligned} \tag{2}$$

New Gumbel-Normal (NGN) distribution.: Considering the cdf F_1 of the Gumbel distribution with parameters $\lambda \in \mathbb{R}$ and $\beta > 0$ and the cdf F_2 of the normal distribution with parameters $\lambda \in \mathbb{R}$ and $\sigma > 0$ the corresponding cdf's are expressed as $F_1(x) = e^{-e^{-\frac{x-\lambda}{\beta}}}$, $F_2(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\lambda)^2}{2\sigma^2}} dt = \Phi(x)$, then we have

$$G(x) = \frac{e^{-e^{-\frac{x-\lambda}{\beta}}} + \Phi(x)}{1 + e^{-e^{-\frac{x-\lambda}{\beta}}}\Phi(x)}$$

and

$$g(x) = \frac{\frac{1}{\beta} e^{-e^{-\frac{x-\lambda}{\beta}} - \frac{x-\lambda}{\beta}} \left(1 - (\Phi(x))^2\right) + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\lambda)^2}{2\sigma^2}} \left(1 - e^{-2e^{-\frac{x-\lambda}{\beta}}}\right)}{\left(1 + e^{-e^{-\frac{x-\lambda}{\beta}}} \Phi(x)\right)^2}, \quad x \in \mathbb{R}. \tag{3}$$

The next section is devoted to applications of these two distributions in a statistical setting.

4. **Applications**

We now propose to check the suitability of the parametric models related to the NTCW and NGN distributions. Two data sets representing different scenario of practical life are considered. Estimations of the different parameters are given via the maximum likelihood method and hence the corresponding log-likelihood $\ell(\Theta)$ is evaluated based on equations (2) and (3) for comparing purposes. In order to compute and measure the compatibility of a random sample with a theoretical probability distribution function, our benchmark is the Kolmogorov-Smirnov (KS) goodness of fit statistic.

Application I: NTCW distribution. Here, we present a real data set taken from [1] for comparing the fits of the NTCW distribution with the new generalized Lindley (NGL) distribution proposed by [1], Lindley distribution and Exponential distribution. The data set represents the breaking stress of carbon fibers (in Gba) and consists of the values: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53. The results are summarized in Table I:

Table I: **NTCW distribution with practical application**

Distribution	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\mu}$	λ	$\ell(\hat{\Theta})$	**KS**
NGL	8.485	7.412	2.788	–	–	-91.107	0.117
Lindley	–	–	0.590	–	–	-122.384	0.711
Exponential	–	–	–	–	0.362	-132.9944	0.282
NTCW	$\hat{\alpha_1} = 8.2544$	$\hat{\beta_1} = 10.2118$	$\hat{\alpha_2} = 3.4404$	$\hat{\beta_2} = 3.0624$	–	-86.0677	0.0791

The above inferences indicates that NTCW not only posses the largest log-likelihood $\ell(\hat{\Theta})$ values but also possess the smallest KS statistic as compared to the competing models. Therefore, NTCW distribution is the best for this data from these criteria.

Application II: NGN distribution. The considered data set shows the lowest 7 days average flows in meter cube per second at gauging station La Parota 1963-1999.The values during this period are 19.8, 15.1, 0.3, 19.1, 19.0, 14.4, 17.5, 15.4, 18.9, 16.5,15.3, 19.3, 19.1, 13.0, 16.4, 15.3, 22.3, 17.4, 16.9, 23.2, 13.1,14.2, 17.1, 15.8, 3.2, 13.4, 17.7, 21.5, 9.8, 21.1, 18.7, 15.0,15.2, 9.8, 21.1, 15.7, 11.9, which are reported by [6]. The comparing models includes the generalized Logistic (GLO), the generalized Pareto (GP) and the Gumbel (Gumb.) distributions. For respective density functions, readers are referred to [6]. The results are given in Table II:

Table II: **NGN distribution with practical application**

Distribution	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\beta}$	α	$\ell(\hat{\Theta})$	**KS**
GLO	–	–	0.267	4.241	-125.06	0.743
GP	–	–	0.999	25.25	-119.47	0.3797
Gumb.	–	–	7.604	17.822	-126.92	0.4062
NGN	16.6998	3.4261	2.8310	–	-100.797	0.2273

In view of above results, we can see that the NGN distribution takes the largest log-likelihood $\ell(\hat{\Theta})$ and the smallest KS statistic. This confirms the capability of the NGN distribution for modeling this data set than other compared models.

5. **Conclusion**

In this note, we introduce a new general family of distributions characterized by a cdf $G(x)$ constructed from m existing cdfs $F_1(x), \ldots, F_m(x)$ via a special transformation called the GM transformation. It can be viewed as a flexible version of the M transformation introduced by [12]. Special cases of this family, generating new distributions, are introduced and discussed. Applications are given to two practical data sets. We explore the estimation of the unknown parameters and show that the new distributions can be used quite effectively to provide better fits than some existing distributions.

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REFERENCES

[1] *A. Al-Babtain, A.M. Eid, A.H.N. Ahmed and F. Merovci,* The Five Parameter Lindley Distribution, Pak. J. Statist., **31**(2015), No. 4, 363-384.

[2] *G.R. Aryal and C.P. Tsokos,* On the transmuted extreme value distribution with application, Nonlinear Analysis: Theory, Methods and Applications, **71**(2009), No. 12, 1401-1407.

[3] *G.R. Aryal,* Transmuted log-logistic distribution, Journal of Statistics Applications and Probability, **2**(2013), No. 1, 11-20.

[4] *R.C. Gupta, P.L. Gupta and R.D. Gupta,* Modeling failure time data by Lehman alternatives, Communications in Statistics, Theory and Methods, **27**(1998), No. 4, 887-904.

[5] *R.D. Gupta and D. Kundu,* Generalized Exponential Distributions, Australian and New Zealand Journal of Statistics, **41**(1999), No. 2, 173-188.

[6] *T. Hussain, H.S. Bakouch and Z. Iqbal,* A New Probability Model for Hydrologic Events: Properties and Applications, Journal of Agricultural, Biological and Environmental Statistics, **17**(2017), 1-20.

[7] *M.S. Khan and R. King,* Transmuted modified Weibull distribution: A generalization of the modified Weibull probability distribution, Europe. J. of Pure Appl. Math., **6**(2013), No. 1, 66-88.

[8] *M.S. Khan, R. King and I.L. Hudson,* Characterizations of the transmuted inverse Weibull distribution, ANZIAM J., **55**(2014), 197-217.

[9] *D. Kumar, U. Singh and S.K. Singh,* A Method of Proposing New Distribution and its Application to Bladder Cancer Patients Data, J. Stat. Appl. Pro. Lett., (2015)**2**, No. 3, 235- 245.

[10] *D. Kumar, U. Singh and S.K. Singh,* A New Distribution Using Sine Function- Its Application to Bladder Cancer Patients Data, J. Stat. Appl. Pro., (2015)**4**, No. 3, 417- 427.

[11] *D. Kumar, U. Singh S.K. and Singh,* Life Time Distributions: Derived from some Minimum Guarantee Distribution, Sohag J. Math., **4**(2017), No. 1, 7-11.

[12] *D. Kumar, U. Singh, S.K. Singh and S. Mukherjee,* The New Probability Distribution: An Aspect to a Life Time Distribution, Math. Sci. Lett., **6**(2017), No. 1, 35-42.

[13] *S. Nadarajah and S. Kotz,* The Exponentiated Type Distributions, Acta Appl Math, **92**(2006), 97-111.

[14] *S. Nadarajah,* The exponentiated Gumbel distribution with climate application, Environmetrics, **17**(2006), 13-23.

[15] *W.T. Shaw and I.R.C Buckley,* The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, (2009), arXiv preprint arXiv:0901.0434.

[16] *M.H. Tahir and G.M. Cordeiro,* Compounding of distributions: a survey and new generalized classes, Journal of Statistical Distributions and Applications, **3**(2016), No. 1, 1-35.