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ON WEAK λ-STATISTICAL CONVERGENCE OF ORDER α

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In the present paper, we give the concept of weak λ-statistical convergence of order α and weakly $[V,\lambda]$-summability of order α. Some relations between of these concepts and also some relations between weak λ-statistical convergence of order α and strong λ-statistical convergence of order α are examined.

Keywords: Weak statistical convergence, Cesàro Summability.

1. Introduction

Zygmund [1] introduced the idea of statistical convergence in the first edition of his monograph published in Warsaw in 1935. The notion of statistical convergence of sequences of real numbers was introduced by Steinhaus [2] and Fast [3]. Later Schoenberg [4] introduced the concept of statistical convergence, independently. In past years, different authors studied properties of statistical convergence and applied this concept in various areas such as number theory, ergodic theory, Fourier analysis, measure theory, trigonometric series, Fuzzy set theory, interval analysis etc.

The statistical convergence depends on density of subsets of $\mathbb{N}$. The natural density of $K \subset \mathbb{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left| \{ k \le n \,:\, k \in K \} \right|$$

where $\left| \{ k \le n \,:\, k \in K \} \right|$ denotes the number of elements of K not exceeding n [5]. Any finite subset of $\mathbb{N}$ have zero natural density and $\delta(K^c) = 1 - \delta(K)$.

A sequence $x = (x_k)$ of numbers is said to be statistically convergent to a number l provided that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \{ k \le n \,:\, |x_k - l| \ge \varepsilon \} \right| = 0.$$

In this case, we write $S - \lim_k x_k = l$.

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Leinder [6] introduced (V,λ)-summability with the help of sequence $\lambda=(\lambda_n)$ as follows: Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ with $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$. The generalized de la Vallee-Poussin mean is defined by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k$$

where $I_n=[n-\lambda_n+1,n]$. A sequence $x=(x_k)$ of numbers is said to be (V,λ)-summable to a number l if $t_n(x)\to l$ as $n\to\infty$.

$$[C,1]=\left\{x=(x_k)\ :\exists l\in \mathrm{R},\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|x_k-l|=0\right\}$$

and

$$[V,\lambda]=\left\{x=(x_k)\ :\exists l\in \mathrm{R},\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}|x_k-l|=0\right\}$$

denotes the sets of sequences $x=(x_k)$ which are strongly Cesáro summable and strongly (V,λ)-summable to l. It is being noted that for $\lambda_n=n$, (V,λ)-summability reduces to $(C,1)$-summability.

Mursaleen [7] introduced the λ-density of $K\subset\mathbb{N}$ is defined by

$$\delta_\lambda(K)=\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{n-\lambda_n+1\leq k\leq n\ :\ k\in K\}\right|$$

and λ-statistical convergence as follows:

A sequence $x=(x_k)$ of numbers is said to be λ-statistically convergent to a number l provided that for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{n-\lambda_n+1\leq k\leq n\ :\ |x_k-l|\geq\varepsilon\}\right|=0.$$

In this case, the number l is called λ-statistical limit of the sequence $x=(x_k)$.

The statistical convergence with degree $0<\beta<1$ was introduced by Gadjiev and Orhan in [8]. Then the statistical convergence of order α and strong p-Cesàro summability of order α were studied by Çolak in [9]. Also Çolak and Bektaş introduced λ- statistical convergence of order α in [10], as follows:

Let the sequence $\lambda=(\lambda_n)$ of real numbers be defined as above and $0<\alpha\leq 1$. The sequence $x=(x_k)\in w$ is said to be λ-statistically convergent of order α if there is a complex number l such that

$$\lim_{n\to\infty}\frac{1}{\lambda_n^{\alpha}}\left|\left\{k\in I_n\ :\ |x_k-l|\geq\varepsilon\right\}\right|=0,$$

where $I_n=[n-\lambda_n+1,n]$. λ_n^{α} denote the αth power $(\lambda_n)^{\alpha}$ of λ_n, that is, $\lambda^{\alpha}=(\lambda_n^{\alpha})=(\lambda_1^{\alpha},\lambda_2^{\alpha},...,\lambda_n^{\alpha},...)$.

A sequence (x_k) in a normed space X is said to be weakly convergent to $l\in X$ provided that $\lim_{k\to\infty}\phi(x_k-l)=0$ for each $\phi\in X^*$, the continuous dual of X. In this case, we write $W-\lim_k x_k=l$.

Et *et al.* introduced the concept $\hat{S}_{\lambda}^{\alpha}$-statistical convergence of order α in [11]. Connor *et al.* introduced weak statistical convergence and used it to give description of Banach spaces with seperable duals in [12].

A sequence (x_k) in a normed space X is said to be weakly statistically convergent to $l\in X$ provided that, for each $\varepsilon>0$, $\delta\left(\left\{k\leq n\ :\ |\phi(x_k-l)|\geq\varepsilon\right\}\right)=0$ for each $\phi\in X^*$, the continuous dual of X. In this case, we write $WS-\lim_k x_k=l$. The set of all weak statisically convergent sequences is denoted by WS.

Bhardwaj *et al.* defined weak statistical Cauchy sequences in a normed space X and studied weak statistical convergence in l_p spaces in [13]. Meenakshi *et al.* studied weak λ- statistical convergence, weak λ-statistically Cauchy and weak (V,λ)- summability in a normed space X in [14]. Throughout this paper we denote the class of all decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$ by Λ. Also, unless stated otherwise, by "for all $n\in\mathrm{N}_{n_0}$" we mean "for all $n\in\mathrm{N}$ except finite numbers of positive integers" where $\mathrm{N}_{n_0}=\{n_0,n_0+1,n_0+2,...\}$ for some $n_0\in\mathrm{N}=\{1,2,3,...\}$.

2. Main Results

In this section we give the main results of the paper. X will denote a normed linear space; X^* its continuous dual and by a sequence $\lambda=(\lambda_n)$ we mean a non-decrasing sequence tending to ∞ with $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$.

Definition 2.1. A sequence (x_k) in X is said to be weakly $[V,\lambda]$-summable of order α to $l \in X$ provided that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} \sum_{k\in I_n} \left|\phi\left(x_k - l\right)\right| = 0,$$

for each $\phi \in X^*$ where $\alpha \in (0,1]$. In this case, we write $W^{\alpha}[V,\lambda] - \lim_k x_k = l$. The set of all weakly $[V,\lambda]$-summable of order α sequences will be denoted by $W^{\alpha}[V,\lambda]$.

Definition 2.2. Let the sequence $\lambda = (\lambda_n)$ of real numbers be defined as above and $0 < \alpha \le 1$ be given. The sequence $x = (x_k) \in w$ is said to be weakly λ-statistically convergent of order α if there is $l \in X$ if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n \ : \ \left|\phi\left(x_k - l\right)\right| \ge \varepsilon\right\}\right| = 0,$$

where $I_n = \left[n - \lambda_n + 1, n\right]$ and for every $\phi \in X^*$. In this case we write $WS_{\lambda}^{\alpha} - \lim_k x_k = l$. The set of all weakly λ- statistically convergent of order α sequences will be denoted by WS_{λ}^{α}.

The weakly λ-statistically convergence of order α is same with the weakly λ-statistically convergence, that is $WS_{\lambda}^{\alpha} = WS_{\lambda}$ for $\alpha = 1$.

Definition 2.3. A sequence (x_k) in X is said to be strongly λ-statistically convergent of order α to $l \in X$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n \ : \ \left\|x_k - l\right\| \ge \varepsilon\right\}\right| = 0$$

where $\alpha \in (0,1]$. In this case, we write $S_{\lambda}^{\alpha} - \lim_k x_k = l$ or $x_k \xrightarrow{S_{\lambda}^{\alpha}} l$. The set of all strongly λ-statistically convergent of order α sequences will be denoted by S_{λ}^{α}.

Theorem 2.4. Let $0 < \alpha \le 1$ and $x = (x_k)$, $y = (y_k)$ be sequences of complex numbers.

(i) For any sequence (x_k) in X, if $WS_{\lambda}^{\alpha} - \lim x_k = l$, then l must be unique.

(ii) If $WS_{\lambda}^{\alpha} - \lim x_k = l_0$ and c being a scalar, then $WS_{\lambda}^{\alpha} - \lim cx_k = cl_0$.

(iii) If $WS_{\lambda}^{\alpha} - \lim x_k = l_0$ and $WS_{\lambda}^{\alpha} - \lim y_k = l_1$, then $WS_{\lambda}^{\alpha} - \lim(x_k + y_k) = l_0 + l_1$.

Theorem 2.5. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \le \mu_n$ for all $n \in \mathrm{N}_{n_0}$ and let α and β be such that $0 < \alpha \le \beta \le 1$.

(i) If

$$\lim_{n\to\infty}\inf\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}>0 \qquad (1)$$

then $WS_{\mu}^{\beta}\subseteq WS_{\lambda}^{\alpha}$,

(ii) If

$$\lim_{n\to\infty}\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}=1 \text{ and } \lim_{n\to\infty}\frac{\mu_n}{\mu_n^{\beta}}=1 \qquad (2)$$

then $WS_{\lambda}^{\alpha}=WS_{\mu}^{\beta}$.

Proof. (i) Let $\lambda_n\le\mu_n$ for all $n\in\mathrm{N}_{n_0}$ and let (1) be satisfied. Since $I_n\subset J_n$, for given $\varepsilon>0$ we have

$$\left\{k\in J_n\ :\ \left|\phi\left(x_k-l\right)\right|\ge\varepsilon\right\}\supset\left\{k\in I_n\ :\ \left|\phi\left(x_k-l\right)\right|\ge\varepsilon\right\}$$

and so

$$\frac{1}{\mu_n^{\beta}}\left|\left\{k\in J_n\ :\ \left|\phi\left(x_k-l\right)\right|\ge\varepsilon\right\}\right|\ge\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}.\frac{1}{\lambda_n^{\alpha}}\left|\left\{k\in I_n\ :\ \left|\phi\left(x_k-l\right)\right|\ge\varepsilon\right\}\right|$$

for all $n\in\mathrm{N}_{n_0}$, where $J_n=[n-\mu_n+1,n]$. By (1) as $n\to\infty$ we get $WS_{\mu}^{\beta}\subseteq WS_{\lambda}^{\alpha}$.

(ii) Let $(x_k)\in WS_{\lambda}^{\alpha}$ and (2) hold. Since $I_n\subset J_n$, for $\varepsilon>0$ we have

$$\begin{aligned}\frac{1}{\mu_n^{\beta}}\left|\left\{k\in J_n\ :\ \left|\phi\left(x_k-l\right)\right|\ge\varepsilon\right\}\right| &=\frac{1}{\mu_n^{\beta}}\left|\left\{n-\mu_n+1\le k\le n-\lambda_n\ :\ \left|\phi\left(x_k-l\right)\right|\ge\varepsilon\right\}\right|\\ &\quad+\frac{1}{\mu_n^{\beta}}\left|\left\{k\in I_n\ :\ \left|\phi\left(x_k-l\right)\right|\ge\varepsilon\right\}\right|\\ &\le\left(\frac{\mu_n-\lambda_n}{\mu_n^{\beta}}\right)+\frac{1}{\mu_n^{\beta}}\left|\left\{k\in I_n\ :\ \left|\phi\left(x_k-l\right)\right|\ge\varepsilon\right\}\right|\\ &\le\left(\frac{\mu_n-\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)+\frac{1}{\mu_n^{\beta}}\left|\left\{k\in I_n\ :\ \left|\phi\left(x_k-l\right)\right|\ge\varepsilon\right\}\right|\\ &\le\left(\frac{\mu_n}{\mu_n^{\beta}}-\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)+\frac{1}{\lambda_n^{\alpha}}\left|\left\{k\in I_n\ :\ \left|\phi\left(x_k-l\right)\right|\ge\varepsilon\right\}\right|\end{aligned}$$

for all $n\in\mathrm{N}_{n_0}$. Since $\lim_n\frac{\mu_n}{\mu_n^{\beta}}=1$ and $\lim_n\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}=1$ by (2) right hand side of last inequality tend to 0 as $n\to\infty$ ($\frac{\mu_n}{\mu_n^{\alpha}}-\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\ge 0$ for all $n\in\mathbb{N}_{n_0}$). This means that $WS_{\lambda}^{\alpha}\subset WS_{\mu}^{\beta}$. Since (2) implies (1) we have $WS_{\lambda}^{\alpha}=WS_{\mu}^{\beta}$.

Theorem 2.6. Let $\lambda=(\lambda_n)\in\Lambda$, $\alpha\in(0,1]$ be any real number and $\liminf_n \lambda_n^\alpha/\lambda_n>0$ holds. If there exists a set $K=\{k_1<k_2<...\}\subset\mathbb{N}$ such that $\delta_\lambda(K)=1$ and $\lim_{k\in K}\phi(x_k-l)=0$ for each $\phi\in X^*$, then (x_k) in X is WS_λ^α-convergent to l.

Proof. It can be seen following similar way in Theorem 2.3 in [14] by taking advantage of Theorem 2.5 (i).

While $0<\alpha\leq 1$ the weakly λ-statistically convergence of order α is well defined. But while $\alpha>1$ it is not well defined in general. Consider the functional defined on X by $\phi(x)>0$. Then both

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n\ :\ \left|\phi\left(x_k-l_0\right)\right|\geq\varepsilon\right\}\right|=\lim_{n\to\infty}\frac{[\lambda_n]}{\lambda_n^\alpha}=0$$

and

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n\ :\ \left|\phi\left(x_k-l_1\right)\right|\geq\varepsilon\right\}\right|=\lim_{n\to\infty}\frac{[\lambda_n]}{\lambda_n^\alpha}=0$$

for $\alpha>1$, such that $x=(x_k)$ WS_λ^α-convergent both to l_0 and l_1. This is impossible.

Note that if we take $\lambda_n=n^\alpha$ for $0<\alpha<1$, then $WS^\alpha\subset WS_\lambda^\alpha$. Also if $\lambda_n=n^\alpha$ with $\alpha=1$ then $WS=WS_\lambda=WS_\lambda^\alpha$.

Theorem 2.7. Let $\lambda=(\lambda_n)\in\Lambda$, $\alpha\in(0,1]$ be any real number. For any sequence in X, if

$$\liminf_{n\to\infty}\lambda_n^\alpha/n>0 \tag{3}$$

holds and $W-\lim_k x_k=l$ then $WS_\lambda^\alpha-\lim_k x_k=l$, however, the converse implication need not be true.

Proof. By Theorem 2.4 in [14] we have that every weak convergence sequence is WS-convergence. Using Theorem 2.5 (i) we obtain desired result. The converse implication need not be true in general. It is clearly following example.

Example 2.8. Let (3) holds and $(x_n)\in\ell_p$ with $1<p<\infty$ be defined by

$$x_n(j)=\begin{cases}\sqrt{n}, & j\in I_n \text{ and } n=m^2;\\ 0, & j\notin I_n \text{ and } n=m^2;\\ \frac{1}{n}, & j\in I_n \text{ and } n\neq m^2;\\ 0, & j\notin I_n \text{ and } n\neq m^2;\end{cases}$$

where $I_n = [n-\lambda_n+1, n]$. Let $K = \{n \in \mathbb{N} : n \neq m^2\}$, then by Theorem 2.6 we can say that (x_k) is WS_λ^α-convergent to 0. For $n \in K^c$, let define the functional by $\phi_j(x) = x(j)$ where $x = (x_n) \in \ell_p$. Clearly, $\phi_j(x_n) = x_n(j) = \sqrt{n} \to \infty$ as $n \to \infty$. Hence $W - \lim_n x_n \neq 0$.

We have the following results from Theorem 2.5.

Corollary 2.9. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathrm{N}_{n_0}$ and (1) holds. Then the following statements hold:

(i) If $\beta = \alpha$, then $WS_\mu^\alpha \subseteq WS_\lambda^\alpha$ for each $\alpha \in (0,1]$.

(ii) If $\beta = 1$, then $WS_\mu \subseteq WS_\lambda^\alpha$ for each $\alpha \in (0,1]$.

Corollary 2.10. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathrm{N}_{n_0}$ and (2) holds. Then the following statements hold:

(i) If $\beta = \alpha$, then $WS_\lambda^\alpha \subseteq WS_\mu^\alpha$ for each $\alpha \in (0,1]$,

(ii) If $\beta = 1$, then $WS_\lambda^\alpha \subseteq WS_\mu$ for each $\alpha \in (0,1]$.

Theorem 2.11. S_λ^α-convergence implies WS_λ^α-convergence with same limit in X but the converse implication need not be true.

Proof. Let (x_k) in X, be a sequence such that $x_k \xrightarrow{S_\lambda^\alpha} l$. Then for evey $\varepsilon > 0$,

$$\frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n : \|x_k - l\| \geq \varepsilon\right\}\right| = 0.$$

Now, for every $\varepsilon > 0$ and each $\phi \in X^*$, the expression

$$\frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n : |\phi(x_k - l)| \geq \varepsilon\right\}\right| \leq \frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n : \|\phi\|\|x_k - l\| \geq \varepsilon\right\}\right|$$
$$= \frac{1}{\lambda_n^\alpha}\left|\left\{k \in I_n : \|x_k - l\| \geq \frac{\varepsilon}{\|\phi\|}\right\}\right|,$$

means S_λ^α-convergence implies WS_λ^α-convergence with same limit. We give an example to show that the converse implication of the above result is not true in general.

Example 2.12. Let consider the space $L_p(-1,1)$ for $p>1$ and (3) holds. Then define $x_n : (-1,1)\to \mathbb{R}$ by

$$\phi_n(x)=\begin{cases} n^{1/p}, & if \quad x\in[0,\frac{1}{n}] \\ 0, & otherwise. \end{cases}$$

Then we have $x_n \xrightarrow{w} 0$ in $L_p(-1,1)$ [15]. By Theorem 2.7 we have $x_n \xrightarrow{WS_\lambda^\alpha} 0$. Next we show that $x_k \xrightarrow{S_\lambda^\alpha} 0$ does not hold. For $0<\varepsilon<1$, since $\|x_k\|_{L_p(-1,1)}=1$ we have

$$\lim_{n\to\infty}\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n : \|x_k-0\|\geq\varepsilon\right\}\right|\neq 0.$$

Hence $x_k \xrightarrow{S_\lambda^\alpha} 0$ does not hold.

Theorem 2.13. Let $0<\alpha\leq\beta\leq 1$. Then the inclusion $WS_\lambda^\alpha\subseteq WS_\lambda^\beta$ is strict for some α and β such that $\alpha<\beta$.

Proof. If $0<\alpha\leq\beta\leq 1$, clearly

$$\frac{1}{\lambda_n^\beta}\left|\left\{k\in I_n : \left|\phi\left(x_k-l\right)\right|\geq\varepsilon\right\}\right|\leq\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n : \left|\phi\left(x_k-l\right)\right|\geq\varepsilon\right\}\right|$$

for every $\varepsilon>0$, which gives that $WS_\lambda^\alpha\subseteq WS_\lambda^\beta$. To show this inclusion is strict we give the following example.

Example 2.14. Let take $\lambda_n=n$ and consider the sequence $x=(x_k)$ defined by

$$x_k=\begin{cases} k, & k=m^2 \\ 0, & k\neq m^2 \end{cases} m=1,2,\ldots\ .$$

Since $x\xrightarrow{S_\lambda^\beta}0$, we have $WS_\lambda^\beta-\lim x_k=0$, i.e., $x\in WS_\lambda^\beta$ for $\frac{1}{2}<\beta\leq 1$. But since $x\notin S_\lambda^\alpha$, $x\notin WS_\lambda^\alpha$ for $0<\alpha\leq\frac{1}{2}$. This means the inclusion $WS_\lambda^\alpha\subseteq WS_\lambda^\beta$ is strict for $\alpha,\beta\in(0,1]$, such that $\alpha\in(0,\frac{1}{2}]$ and $\beta\in(\frac{1}{2},1]$.

Corollary 2.15. If a sequence is weak λ-statistical convergent of order α to l, then it is weak λ-statistical convergent to l, that is, $WS_\lambda^\alpha\subseteq WS_\lambda$ is strict for each $\alpha\in(0,1]$.

Corollary 2.16. (i) $WS_{\lambda}^{\alpha}=WS_{\lambda}^{\beta}$ if and only if $\alpha=\beta$.

(ii) $WS_{\lambda}^{\alpha}=WS_{\lambda}$ if and only if $\alpha=1$.

Theorem 2.17. Given for $\lambda=(\lambda_n)$, $\mu=(\mu_n)\in\Lambda$ suppose that $\lambda_n\leq\mu_n$ for all $n\in\mathrm{N}_{n_0}$ and let $0<\alpha\leq\beta\leq1$. Then the following statements hold:

(i) If (1) holds, then $W^{\beta}[V,\mu]\subseteq W^{\alpha}[V,\lambda]$,

(ii) If (2) holds, then $W^{\beta}[V,\mu]=W^{\alpha}[V,\lambda]$.

Proof. (i) Suppose that $\lambda_n\leq\mu_n$ for all $n\in\mathrm{N}_{n_0}$. Then $I_n\subset J_n$ so that we may write

$$\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n}\left|\phi(x_k-l)\right|\geq\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}\left|\phi(x_k-l)\right|$$

for all $n\in\mathrm{N}_{n_0}$. This gives that

$$\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n}\left|\phi(x_k-l)\right|\geq\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\frac{1}{\lambda_n^{\alpha}}\sum_{k\in I_n}\left|\phi(x_k-l)\right|.$$

As $n\to\infty$ by (1) we have $W^{\beta}[V,\mu]\subset W^{\alpha}[V,\lambda]$.

(ii) Let $x=(x_k)\in W^{\alpha}[V,\lambda]$ and suppose that (2) holds. Since ϕ is bounded there exists some $M>0$ such that $\left|\phi(x_k-l)\right|\leq M$ for all k. Now, since $\lambda_n\leq\mu_n$ and so that $\frac{1}{\mu_n^{\beta}}\leq\frac{1}{\lambda_n^{\alpha}}$, and $I_n\subset J_n$ for each $n\in\mathrm{N}_{n_0}$, we have

$$\begin{aligned}\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n}\left|\phi(x_k-l)\right|&=\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n-I_n}\left|\phi(x_k-l)\right|+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}\left|\phi(x_k-l)\right|\\&\leq\left(\frac{\mu_n-\lambda_n}{\mu_n^{\beta}}\right)M+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}\left|\phi(x_k-l)\right|\\&\leq\left(\frac{\mu_n-\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)M+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}\left|\phi(x_k-l)\right|\\&\leq\left(\frac{\mu_n}{\mu_n^{\beta}}-\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)M+\frac{1}{\lambda_n^{\alpha}}\sum_{k\in I_n}\left|\phi(x_k-l)\right|\end{aligned}$$

for every $n \in \mathrm{N}_{n_0}$. Therefore $W^{\alpha}[V,\lambda] \subseteq W^{\beta}[V,\mu]$. Since (2) implies (1) we have the equality $W^{\alpha}[V,\lambda] = W^{\beta}[V,\mu]$.

Corollary 2.18. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \le \mu_n$ for all $n \in \mathrm{N}_{n_0}$ and (1) holds. Then the following statements hold:

(i) If $\beta = \alpha$, then $W^{\alpha}[V,\mu] \subset W^{\alpha}[V,\lambda]$ for each $\alpha \in (0,1]$,

(ii) If $\beta = 1$, then $W[V,\mu] \subset W^{\alpha}[V,\lambda]$ for each $\alpha \in (0,1]$.

Corollary 2.19. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ belong to Λ such that $\lambda_n \le \mu_n$ for all $n \in \mathrm{N}_{n_0}$ and (2) holds. Then the following statements hold:

(i) If $\beta = \alpha$, then $W^{\alpha}[V,\lambda] \subseteq W^{\alpha}[V,\mu]$ for each $\alpha \in (0,1]$,

(ii) If $\beta = 1$, then $W^{\alpha}[V,\lambda] \subseteq W[V,\mu]$ for each $\alpha \in (0,1]$.

Theorem 2.20. Let $\alpha, \beta \in (0,1]$ be real numbers such that $\alpha \le \beta$, and $\lambda = (\lambda_n)$, $\mu = (\mu_n) \in \Lambda$ such that $\lambda_n \le \mu_n$ for all $n \in \mathrm{N}_{n_0}$.

(i) Let (1) holds, then if a sequence is $W^{\beta}[\lambda,\mu]$-summable of order β, to l, then it is WS_{λ}^{α}-statistically convergent of order α, to l,

(ii) Let (2) holds, then if a sequence is WS_{λ}^{α}-statistically convergent of order α, to l, then it is $W^{\beta}[V,\mu]$-summable of order β, to l.

Proof. (i) For any sequence $x = (x_k)$ and $\varepsilon > 0$, we have

$$\begin{aligned}\sum_{k \in J_n} \left|\phi(x_k - l)\right| &= \sum_{\substack{k \in J_n \\ |\phi(x_k - l)| \ge \varepsilon}} \left|\phi(x_k - l)\right| + \sum_{\substack{k \in J_n \\ |\phi(x_k - l)| < \varepsilon}} \left|\phi(x_k - l)\right| \\ &\ge \sum_{\substack{k \in I_n \\ |\phi(x_k - l)| \ge \varepsilon}} \left|\phi(x_k - l)\right| + \sum_{\substack{k \in I_n \\ |\phi(x_k - l)| < \varepsilon}} \left|\phi(x_k - l)\right| \\ &\ge \sum_{\substack{k \in I_n \\ |\phi(x_k - l)| \ge \varepsilon}} \left|\phi(x_k - l)\right| \\ &\ge \left|\left\{k \in I_n \ : \ \left|\phi(x_k - l)\right| \ge \varepsilon\right\}\right| .\varepsilon\end{aligned}$$

and so that

$$\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n}\left|\phi\left(x_k-l\right)\right|\geq\frac{1}{\mu_n^{\beta}}\left|\left\{k\in I_n\ :\ \left|\phi\left(x_k-l\right)\right|\geq\varepsilon\right\}\right|.\varepsilon$$
$$\geq\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\frac{1}{\lambda_n^{\alpha}}\left|\left\{k\in I_n\ :\ \left|\phi\left(x_k-l\right)\right|\geq\varepsilon\right\}\right|.\varepsilon.$$

Since (1) holds it follows that if $x=\left(x_k\right)$ is $W^{\beta}[V,\mu]$-summable of order β, to l, then it is WS_{λ}^{α}-statistically convergent of order α, to l.

(ii) Suppose that $WS_{\lambda}^{\alpha}-\lim x_k=l$. Since ϕ is bounded there exists some $M>0$ such that $\left|\phi\left(x_k-l\right)\right|\leq M$ for all k, then for every $\varepsilon>0$ we have

$$\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n}\left|\phi\left(x_k-l\right)\right|=\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n-I_n}\left|\phi\left(x_k-l\right)\right|+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}\left|\phi\left(x_k-l\right)\right|$$
$$\leq\left(\frac{\mu_n-\lambda_n}{\mu_n^{\beta}}\right)M+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}\left|\phi\left(x_k-l\right)\right|$$
$$\leq\left(\frac{\mu_n-\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)M+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}\left|\phi\left(x_k-l\right)\right|$$
$$=\left(\frac{\mu_n}{\mu_n^{\beta}}-\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)M+\frac{1}{\mu_n^{\beta}}\sum_{\substack{k\in I_n\\ \left|\phi\left(x_k-l\right)\right|\geq\varepsilon}}\left|\phi\left(x_k-l\right)\right|+\frac{1}{\mu_n^{\beta}}\sum_{\substack{k\in I_n\\ \left|\phi\left(x_k-l\right)\right|<\varepsilon}}\left|\phi\left(x_k-l\right)\right|$$
$$\leq\left(\frac{\mu_n}{\mu_n^{\beta}}-\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)M+\frac{M}{\lambda_n^{\alpha}}\left|\left\{k\in I_n\ :\ \left|\phi\left(x_k-l\right)\right|\geq\varepsilon\right\}\right|+\frac{\lambda_n}{\mu_n^{\beta}}\varepsilon$$

for all $n\in\mathrm{N}_{n_0}$. Using (2) we obtain that $W^{\beta}[V,\lambda]-\lim x_k=l$, whenever $WS_{\lambda}^{\alpha}-\lim x_k=l$.

Corollary 2.21. Let $\lambda=\left(\lambda_n\right)$ and $\mu=\left(\mu_n\right)$ belong to Λ such that $\lambda_n\leq\mu_n$ for all $n\in\mathrm{N}_{n_0}$ and (1) holds. Then the following statements hold:

(i) If $\beta=\alpha$, then $W^{\alpha}[V,\mu]\subset WS_{\lambda}^{\alpha}$ for each $\alpha\in(0,1]$,

(ii) If $\beta=1$, then $W[V,\mu]\subset WS_{\lambda}^{\alpha}$ for each $\alpha\in(0,1]$.

Corollary 2.22. Let $\lambda=\left(\lambda_n\right)$ and $\mu=\left(\mu_n\right)$ belong to Λ such that $\lambda_n\leq\mu_n$ for all $n\in\mathrm{N}_{n_0}$ and (2) holds. Then the following statements hold:

(i) If $\beta=\alpha$, then $WS_{\lambda}^{\alpha}\subset W^{\alpha}[V,\mu]$ for each $\alpha\in(0,1]$,

(ii) If $\beta=1$, then $WS_{\lambda}^{\alpha}\subset W[V,\mu]$ for each $\alpha\in(0,1]$.

REFERENCES

[1] *A. Zygmund*, Trigonometric Series, Cambridge University Press, Cambridge UK, 1979
[2] *H. Steinhaus*, Sur la convergence ordinaire et la convergence asymptotique, Colloquium Mathematicum, 1951, 2:73-74
[3] *H. Fast*, Sur la convergence statistique, Colloquium Math, 1951, 2, 241-244
[4] *I. J. Schoenberg*, The integrability of certain functions and related summability methods, American Mathematical Monthly, 1959, 66:361-375
[5] *I. Niven*, *H. S. Zuckerman*, An introduction to the Theory of Numbers, Fourth Ed. John Wiley & Sons New York, 1980
[6] *L. Leinder*, Über die la Vallee-Pousinsche Summierbarkeit Allgemeiner Orthogonalreihen, Acta Mathematica Scientiarum Hungaricae, 1965, 16:375-387
[7] *M. Mursaleen*, λ-statistical convergence, Math. Slovaca, 2000, 50(1):111-115
[8] *A. D. Gadjiev*, *C. Orhan*, Some approximation theorems via statistical convergence, Rocky Mountain J. Math., 2002, 32(1), 129-138
[9] *R. Çolak*, Statistical convergence of order α, Modern Methods in Analysis and Its Applications, Anamaya Pub., New Delhi India, 2010
[10] *R. Çolak, Ç. A. Bektaş*, λ-statistical convergence of order α, Acta Math. Sci. Ser. B Engl. Ed., 2011, 31, no.3, 953-959
[11] *M. Et, R. Çolak, Y. Altin* , Strongly almost summable sequences of order α, Kuwait J. Sci., 2014, 41, no.2, 35-47
[12] *J. Connor, M. Ganichev, V. Kadets*, A characterization of Banach spaces with seperable duals via weak statistical convergence, J. Math. Anal. Appl., 2000, 244-251
[13] *V. K. Bhardwaj*, *I. Bala*, On weak statistical convergence, Int. J. Math. Math. Sci., 2007, Art. ID 38530 9 pp.
[14] *Meenakshi*, *M. S. Saroa*, *V. Kumar*, Weak statistical convergence defined by de la Vallée-Poussin mean, Bull. Calcutta Math. Soc., 2014, 106, no.3, 215-224
[15] *K. H. Karlsen*, Notes on weak convergence, University of Oslo, Norway, *2006.*