CONFORMAL ANTI-INVARIANT RIEMANNIAN MAPS TO KÄHLER MANIFOLDS

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We introduce conformal anti-invariant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds and show that they include both anti-invariant submanifolds and anti-invariant Riemannian maps. We give non-trivial examples, investigate the geometry of certain distributions and obtain decomposition theorems for the base manifold. The harmonicity and totally geodesicity of conformal anti-invariant Riemannian maps are also obtained. Moreover, we study weakly umbilical conformal Riemannian maps and obtain a classification theorem for umbilical conformal anti-invariant Riemannian maps.

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1. Introduction

Let $(\mathcal{M}, J_{\mathcal{M}})$ be an almost complex manifold with almost complex structure $J_{\mathcal{M}}$. A totally real submanifold (anti-invariant submanifold) M is a submanifold such that the almost complex structure $J_{\mathcal{M}}$ of the ambient manifold $\mathcal{M}$ carries a tangent space of M into the corresponding normal space of M. A totally real submanifold is called *Lagrangian* if $dim_{\mathbb{R}}M = dim_{\mathbb{C}}\mathcal{M}$. Real curves of Kähler manifolds are examples of totally real submanifolds. The first contribution to the geometry of totally real submanifolds was given in the early $1970'$s [3]. For details, see [13].

As a generalization of isometric immersions and Riemannian submersions, Riemannian maps were introduced in [4] as follows. Let $F : (M, g_{_M}) \longrightarrow (N, g_{_N})$ be a smooth map between Riemannian manifolds such that $0 < rankF < min\{m, n\}$, where $dimM = m$ and $dimN = n$. Then we denote the kernel space of F_* by $kerF_*$ and consider the orthogonal complementary space $\mathcal{H} = (kerF_*)^{\perp}$ to $kerF_*$ in TM. Then the tangent bundle of M has the following decomposition

$$TM = kerF_* \oplus \mathcal{H}.$$

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We denote the range of F_* by $rangeF_*$ and consider the orthogonal complementary space $(rangeF_*)^\perp$ to $rangeF_*$ in the tangent bundle TN of N. Since $rankF < min\{m,n\}$, we always have $(rangeF_*)^\perp$. Thus the tangent bundle TN of N has the following decomposition

$$F^{-1}(TN) = (rangeF_*) \oplus (rangeF_*)^\perp.$$

Now, a smooth map $F : (M_1^m, g_M) \longrightarrow (M_2^n, g_N)$ is called Riemannian map at $p_1 \in M$ if the horizontal restriction $F^h_{*p_1} : (kerF_{*p_1})^\perp \longrightarrow (rangeF_{*p_1})$ is a linear isometry between the inner product spaces $((kerF_{*p_1})^\perp, g_M(p_1) \mid_{(kerF_{*p_1})^\perp})$ and $(rangeF_{*p_1}, g_N(p_2) \mid_{(rangeF_{*p_1})})$, $p_2 = F(p_1)$. Thus F_* satisfies the equation

$$g_N(F_*\tilde{X}, F_*\tilde{Y}) = g_M(\tilde{X}, \tilde{Y}) \tag{1}$$

for $\tilde{X}, \tilde{Y}$ vector fields tangent to $\mathcal{H}$. Indeed, it follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $kerF_* = \{0\}$ and $(rangeF_*)^\perp = \{0\}$. It is known that a Riemannian map is a subimmersion [4] and this fact implies that the rank of the linear map $F_{*p} : T_pM \longrightarrow T_{F(p)}N$ is constant for p in each connected component of M, [1] and [4]. It is also important to note that Riemannian maps satisfy the eikonal equation. Different properties of Riemannian maps have been studied widely by many authors, see: [5], [6], [8], and [9]. Recently, conformal Riemannian maps as a generalization of Riemannian maps have been defined in [12] and the harmonicity of such maps have been also obtained.

On the other hand, as a generalization of totally real submanifolds, anti-invariant Riemannian maps from Riemannian manifolds to almost complex manifolds were defined and studied in [11]. In this paper, we are going to introduce and study conformal anti-invariant Riemannian maps from Riemannian manifolds to almost complex manifolds as a generalization of totally real submanifolds and anti-invariant Riemannian maps.

2. **Preliminaries**

In this section, we recall some basic materials from [2, 14]. A $2n-$dimensional Riemannian manifold $(\mathcal{M}, g, J)$ is called an almost Hermitian manifold if there exists a tensor field J of type $(1,1)$ on $\mathcal{M}$ such that $J^2 = -I$ and

$$g(\tilde{X}, \tilde{Y}) = g(J\tilde{X}, J\tilde{Y}), \quad \forall \tilde{X}, \tilde{Y} \in \Gamma(T\mathcal{M}), \tag{2}$$

where I denotes the identity transformation of $T_p\mathcal{M}$. Consider an almost Hermitian manifold $(\mathcal{M}, g, J)$ and denote by ∇ the Levi-Civita connection on $\mathcal{M}$ with respect to g. Then $\mathcal{M}$ is called a Kähler manifold [14] if J is parallel with respect to ∇, i.e.

$$(\nabla_{\tilde{X}} J)\tilde{Y} = 0, \tag{3}$$

$\forall \tilde{X}, \tilde{Y} \in \Gamma(T\mathcal{M})$.

We now recall the notion of harmonic maps between Riemannian manifolds. Let $(\mathcal{M}, g_{\mathcal{M}})$ and (N, g_N) be Riemannian manifolds and suppose that $\varphi : \mathcal{M} \to N$ is a smooth map between them. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(\tilde{X},\tilde{Y}) = \nabla^{\varphi}_{\tilde{X}}\varphi_*(\tilde{Y}) - \varphi_*(\nabla^{\mathcal{M}}_{\tilde{X}}\tilde{Y}) \quad (4)$$

for $\tilde{X},\tilde{Y} \in \Gamma(TM)$, where ∇^{φ} is the pullback connection. It is known that the second fundamental form is symmetric. The tension field of φ is the section $\tau(\varphi)$ of the pullback bundle $\Gamma(\varphi^{-1}TN)$ defined by$\tau(\varphi) = \operatorname{div}\varphi_* = \sum_{i=1}^{m}(\nabla\varphi_*)(e_i,e_i)$, where $\{e_1,...,e_m\}$ is the orthonormal frame on $\mathcal{M}$. A smooth map φ satisfying $\tau(\varphi)=0$ is called a harmonic map, see [2].

We denote by ∇^2 both the Levi-Civita connection of (N,g_N) and its pullback along F. Then according to [7], for any vector field $\tilde{X}$ on $\mathcal{M}$ and any section V of $(rangeF_*)^{\perp}$, where $(rangeF_*)^{\perp}$ is the subbundle of $F^{-1}(TN)$ with fiber $(F_*(T_p\mathcal{M}))^{\perp}-$ orthogonal complement of $F_*(T_p\mathcal{M})$ for g_N over p, we have $\nabla^{F\perp}_{\tilde{X}}V$ which is the orthogonal projection of $\nabla^2_{\tilde{X}}V$ on $(F_*(T_p\mathcal{M}))^{\perp}-$ such that $\nabla^{F\perp}g_2=0$. We now define A_V as

$$\nabla^2_{\tilde{X}}V = -A_V F_*\tilde{X} + \nabla^{F\perp}_{\tilde{X}}V \quad (5)$$

where $A_V F_*\tilde{X}$ is tangential component (a vector field along F) of $\nabla^2_{\tilde{X}}V$. It is easy to see that $A_V F_*\tilde{X}$ is bilinear in V and F_* and $A_V F_*\tilde{X}$ at p depends only on V_p and $F_{*p}\tilde{X}_p$. By direct computations, we obtain

$$g_2(A_V F_*\tilde{X}, F_*\tilde{Y}) = g_2(V, (\nabla F_*)(\tilde{X},\tilde{Y})) \quad (6)$$

for $\tilde{X},\tilde{Y} \in \Gamma((kerF_*)^{\perp})$ and $V \in \Gamma((rangeF_*)^{\perp})$. Since (∇F_*) is symmetric, it follows that A_V is a symmetric linear transformation of $rangeF_*$.

3. **Conformal anti-invariant Riemannian maps**

In this section, we define and study conformal anti-invariant Riemannian maps, give examples, investigate the geometry of leaves of the distributions which are defined on the target manifolds. We also give a decomposition theorem and obtain necessary and sufficient conditions for such conformal Riemannian maps to be totally geodesic. We first recall that, in [12], the second author of the present paper showed that the second fundamental form $(\nabla F_*)(\tilde{X},\tilde{Y}), \forall\tilde{X},\tilde{Y} \in \Gamma((\ker F_*)^{\perp})$, of a conformal Riemannian map is in the following form

$$(\nabla F_*)(\tilde{X},\tilde{Y})^{rangeF_*} = \tilde{X}(\ln\lambda)F_*\tilde{Y} + \tilde{Y}(\ln\lambda)F_*\tilde{X} - g_1(\tilde{X},\tilde{Y})F_*(grad\ln\lambda). \quad (7)$$

Thus if we denote the $(rangeF_*)^{\perp}-$ component of $(\nabla F_*)(\tilde{X},\tilde{Y})$ by $(\nabla F_*)(\tilde{X},\tilde{Y})^{(rangeF_*)^{\perp}}$, we can write $(\nabla F_*)(\tilde{X},\tilde{Y})$ as

$$(\nabla F_*)(\tilde{X},\tilde{Y}) = (\nabla F_*)(\tilde{X},\tilde{Y})^{rangeF_*} + (\nabla F_*)(\tilde{X},\tilde{Y})^{(rangeF_*)^{\perp}}, \quad (8)$$

for $\tilde{X},\tilde{Y} \in \Gamma((\ker F_*)^{\perp})$. Hence we have

$$\begin{aligned}(\nabla F_*)(\tilde{X},\tilde{Y}) &= \tilde{X}(\ln\lambda)F_*\tilde{Y} + \tilde{Y}(\ln\lambda)F_*\tilde{X} - g_1(\tilde{X},\tilde{Y})F_*(grad\ln\lambda)\\ &\quad +(\nabla F_*)(\tilde{X},\tilde{Y})^{(rangeF_*)^{\perp}},\end{aligned} \quad (9)$$

We now present the following definition for conformal anti-invariant Riemannian maps as a generalization of totally real submanifolds and anti-invariant Riemannian maps.

Definition 3.1. *Let F be a conformal Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to an almost Hermitian manifold $(\mathcal{M}_2, g_2, J)$. Then we say that F is a conformal anti-invariant Riemannian map at $p \in \mathcal{M}$ if $J(rangeF_*)_p \subseteq (rangeF_{*p})^{\perp}$. If F is a conformal anti-invariant Riemannian map for any $p \in \mathcal{M}$, then F is called a conformal anti-invariant Riemannian map.*

We are going to give some examples of conformal anti-invariant Riemannian maps.

Example 3.1. [13] *Every anti-invariant submanifold of an almost Hermitian manifold is a conformal anti-invariant Riemannian map with $\lambda = 1$ and $kerF_* = \{0\}$.*

Example 3.2. [11] Every anti-invariant Riemannian map from a Riemannian manifold to an almost Hermitian manifold is a conformal anti-invariant Riemannian map with $\lambda = 1$.

We say that a conformal anti-invariant Riemannian map is proper if $\lambda \neq I$. We now present an example of a proper conformal anti-invariant Riemannian map. In the following $\mathbb{R}^{2m}$ denotes the Euclidean $2m$-space with the standard metric. An almost complex structure J on $\mathbb{R}^{2m}$ is said to be compatible if $(\mathbb{R}^{2m}, J)$ is complex analytically isometric to the complex number space $\mathbb{C}^m$ with the standard flat Kählerian metric. We denote by J the compatible almost complex structure on $\mathbb{R}^{2m}$ defined by

$$J(\overline{a}_1, ..., \overline{a}_{2m}) = (-\overline{a}_2, \overline{a}_1, ..., -\overline{a}_{2m}, \overline{a}_{2m-1}).$$

Example 3.3. *Consider the following map defined by*

$$\begin{array}{cccc} F: & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \\ & (\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4) & & (e^{\overline{x}_1}\sin\overline{x}_2, 0, e^{\overline{x}_1}\cos\overline{x}_2, 0). \end{array}$$

We have

$$kerF_* = span\{Z_1 = \partial\overline{x}_3,\ Z_2 = \partial\overline{x}_4\}$$

and

$$(kerF_*)^{\perp} = span\{H_1 = e^{\overline{x}_1}\sin\overline{x}_2\partial\overline{x}_1 + e^{\overline{x}_1}\cos\overline{x}_2\partial\overline{x}_2, H_2 = e^{\overline{x}_1}\cos\overline{x}_2\partial\overline{x}_1 - e^{\overline{x}_1}\sin\overline{x}_2\partial\overline{x}_2\}.$$

By direct computations, we have $rangeF_* = span\{F_*H_1 = e^{2\overline{x}_1}\partial\overline{y}_1,\ F_*H_2 = e^{2\overline{x}_1}\partial\overline{y}_3\}$ and $(rangeF_*)^{\perp} = \{\frac{\partial}{\partial\overline{y}_2}, \frac{\partial}{\partial\overline{y}_4}\}$. It is also easy to check that

$$g_2(F_*H_1, F_*H_1) = e^{2\overline{x}_1}g_1(H_1, H_1)$$

and

$$g_2(F_*H_2, F_*H_2) = e^{2\overline{x}_1}g_1(H_2, H_2),$$

which show that F is a conformal Riemannian map with $\lambda = e^{\overline{x}_1}$. Moreover, it is easy to see that $JF_*H_1 = e^{2\overline{x}_1}\frac{\partial}{\partial \overline{y}_2}$ and $JF_*H_2 = e^{2\overline{x}_1}\frac{\partial}{\partial \overline{y}_4}$, where J is the canonical complex structure of $\mathbb{R}^4$ defined by

$$J(\overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4) = (-\overline{y}_2, \overline{y}_1, -\overline{y}_3, \overline{y}_4).$$

As a result, F is a conformal anti-invariant Riemannian map.

Let F be a conformal anti-invariant Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to an almost Hermitian manifold $(\mathcal{M}_2, g_2, J)$. First of all, from Definition 3.1, we have $J(rangeF_*) \cap (rangeF_*)^\perp \neq \{0\}$. We denote the complementary orthogonal distribution to $J(rangeF_*)$ in $((rangeF_*)^\perp)$ by μ. Then we have

$$(rangeF_*)^\perp = J(rangeF_*) \oplus \mu. \tag{10}$$

It is easy to see that μ is an invariant distribution of $(rangeF_*)^\perp$, under the endomorphism J_2. Thus, for $V \in \Gamma((rangeF_*)^\perp)$, we have

$$JV = \mathcal{B}V + \mathcal{C}V \tag{11}$$

where $\mathcal{B}V \in \Gamma(rangeF_*)$ and $\mathcal{C}V \in \Gamma((rangeF_*)^\perp)$.

We now investigate the geometry of the leaves of $(rangeF_*)$ and $(rangeF_*)^\perp$. First, we give the following result.

Theorem 3.1. *Let F be a conformal anti-invariant Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Kähler manifold $(\mathcal{M}_2, g_2, J)$. Then $(rangeF_*)$ defines a totally geodesic foliation on $\mathcal{M}_2$ if and only if*

$$g_2((\nabla F_*)(\tilde{X}, \tilde{Y}')^{(rangeF_*)^\perp}, JF_*\tilde{Y}) = g_2(\nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}, \mathcal{C}W) \tag{12}$$

for any $W \in \Gamma((rangeF_*)^\perp)$ and $\tilde{X}, \tilde{Y}, \tilde{Y}' \in \Gamma((kerF_*)^\perp)$, such that $F_*\tilde{Y}' = \mathcal{B}V$.

Proof. For $\tilde{X}, \tilde{Y} \in \Gamma((kerF_*)^\perp)$ and $W \in \Gamma((rangeF_*)^\perp)$, using (2) we have

$$g_2(\nabla^2_{\tilde{X}} F_*\tilde{Y}, W) = g_2(\nabla^2_{\tilde{X}} JF_*\tilde{Y}, JW).$$

Thus from (11) we obtain

$$g_2(\nabla^2_{\tilde{X}} F_*\tilde{Y}, W) = -g_2(\nabla^2_{\tilde{X}} F_*\tilde{Y}', JF_*\tilde{Y}) + g_2(\nabla^2_{\tilde{X}} JF_*\tilde{Y}, \mathcal{C}W),$$

where $F_*\tilde{Y}' = \mathcal{B}W$ for $\tilde{Y}' \in \Gamma((kerF_*)^\perp)$. Since F is a conformal Riemannian map, using (4), (5) and (8) we obtain

$$\begin{aligned} g_2(\nabla^2_{\tilde{X}} F_*\tilde{Y}, W) = & -g_2((\nabla F_*)(\tilde{X}, \tilde{Y}')^{rangeF_*} + (\nabla F_*)(\tilde{X}, \tilde{Y}')^{(rangeF_*)^\perp} + F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}} \tilde{Y}'), JF_*\tilde{Y}) \\ & + g_2(-A_{JF_*\tilde{Y}}\tilde{X} + \nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}, \mathcal{C}W). \end{aligned}$$

Hence, we arrive at

$$g_2(\nabla^2_{\tilde{X}} F_*\tilde{Y}, W) = -g_2((\nabla F_*)(\tilde{X}, \tilde{Y}')^{(rangeF_*)^\perp}, JF_*\tilde{Y}) + g_2(\nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}, \mathcal{C}W).$$

From above equation, $(rangeF_*)$ defines a totally geodesic foliation on $\mathcal{M}_2$ if and only if (12) is satisfied. □

In a similar way, we obtain the following Theorem:

Theorem 3.2. *Let F be a conformal anti-invariant Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Kähler manifold $(\mathcal{M}_2, g_2, J)$. Then $(rangeF_*)^\perp$ defines a totally geodesic foliation on $\mathcal{M}_2$ if and only if*

(i) $(rangeF_*)^\perp$ *defines a totally geodesic foliation on* $\mathcal{M}_2$.

(ii) F *is a horizontally homothetic conformal Riemannian map.*

(iii) $$g_2(\mathcal{B}V, A_{\mathcal{C}V}F_*\tilde{X} + F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}}Z')) = -g_2(\mathcal{C}W, (\nabla F_*)(\tilde{X}, Z')^{(rangeF_*)^\perp} + \nabla^{F\perp}_{\tilde{X}}\mathcal{C}V) - g_2(W, [V, F_*\tilde{X}])$$

for any $V, W \in \Gamma((rangeF_*)^\perp)$ and $\tilde{X}, Z' \in \Gamma((kerF_*)^\perp)$ such that $F_*Z' = \mathcal{B}V$.

Proof. For $\tilde{X} \in \Gamma((kerF_*)^\perp)$ and $V, W \in \Gamma((rangeF_*)^\perp)$, since $\mathcal{M}_2$ is a Kähler manifold, using (2) we have

$$g_2(\nabla^2_V W, F_*\tilde{X}) = -g_2(W, [V, F_*\tilde{X}]) - g_2(JW, \nabla^2_{F_*\tilde{X}} JV).$$

Then using from (11), (4) and (5) we obtain

$$\begin{aligned} g_2(\nabla^2_V W, F_*\tilde{X}) = {} & -g_2(W, [V, F_*\tilde{X}]) - g_2(\mathcal{B}W, (\nabla F_*)(\tilde{X}, Z') + F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}} Z')) \\ & - g_2(\mathcal{B}W, -A_{\mathcal{C}V}F_*\tilde{X} + \nabla^{F\perp}_{\tilde{X}}\mathcal{C}V) - g_2(\mathcal{C}W, (\nabla F_*)(\tilde{X}, Z') + F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}} Z')) \\ & - g_2(\mathcal{C}W, -A_{\mathcal{C}V}F_*\tilde{X} + \nabla^{F\perp}_{\tilde{X}}\mathcal{C}V), \end{aligned}$$

where $F_*Z' = \mathcal{B}V \in \Gamma(rangeF_*)$ for $Z' \in \Gamma((kerF_*)^\perp)$. Since F is a conformal Riemannian map, using (8), we arrive at

$$\begin{aligned} g_2(\nabla^2_V W, F_*\tilde{X}) = {} & -g_2(W, [V, F_*\tilde{X}]) - g_2(\mathcal{B}W, (\nabla F_*)(\tilde{X}, Z')^{rangeF_*}) - g_2(\mathcal{B}W, F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}} Z')) \\ & + g_2(\mathcal{B}W, A_{\mathcal{C}V}F_*\tilde{X}) - g_2(\mathcal{C}W, (\nabla F_*)(\tilde{X}, \tilde{Y}')^{(rangeF_*)^\perp}) - g_2(\mathcal{C}W, \nabla^{F\perp}_{\tilde{X}}\mathcal{C}V) \end{aligned}$$

Then from (9), we get

$$\begin{aligned} g_2(\nabla^2_V W, F_*\tilde{X}) = {} & -g_2(W, [V, F_*\tilde{X}]) - g_2(\mathcal{B}W, F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}} Z')) + g_2(\mathcal{B}W, A_{\mathcal{C}V}F_*\tilde{X}) \\ & - g_2(\mathcal{C}W, (\nabla F_*)(\tilde{X}, \tilde{Y}')^{(rangeF_*)^\perp}) - g_2(\mathcal{C}W, \nabla^{F\perp}_{\tilde{X}}\mathcal{C}V) \\ & - g_2(\mathcal{B}W, \tilde{X}(\ln\lambda)F_*Z' + Z'(\ln\lambda)F_*\tilde{X} - g_1(\tilde{X}, Z')F_*(grad\ln\lambda)) \end{aligned}$$

or

$$\begin{aligned} g_2(\nabla^2_V W, F_*\tilde{X}) = {} & -g_2(W, [V, F_*\tilde{X}]) - g_2(\mathcal{B}W, F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}} Z')) + g_2(\mathcal{B}W, A_{\mathcal{C}V}F_*\tilde{X}) \\ & - g_2(\mathcal{C}W, (\nabla F_*)(\tilde{X}, \tilde{Y}')^{(rangeF_*)^\perp}) - g_2(\mathcal{C}W, \nabla^{F\perp}_{\tilde{X}}\mathcal{C}V) \\ & - g_1(\tilde{X}, grad\ln\lambda)g_2(\mathcal{B}W, F_*Z') - g_1(Z', grad\ln\lambda)g_2(\mathcal{B}W, F_*\tilde{X}) \\ & + g_1(\tilde{X}, Z')g_2(\mathcal{B}W, F_*(grad\ln\lambda)) \end{aligned}$$

Hence we have

$$\begin{aligned} g_2(\nabla^2_V W, F_*\tilde{X}) = & -g_2(W,[V,F_*\tilde{X}]) - g_2(\mathcal{B}W, F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}} Z')) + g_2(\mathcal{B}W, A_{\mathcal{C}V} F_*\tilde{X}) \\ & - g_2(\mathcal{C}W, (\nabla F_*)(\tilde{X},\tilde{Y}')^{(rangeF_*)^\perp}) - g_2(\mathcal{C}W, \nabla^{F\perp}_{\tilde{X}} \mathcal{C}V) \\ & - g_1(\tilde{X}, \mathcal{H}grad\ln\lambda) g_2(\mathcal{B}W, F_*Z') - g_1(Z', \mathcal{H}grad\ln\lambda) g_2(\mathcal{B}W, F_*\tilde{X}) \\ & + g_1(\tilde{X}, Z') g_2(\mathcal{B}W, F_*(grad\ln\lambda)) \end{aligned}$$

From above equation, we can conclude that the two assertions in Theorem 3.2 imply the third. □

We now recall the following characterization for locally (usual) product Riemannian manifold from [10]. Let g be a Riemannian metric tensor on the manifold $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ and assume that the canonical foliations $D_{\mathcal{M}_1}$ and $D_{\mathcal{M}_2}$ intersect perpendicularly everywhere. Then g is the metric tensor of a usual product of Riemannian manifolds if and only if $D_{\mathcal{M}_1}$ and $D_{\mathcal{M}_2}$ are totally geodesic foliations. From Theorem 3.1 and Theorem 3.2, we have the following theorem;

Theorem 3.3. *Let F be a horizontally homothetic conformal anti-invariant Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Kähler manifold $(\mathcal{M}_2, g_2, J)$. Then the base manifold is a locally product manifold $\mathcal{M}_{2(rangeF_*)} \times \mathcal{M}_{2(rangeF_*)^\perp}$ if and only if*

$$g_2((\nabla F_*)(\tilde{X},\tilde{Y}')^{(rangeF_*)^\perp}, JF_*\tilde{Y}) = g_2(\nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}, \mathcal{C}V)$$

and

$$\begin{aligned} g_2(\mathcal{B}V, A_{\mathcal{C}V}F_*\tilde{X} + F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}} Z')) = & -g_2(\mathcal{C}W, (\nabla F_*)(\tilde{X}, Z')^{(rangeF_*)^\perp} + \nabla^{F\perp}_{\tilde{X}} \mathcal{C}V) \\ & - g_2(W,[V,F_*\tilde{X}]) \end{aligned}$$

for any $V, W \in \Gamma((rangeF_*)^\perp)$ and $\tilde{X}, \tilde{Y}, \tilde{Y}', Z' \in \Gamma((kerF_*)^\perp)$ such that $F_*\tilde{Y}' = \mathcal{B}W$ and $F_*Z' = \mathcal{B}V$.

In the sequel we are going to investigate the harmonicity of conformal anti-invariant Riemannian map. We first have the following general result.

Theorem 3.4. *Let F be a conformal anti-invariant Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Kähler manifold $(\mathcal{M}_2, g_2, J)$. Then F is harmonic if and only if the following conditions are satisfied;*

(a) *the fibres are minimal,*
(b) $trace\mathcal{B}\nabla^{F\perp}_{(.)} JF_*(.) - F_*(\nabla^{\mathcal{M}_1}_{(.)}(.)) = 0,$
(c) $traceJA_{JF_*(.)}(.) - \mathcal{C}\nabla^{F\perp}_{(.)} JF_*(.) = 0.$

Proof. For $U \in \Gamma(kerF_*)$, using (4), we have

$$(\nabla F_*)(U,U) = -F_*(\nabla^{\mathcal{M}_1}_U U). \qquad (13)$$

For $\tilde{X} \in \Gamma((kerF_*)^\perp)$, using (4) and (3), we have

$$(\nabla F_*)(\tilde{X},\tilde{X}) = \nabla^2_{\tilde{X}} F_*\tilde{X} - F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{X}) = -J\nabla^2_{\tilde{X}} JF_*\tilde{X} - F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{X}).$$

From (5),(8) and (11) we obtain

$$(\nabla F_*)(\tilde{X},\tilde{X})^{(rangeF_*)} + (\nabla F_*)(\tilde{X},\tilde{X})^{(rangeF_*)^\perp} = JA_{JF_*\tilde{X}}\tilde{X} - \mathcal{B}\nabla^{F\perp}_{\tilde{X}}JF_*\tilde{X} - \mathcal{C}\nabla^{F\perp}_{\tilde{X}}JF_*\tilde{X} - F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{X}). \quad (14)$$

Then taking the $(rangeF_*)-$ components and $((rangeF_*)^\perp)-$ components of above expression (14), we arrive at

$$(\nabla F_*)(\tilde{X},\tilde{X})^{(rangeF_*)} = -\mathcal{B}\nabla^{F\perp}_{\tilde{X}}JF_*\tilde{X} - F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{X}) \quad (15)$$

and

$$(\nabla F_*)(\tilde{X},\tilde{X})^{(rangeF_*)^\perp} = JA_{JF_*\tilde{X}}\tilde{X} - \mathcal{C}\nabla^{F\perp}_{\tilde{X}}JF_*\tilde{X}. \quad (16)$$

Then proof follows from (13), (15) and (16). □

Definition 3.2. *Let F be a conformal Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$. Then we say that F is a horizontally homothetic conformal Riemannian map if the gradient of its dilation λ is vertical, i.e., $\mathcal{H}(grad\lambda) = 0$.*

From Theorem 3.4, we have the following result.

Corollary 3.1. *Let $F : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, g_2, J)$ be a conformal anti-invariant Riemannian map such that $n \neq \frac{2}{\lambda^2}$, where $(\mathcal{M}_1, g_1)$ is a Riemannian manifold and $(\mathcal{M}_2, g_2, J)$ is a Kähler manifold. If F satisfies*

$$trace\mathcal{B}\nabla^{F\perp}_{(.)}JF_*(.) - F_*(\nabla^{\mathcal{M}_1}_{(.)}(.)) = 0,$$

then F is a horizontally homothetic conformal Riemannian map.

We recall that a differentiable map F between Riemannian manifold $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ is called a totally geodesic map if $(\nabla F_*)(\tilde{X},\tilde{Y}) = 0$ for all $\tilde{X},\tilde{Y} \in \Gamma(TM_1)$. We have the following theorem.

Theorem 3.5. *Let F be a conformal anti-invariant Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Kähler manifold $(\mathcal{M}_2, g_2, J)$. Then F is totally geodesic if and only if*

(a) $g_2(B\nabla^{F\perp}_{\tilde{X}}JF_*\tilde{Y}_2, F_*(Z)) = \lambda^2 g_1(\nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{Y}, Z)$

(b) $JA_{JF_*\tilde{Y}_2}\tilde{X} = C\nabla^{F\perp}_{\tilde{X}}JF_*\tilde{Y}_2$

for any $\tilde{X},\tilde{Y} = \tilde{Y}_1 + \tilde{Y}_2, Z \in \Gamma(TM_1)$, where $\tilde{Y}_1 \in \Gamma(kerF_*), \tilde{Y}_2 \in \Gamma((kerF_*)^\perp)$.

Proof. For $\tilde{X},\tilde{Y} \in \Gamma(TM_1)$ and $\tilde{Y}_1 \in \Gamma(kerF_*), \tilde{Y}_2 \in \Gamma((kerF_*)^\perp)$, using (4), (3) and (5), we have

$$(\nabla F_*)(\tilde{X},\tilde{Y}) = -J(-A_{JF_*\tilde{Y}_2}\tilde{X} + \nabla^{F\perp}_{\tilde{X}}JF_*\tilde{Y}_2) - F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{Y}).$$

Then from (11) we get

$$(\nabla F_*)(\tilde{X},\tilde{Y}) = JA_{JF_*\tilde{Y}_2}\tilde{X} - B\nabla^{F\perp}_{\tilde{X}}JF_*\tilde{Y}_2 - C\nabla^{F\perp}_{\tilde{X}}JF_*\tilde{Y}_2 - F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{Y}).$$

Since F is conformal Riemannian map, using (8), we get

$$(\nabla F_*)(\tilde{X},\tilde{Y})^{rangeF_*} + (\nabla F_*)(\tilde{X},\tilde{Y})^{(rangeF_*)^\perp} = JA_{JF_*\tilde{Y}_2}\tilde{X} - B\nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}_2 - C\nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}_2 - F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{Y}).$$

Then taking the $(rangeF_*)$ and $((rangeF_*)^\perp)$ components we arrive at

$$(\nabla F_*)(\tilde{X},\tilde{Y})^{(rangeF_*)} = B\nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}_2 + F_*(\nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{Y})$$

and

$$(\nabla F_*)(\tilde{X},\tilde{Y})^{(rangeF_*)^\perp} = JA_{JF_*\tilde{Y}_2}\tilde{X} - C\nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}_2.$$

Thus $(\nabla F_*)(\tilde{X},\tilde{Y}) = 0$ if and only if $(\nabla F_*)(\tilde{X},\tilde{Y})^{rangeF_*} = 0$ and $(\nabla F_*)(\tilde{X},\tilde{Y})^{(rangeF_*)^\perp} = 0$. Hence we have

$$g_2(\mathcal{B}\nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}_2, F_*(Z)) = -\lambda^2 g_1(\nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{Y}, Z)$$

and

$$JA_{JF_*\tilde{Y}_2}\tilde{X} - C\nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}_2 = 0,$$

which complete the proof. □

We also have the following result for totally geodesic conformal anti-invariant Riemannian maps.

Theorem 3.6. *Let F be a conformal anti-invariant Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Kähler manifold $(\mathcal{M}_2, g_2, J)$. Then F is totally geodesic if and only if*

(a) *The horizontal distribution $(kerF_*)^\perp$ defines a totally geodesic foliation on $\mathcal{M}_1$.*
(b) *all the fibres $F^{-1}(y)$ are totally geodesic for $y \in \mathcal{M}_2$.*
(c) *$(rangeF_*)^\perp$ defines a totally geodesic foliation on $\mathcal{M}_2$.*

for any $\tilde{X},\tilde{Y} \in \Gamma(kerF_*)^\perp$ and $V \in \Gamma(rangeF_*)$.

Proof. For $\tilde{X},\tilde{Y} \in \Gamma(kerF_*)^\perp$ and $U \in \Gamma(kerF_*)$, using (4), we have

$$g_2((\nabla F_*)(\tilde{X},U), F_*\tilde{Y}) = -\lambda^2 g_1(\nabla^{\mathcal{M}_1}_{\tilde{X}} U, \tilde{Y}).$$

Since $\nabla^{\mathcal{M}_1}$ is a Levi-Civita connection, we obtain

$$g_2((\nabla F_*)(\tilde{X},U), F_*\tilde{Y}) = \lambda^2 g_1(U, \nabla^{\mathcal{M}_1}_{\tilde{X}}\tilde{Y}),\ (\lambda \neq 0).$$

Hence $(\nabla F_*)(\tilde{X},U) = 0$ for $\tilde{X} \in \Gamma(kerF_*)^\perp$ and $U \in \Gamma(kerF_*)$ if and only if (a).

For $U, V \in \Gamma(kerF_*)$ and $\tilde{X} \in \Gamma(kerF_*)^\perp$, we have

$$g_2((\nabla F_*)(U,V), F_*\tilde{X}) = -\lambda^2 g_1(\nabla^{\mathcal{M}_1}_U V, \tilde{Y}),\ (\lambda \neq 0)$$

Thus $(\nabla F_*)(U,V) = 0$ for $U, V \in \Gamma(kerF_*)$ if and only if (b).

For $\tilde{X},\tilde{Y} \in \Gamma(kerF_*)^\perp$ and $V \in \Gamma(rangeF_*)$, since $\mathcal{M}_2$ is a Kähler manifold, using (2), (4), (11) we have

$$g_2((\nabla F_*)(\tilde{X},\tilde{Y}), V) - g_2(\nabla^2_{\tilde{X}} F_*\tilde{Y}', JF_*\tilde{Y}) + g_2(\nabla^2_{\tilde{X}} JF_*\tilde{Y}, \mathcal{C}V),$$

where $F_*\tilde{Y}' = \mathcal{B}V$ for $\tilde{Y}' \in \Gamma((kerF_*)^\perp)$. Since F is a conformal Riemannian map, using (4), (5) and (8) we obtain

$$g_2((\nabla F_*)(\tilde{X},\tilde{Y}),V) = -g_2((\nabla F_*)(\tilde{X},\tilde{Y}')^{(rangeF_*)^\perp}, JF_*\tilde{Y}) + g_2(\nabla^{F\perp}_{\tilde{X}} JF_*\tilde{Y}, \mathcal{C}V).$$

Thus, $(\nabla F_*)(\tilde{X},\tilde{Y}) = 0$ for $\tilde{X},\tilde{Y} \in \Gamma((kerF_*)^\perp)$ if and only if (c). □

4. Umbilical conformal anti-invariant Riemannian maps

In this section, we investigate the umbilical case for the conformal anti-invariant Riemannian maps. We first recall the following definition.

Definition 4.1. [7] *Let F be a map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$. Then F is called a weakly g_1-umbilical if there exist*

(1) *a field ξ along F, nowhere 0, with values in $(kerF_*)^\perp$,*
(2) *a field Z on $\mathcal{M}$ such that for every $\tilde{X},\tilde{Y}$ on $\Gamma(TM)$ we have*

$$(\nabla F_*)(\tilde{X},\tilde{Y}) = g_1(\tilde{X},\tilde{Y})[F_*(Z)+\xi]. \tag{17}$$

F is called strong g_1-umbilical if $Z = 0$.

Using the above definition, we can give the following theorem.

Theorem 4.1. *Let F be a g_1-umbilical conformal Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Riemannian manifold $(\mathcal{M}_2, g_2)$ such that $dim(\mathcal{H}) \geq 2$. Then F is a totally geodesic map.*

Proof. We suppose that F is a weakly g_1- umbilical conformal Riemannian map such that $dim(\mathcal{H}) \geq 2$. Then from (9) and (17) we have

$$\tilde{X}(\ln\lambda)F_*\tilde{Y} + \tilde{Y}(\ln\lambda)F_*\tilde{X} - g_1(\tilde{X},\tilde{Y})F_*(grad\ln\lambda) = g_1(\tilde{X},\tilde{Y})F_*Z \tag{18}$$

and

$$(\nabla F_*)(\tilde{X},\tilde{Y})^{(rangeF_*)^\perp} = g_1(\tilde{X},\tilde{Y})\xi. \tag{19}$$

Since $dim(\mathcal{H}) \geq 2$, we can choose $\tilde{X}$ and $\tilde{Y}$ such that $g_1(\tilde{X},\tilde{Y}) = 0$. Then we get

$$\tilde{X}(\ln\lambda)F_*\tilde{Y} + \tilde{Y}(\ln\lambda)F_*\tilde{X} = 0.$$

Since $\tilde{X}$ and $\tilde{Y}$ are orthogonal and F is a conformal Riemannian map, we have

$$g_2(F_*\tilde{X}, F_*\tilde{Y}) = \lambda^2 g_1(\tilde{X},\tilde{Y}) = 0.$$

$F_*\tilde{X}$ and $F_*\tilde{Y}$ are also orthogonal. Then we get

$$\tilde{X}(\ln\lambda)F_*\tilde{Y} = 0, \quad \tilde{Y}(\ln\lambda)F_*\tilde{X} = 0.$$

Thus F is a horizontally homothetic Riemannian map. Since F is horizontally homothetic, from (18), we get $Z = 0$. Thus $(\nabla F_*)(\tilde{X},\tilde{Y}) = g_1(\tilde{X},\tilde{Y})\xi$ for $\tilde{X},\tilde{Y} \in \Gamma(TM)$. In particular, for $U,V \in \Gamma(kerF_*)$, we get

$$-F_*(\nabla_U V) = g_1(U,V)\xi.$$

The right side of this equation belongs to $\Gamma((rangeF_*)^\perp)$ while the left side of this equation belongs to $\Gamma(rangeF_*)$. Hence $F_*(\nabla_U V) = 0$ and $\xi = 0$ which proves our assertion. □

From Theorem 3.6 and Theorem 4.1, we have the following result.

Corollary 4.1. *Let F be a g_1-umbilical conformal anti-invariant Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Kähler manifold $(\mathcal{M}_2, g_2, J)$ such that $dim(\mathcal{H}) \geq 2$. Then we have the following assertions:*

(a) *The horizontal distribution $(kerF_*)^\perp$ defines a totally geodesic foliation on $\mathcal{M}_1$.*
(b) *all the fibres $F^{-1}(y)$ are totally geodesic for $y \in \mathcal{M}_2$.*
(c) *$(rangeF_*)^\perp$ defines a totally geodesic foliation on $\mathcal{M}_2$.*

From the above Theorem 4.1, we can give the following;

Theorem 4.2. *Let $F : (\mathcal{M}_1, g_1) \to (\mathcal{M}_2, J, g_2)$ be a g_1-umbilical conformal anti-invariant Riemannian map from a Riemannian manifold $(\mathcal{M}_1, g_1)$ to a Kähler manifold $(\mathcal{M}_2, g_2, J)$. Then at least one of the following is satisfied:*

(a) *The horizontal distribution $(kerF_*)^\perp$ is 1 dimensional distribution.*
(b) *F is a totally geodesic conformal Riemannian map.*

Proof. We suppose that F is not a totally geodesic g_1- umbilical conformal Riemannian map. Then for $w_1, w_2 \in \Gamma((kerF_*)^\perp)$, since $\mathcal{M}_2$ is a Kähler manifold, using (6), (4) and (17) we obtain

$$-A_{JF_*(w_1)}F_*(w_2) + \nabla^\perp_{F_*(w_2)} JF_*(w_1) = g_1(w_1, w_2)J\xi + g_1(w_1, w_2)JF_*(Z) + JF_*(\nabla^1_{w_2} w_1).$$

Taking inner product with $F_*(w_2)$ in the above equation, we get

$$-g_2(A_{JF_*(w_1)}F_*(w_2), F_*(w_2) = -g_1(w_1, w_2)g_2(\xi, JF_*(w_2)). \tag{20}$$

From (6), (17) and (20), we get

$$g_1(w_2, w_2)g_2(\xi, JF_*(w_1)) = g_1(w_1, w_2)g_2(\xi, JF_*(w_2)). \tag{21}$$

Interchanging the role of w_1 and w_2 in (21), we obtain

$$g_1(w_1, w_1)g_2(\xi, JF_*(w_2)) = g_1(w_1, w_2)g_2(\xi, JF_*(w_1)). \tag{22}$$

From (21) and (22), we get

$$g_2(\xi, JF_*(w_2)) = \frac{g_1(w_1, w_2)^2}{g_1(w_1, w_1)g_1(w_2, w_2)} g_2(\xi, JF_*(w_2)). \tag{23}$$

From (23), w_1 and w_2 are linear dependent, which gives the proof. □

5. **Conclusions**

In this paper, we just introduce a general Riemannian map from a Riemannian manifold to an almost Hermitian manifold. From the theory of submanifolds of almost Hermitian manifolds, one can see that there are many new research problems to be investigated.

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