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FRACTIONAL VECTOR CALCULUS IN THE FRAME OF A GENERALIZED CAPUTO FRACTIONAL DERIVATIVE

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The authors in [1] recently introduced a new generalized fractional derivative on $AC_\gamma^n[a,b]$ and $C_\gamma^n[a,b]$, and defined their Caputo version. This derivative contains two parameters and reduces to the classical Caputo derivatives if one of these parameters tend to certain values. From here and after, by generalized Caputo fractional derivative, we refer to the Caputo version of the generalized fractional derivative. This paper studies the generalized Caputo fractional derivative and establishes the Fundamental Theorem of Fractional Calculus (FTFC) in the sense of this derivative. The fundamental results are used in establishing some vital theorems and then applied to vector calculus.

Keywords: Generalized Caputo fractional derivative, fundamental theorem of fractional calculus (FTFC), fractional vector calculus, fractional Green's theorem, fractional Gauss' theorem.

1. Introduction and Auxiliary Results

The popularity of fractional calculus (calculus of derivatives and integrals of any arbitrary order) and the interest for the subject have grown astoundingly during the past three decades or so [3,4,9,10].

Of the many definitions of fractional derivatives, the Caputo derivative seems to have more demonstrated advantages and numerous seemingly diverse applications than the others. Such advantages allow the use of the derivative (Caputo derivative) in modifying other fractional derivatives with some shortcomings. For example, authors in [2] and [4] have modified the Hadamard

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fractional derivatives into a more convenient one that has initial conditions that can be physically interpretable similar to the ones in the Caputo settings.

Several real life problems have been studied using the fractional derivatives, specifically with the Caputo fractional derivative which is widely applied in various areas of sciences and engineering [11,12]. For instance, it is known that due to their non-locality, fractional differential operators give a better description of systems with memory effect even though the non-locality takes different forms [3,9,11,12]. Thus, fractional operators are generalized in order to get the real non-local phenomena while numerous works are being carried out on fractional integrals and derivatives with non-local and non-singular kernels [10,13,14].

Recently, the authors in [1] defined a generalized fractional derivative on the space $AC_\gamma^n[a,b]$ (the space of functions defined on $[a,b]$ such that $\gamma^{n-1}f \in AC[a,b]$, where $\gamma = x^{1-\rho}\frac{d}{dx}$) and defined their Caputo version.

Authors in [6] presented FTFC in the sense of Caputo fractional derivative while developing FTFC using the same fractional derivative and applying it to fractional vector calculus can be seen in [7-8]. However, in this paper, we present new and generalized results using a generalized Caputo fractional derivative that includes two parameters and curtails to the classical Caputo derivative when one the parameters is replaced by 1 and to Caputo-Hadamard fractional derivative approaches 0 . The derivative is used to develop a generalized FTFC thereby using the new results in formulating other theorems. The fundamental result of the FTFC is applied to vector calculus incorporating the formulations and proofs of Green's and Gauss' theorems. In the present section, we give some fundamental definitions and known results which are used in this article. Section 2 presents FTFC in the sense of $Caputo - Katugampola$ fractional derivative and some consequent results. Applications of the generalized FTFC are given in Section 3 while Section 4 concludes the paper.

1.1 Preliminary definitions

Let $[a,b]$ be a finite interval, $0 \leq \epsilon \leq 1, \rho \geq 0$ and $AC[a,b]$ be the set of absolute continuous functions on $[a,b]$. Then we define

$$AC_\gamma^n[a,b] = \left\{f:[a,b] \to \mathbb{C} \text{ and } \gamma^{n-1}f \in AC[a,b], \gamma = x^{1-\rho}\frac{d}{dx}\right\}, AC_\gamma^1[a,b] = AC[a,b] \quad (1)$$

$$C_{\gamma,\epsilon}^n[a,b] = \left\{f:[a,b] \to \mathbb{C} \text{ and } \gamma^{n-1}f \in C[a,b], \gamma^n f \in C_{\epsilon,\rho}[a,b], \gamma = x^{1-\rho}\frac{d}{dx}\right\} \quad (2)$$

endowed with the norm $\|f\|_{C_{\gamma,\epsilon}^n} = \sum_{k=0}^{n-1}\|\gamma^k f\|_C + \|\gamma^n f\|_{C_{\epsilon,\rho}}$, where $C_{\gamma,0}^n[a,b] = C_\gamma^n[a,b]$ endowed with the norm $\|f\|_{C_\gamma^n} = \sum_{k=0}^{n}\|\gamma^k f\|_C$.

Here, $C_{\epsilon,\rho}[a,b]=\left\{f:[a,b]\to\mathbb{R}:\left(\frac{x^\rho-a^\rho}{\rho}\right)^\epsilon f(x)\in C[a,b]\right\},\rho\neq 0$ equipped with the norm $\|f\|_{C_{\epsilon,\rho}}=\left\|\left(\frac{x^\rho-a^\rho}{\rho}\right)^\epsilon f(x)\right\|_C$, while $C_{\epsilon,\rho}[a,b]=\left\{f:[a,b]\to\mathbb{R}:\left(\ln\frac{x}{a}\right)^\epsilon f(x)\in C[a,b]\right\}$ when $\rho=0$ equipped with the norm $\|f\|_{C_{\epsilon,\rho}}=\left\|\left(\ln\frac{x}{a}\right)^\epsilon f(x)\right\|_C$.

The generalized left and right fractional integrals of order α, $(Re(\alpha)\geq 0)$ in Katugampola settings are defined [5] respectively by

$$({}_aI^{\alpha,\rho}f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\frac{x^\rho-y^\rho}{\rho}\right)^{\alpha-1}f(y)\frac{dy}{y^{1-\rho}} \tag{3}$$

$$\left(I_b^{\alpha,\rho}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\frac{y^\rho-x^\rho}{\rho}\right)^{\alpha-1}f(y)\frac{dy}{y^{1-\rho}}. \tag{4}$$

The generalized fractional derivatives of functions in the space $AC_\gamma^n[a,b]$ or $C_\gamma^n[a,b]$ with $n=[Re(\alpha)]+1,\ Re(\alpha)>0$ can be defined [1] as

$$({}_aD^{\alpha,\rho}f)(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\frac{x^\rho-y^\rho}{\rho}\right)^{n-\alpha-1}\frac{(\gamma^nf)(y)dy}{y^{1-\rho}}+\sum_{k=0}^{n-1}\frac{\gamma^kf(a)}{\Gamma(k-\alpha-1)}\left(\frac{y^\rho-a^\rho}{\rho}\right)^{k-\alpha} \tag{5}$$

$$\left(D_b^{\alpha,\rho}f\right)(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b\left(\frac{y^\rho-x^\rho}{\rho}\right)^{n-\alpha-1}\frac{(\gamma^nf)(y)dy}{t^{1-\rho}}+\sum_{k=0}^{n-1}\frac{(-\gamma)^kf(b)}{\Gamma(k-\alpha-1)}\left(\frac{b^\rho-x^\rho}{\rho}\right)^{k-\alpha} \tag{6}$$

Definition 1

Let $Re(\alpha)\geq 0$ and $n=[Re(\alpha)]+1$. If $f\in AC_\gamma^n[a,b]$, where $0<a<b<\infty$, the left generalized Caputo fractional derivative of f of order α is defined by

$$({}_a^CD^{\alpha,\rho}f)(x)=\left({}_aD^{\alpha,\rho}\left[f(y)-\sum_{k=0}^{n-1}\frac{\gamma^kf(a)}{k!}\left(\frac{y^\rho-a^\rho}{\rho}\right)^k\right]\right)(x). \tag{7}$$

If $0<Re(\alpha)<1$, (13) becomes

$$({}_a^CD^{\alpha,\rho}f)(x)=({}_aD^{\alpha,\rho}[f(y)-f(a)])(x). \tag{8}$$

The following theorem gives an alternative definition of the derivative in (13).

Theorem 1 [1]

Let $Re(\alpha) \geq 0,\ \ n = [Re(\alpha)] + 1$ and $f \in AC_\gamma^n[a,b]$, where $0 < a < b < \infty$.

1. If $\alpha \notin \mathbb{N}_0$ $({}_a^C D^{\alpha,\rho} f)(x) = {}_a I^{n-\alpha,\rho}(\gamma^n f)(x)$. (9)
2. If $\alpha \in \mathbb{N}$, ${}_a^C D^{\alpha,\rho} f = \gamma^n f$. (10)

$$ {}_a^C D^{0,\rho} f = f. \tag{11}$$

Theorem 2 [1]

Let $Re(\alpha) \geq 0,\ \ n = [Re(\alpha)] + 1$ and $f \in C_\gamma^n[a,b]$, where $0 < a < b < \infty$. Then, ${}_a^C D^{\alpha,\rho} f$ is continuous on $[a,b]$ and

$$({}_a^C D^{\alpha,\rho} f)(a) = 0. \tag{12}$$

2. Generalized FTFC

The first fundamental theorem of calculus states that if F is defined by

$$F(x) = \int_a^x f(t)dt \tag{13}$$

then

$$F'(x) = f(x) \tag{14}$$

at each point in the closed interval.

The second FTC guarantees that if F is the indefinite integral of a continuous function f on $[a,b]$, then

$$\int_a^b f(t)dt = F(b) - F(a) = F(t)|_a^b. \tag{15}$$

It is obvious that the generalized fractional derivatives do not have generalization of the FTFC in the form of (15). Thus,

$$({}_a I^{\alpha,\rho}\, {}_a D^{\alpha,\rho} f)(b) \neq f(b) - f(a). \tag{16}$$

The reasons of this are the facts the differential operators $\gamma^n = \left(x^{1-\rho}\frac{d}{dx}\right)^n$ used in the definition of the generalized fractional derivatives appear outside the integrals and these operators do not commute with the integrals. That is,

$$({}_a I^{\alpha,\rho}\, {}_a D^{\alpha,\rho} f)(x) = {}_a I^{\alpha,\rho} \gamma^n\, {}_a I^{n-\alpha,\rho} f(x) \neq {}_a I^{\alpha,\rho}\, {}_a I^{n-\alpha,\rho} \gamma^n f(x) = f(x) \tag{17}$$

However, it has been proven (see [6]-[8]) that

$$({}_a I^{\alpha}\, {}_a^C \mathcal{D}^{\alpha} f)(b) = f(b) - f(a), \tag{18}$$

where ${}_a^C \mathcal{D}^{\alpha}$ is the Caputo fractional derivative.

Lemma 1

Let $Re(\alpha) \geq 0$ and $Re(\beta) \geq 0$, then

$$\left({}_a I^{\alpha,\rho}\left(\frac{y^\rho - a^\rho}{\rho}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\beta+\alpha-1}. \tag{19}$$

The proof of Lemma 1 can be done by using Definition 1 and properties of the gamma function. Now, using the generalized Caputo fractional derivative, we present the generalized FTFC.

Theorem 3 (FTFC)

Let $0 < Re(\alpha) \leq 1,\ \ n = [Re(\alpha)] + 1$ and $f \in AC_{\gamma}^{n}[a,b]$, where $0 < a < b < \infty$. Then,

a. If $F(x) = {}_{a}I^{\alpha,\rho} f(x)$, then

$$({}_{a}^{C}D^{\alpha,\rho}F)(x) = f(x). \tag{20}$$

b.

$$({}_{a}I^{\alpha,\rho}\,{}_{a}^{C}D^{\alpha,\rho}F)(b) = F(b) - F(a), \tag{21}$$

where $({}_{a}I^{\alpha,\rho}f)(b) = \frac{1}{\Gamma(\alpha)}\int_{a}^{b}\left(\frac{x^{\rho}-y^{\rho}}{\rho}\right)^{\alpha-1} f(y)\frac{dy}{y^{1-\rho}}$.

Proof. Assertion a. can be proved using Theorem 3.5 in [1] while assertion b. can be proved using Theorem 1, Theorem 4.1 in [5] and Theorem 3.6 in [1].

Lemma 2

Let $Re(\alpha) \geq 0,\ n = [Re(\alpha)] + 1$ and $f \in AC_{\gamma}^{n}[a,b]$, where $0 < a < b < \infty$. Then,

$$f(x) = f(a) + \frac{{}_{a}^{C}D^{\alpha,\rho}f(\lambda)}{\Gamma(1+\alpha)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\alpha},\quad \lambda \in (a,x). \tag{22}$$

Proof. Evaluating the integral in (21) using (3) we obtain

$$f(x) - f(a) = ({}_{a}I^{\alpha,\rho}\,{}_{a}^{C}D^{\alpha,\rho})f(x)$$

$$= \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\left(\frac{x^{\rho}-y^{\rho}}{\rho}\right)^{\alpha-1} ({}_{a}^{C}D^{\alpha,\rho}f)(y)\frac{dy}{y^{1-\rho}}. \tag{23}$$

At this point, we apply the mean value theorem for integrals. Thus,

$$f(x) - f(a) = ({}_{a}^{C}D^{\alpha,\rho}f)(\lambda)\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\left(\frac{x^{\rho}-y^{\rho}}{\rho}\right)^{\alpha-1}\frac{dy}{y^{1-\rho}},\quad \lambda \in (a,x). \tag{24}$$

It can be observed that the right-hand side of (24) involves the generalized fractional integral, where the function $f(t) = 1$, that is, ${}_{a}I^{\alpha,\rho}(1)$. Therefore, using Lemma 1 with $\beta = 1$ gives ${}_{a}I^{\alpha,\rho}(1) = \frac{1}{\Gamma(1+\alpha)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}$.

Therefore (24) becomes $f(x) - f(a) = \frac{({}_{a}^{C}D^{\alpha,\rho}f)(\lambda)}{\Gamma(1+\alpha)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\alpha},\quad \lambda \in (a,x)$

Lemma 3

Let $Re(\alpha) \geq 0,\ n = [Re(\alpha)] + 1$ and $f \in AC_{\gamma}^{n}[a,b]$, with $0 < a < b < \infty$. For $k,m \in \mathbb{N}$,

$$({}_{a}I^{\alpha,\rho})^{k}({}_{a}^{C}D^{\alpha,\rho})^{m}f(x) = \frac{({}_{a}^{C}D^{\alpha,\rho})^{m}f(\tau)}{\Gamma(k\alpha+1)}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{k\alpha},\quad \tau \in (a,x). \tag{25}$$

Proof. The proof can be done using semi-group property for integrals in [3].

Theorem 4

Let $f(x) \in C_\gamma^n[a,b]$, $0 < a < b < \infty$ and $\alpha, \beta \in \mathbb{C}$ such that $Re(\alpha) \geq 0, Re(\beta) \geq 0$. Then

$$ {}_a^C D^{\alpha,\rho}\, {}_a I^{\beta,\rho} f(x) = {}_a I^{\beta-\alpha,\rho} f(x). \tag{26} $$

Proof. The proof can be done using (3), Theorem 1 and semi-group properties for fractional integrals.

3. Applications of the Generalized FTFC in Vector Calculus

We define the generalized Caputo fractional differential operator on $[a,b]$ with $\alpha \in \mathbb{C}, Re(\alpha) \geq 0$ by

$$ {}_a^C D_x^{\alpha,\rho}[t] = \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} \frac{dt}{t^{1-\rho}} \left(t^{1-\rho} \frac{\partial}{\partial t}\right)^n = {}_a I_x^{n-\alpha,\rho}[t] \gamma^n[t], \quad (27) $$

where $n = [Re(\alpha)] + 1,\ \rho > 0$. Note that when $\rho = 1$, then (27) corresponds to the definition of Caputo fractional differential operator given in [8]. Therefore, the operator notations in (20) and (21) become

$$ {}_a^C D_x^{\alpha,\rho}[t]\, {}_a I_t^{\alpha,\rho}[s] f(s) = f(x) \tag{28} $$

$$ {}_a I_b^{\alpha,\rho}[x]\, {}_a^C D_x^{\alpha,\rho}[t] f(t) = f(b) - f(a). \tag{29} $$

Let $\Omega \subset \mathbb{R}^3$. Then the following defines a fractional generalization of nabla operator

$$ \nabla_\Omega^{\alpha,\rho} = {}^C\boldsymbol{D}_\Omega^{\alpha,\rho} = \boldsymbol{e}_1\, {}^C D_\Omega^{\alpha,\rho}[x_1] + \boldsymbol{e}_2\, {}^C D_\Omega^{\alpha,\rho}[x_2] + \boldsymbol{e}_3\, {}^C D_\Omega^{\alpha,\rho}[x_3] \tag{30} $$

where ${}^C D_\Omega^{\alpha,\rho}[x_j]$ is the generalized Caputo fractional derivative with respect to the coordinates x_j. In the case of parallelpiped domain $\Omega = \{a_j < x_j < b_j; \quad j = 1,2,3\}$,

$$ {}^C D_\Omega^{\alpha,\rho}[x_j] = {}_{a_j}^C D_{b_j}^{\alpha,\rho}[x_j], \qquad j = 1,2,3. \tag{31} $$

Definition 2

Let $\Omega \subset \mathbb{R}^3$ and $f \in AC_\gamma^n(\Omega)$. Then we define fractional gradient of f by

$$ \begin{aligned} \mathrm{Grad}_\Omega^{\alpha,\rho} f = {}^C\boldsymbol{D}_\Omega^{\alpha,\rho} f &= \boldsymbol{e}_j\, {}^C D_\Omega^{\alpha,\rho}[x_j] f(x_1,x_2,x_3) \\ &= \boldsymbol{e}_1\, {}^C D_\Omega^{\alpha,\rho}[x_1] f(x_1,x_2,x_3) + \boldsymbol{e}_2\, {}^C D_\Omega^{\alpha,\rho}[x_2] f(x_1,x_2,x_3) \\ &\quad + \boldsymbol{e}_3\, {}^C D_\Omega^{\alpha,\rho}[x_3] f(x_1,x_2,x_3), \end{aligned} \tag{32} $$

where $0 < Re(\alpha) \leq 1$.

Definition 3

If $\Omega \subset \mathbb{R}^3$ and $\boldsymbol{F} \in \left[AC_\gamma^n(\Omega); \mathbb{R}^3\right]$, then we define its fractional divergence as
$\mathrm{Div}_\Omega^{\alpha,\rho}\boldsymbol{F} = \left({}^C\boldsymbol{D}_\Omega^{\alpha,\rho}, \boldsymbol{F}\right) = {}^C D_\Omega^{\alpha,\rho}\left[x_j\right]F_j(x_1,x_2,x_3) = {}^C D_\Omega^{\alpha,\rho}[x_1]F_1(x_1,x_2,x_3) +$
${}^C D_\Omega^{\alpha,\rho}[x_2]F_2(x_1,x_2,x_3) +$
${}^C D_\Omega^{\alpha,\rho}[x_3]F_3(x_1,x_2,x_3).$ (33)

Definition 4

Let $\Omega \subset \mathbb{R}^3\Omega \in \mathbb{R}^3$ such that $\boldsymbol{F} \in \left[AC_\gamma^n(\Omega); \mathbb{R}^3\right]$. The fractional curl operator can be given by
$\mathrm{Curl}_\Omega^{\alpha,\rho}\boldsymbol{F} = \left({}^C\boldsymbol{D}_\Omega^{\alpha,\rho}, \boldsymbol{F}\right) = \boldsymbol{e_l}\varepsilon_{lmk}\,{}^C D_\Omega^{\alpha,\rho}[x_m]F_k(x_1,x_2,x_3) = \boldsymbol{e_1}\left({}^C D_\Omega^{\alpha,\rho}[x_2]F_3 -\right.$
$\left.{}^C D_\Omega^{\alpha,\rho}[x_3]F_2\right) + \boldsymbol{e_2}\left({}^C D_\Omega^{\alpha,\rho}[x_3]F_1 - {}^C D_\Omega^{\alpha,\rho}[x_1]F_3\right) + \boldsymbol{e_3}\left({}^C D_\Omega^{\alpha,\rho}[x_1]F_2 \; -\right.$
$\left.{}^C D_\Omega^{\alpha,\rho}[x_2]F_1\right),$ (34)

where ε_{lmk} is Levi-Civita symbol which is defined by

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{for } i = j, j = k \text{ or } k = i \\ +1 & \text{for } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & \text{for } (i,j,k) \in \{(1,3,2),(3,2,1),(2,1,3)\}. \end{cases} \quad (35)$$

3.1 Fractional Green's Theorem

The following theorem gives a fractional generalization of Green's theorem. The main step in the proof is the application of FTFC.

Theorem 5 (Fractional Green's Theorem)

Let Ω be a rectangular domain

$$\Omega = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, c \le y \le d\} \quad (36)$$

with boundary $\partial\Omega$. If $P$and $Q$are absolutely continuous or continuously differentiable on $\overline{\Omega}$ and $0 < Re(\alpha) \le 1$, then

$$I_{\partial\Omega}^{\alpha,\rho}[x]P(x,y) + I_{\partial\Omega}^{\alpha,\rho}[y]Q(x,y) =$$
$$= I_\Omega^{\alpha,\rho}[x,y]\left({}^C D_\Omega^{\alpha,\rho}[s]Q(s,y) - {}^C D_\Omega^{\alpha,\rho}[t]P(x,t)\right). \quad (37)$$

Note that $I_\Omega^{\alpha,\rho}[x,y]$ denotes the double integral as in the classical Green's Theorem

$$\int_{\partial\Omega}(Pdx + Qdy) = \int\int_\Omega (D_xQ - D_yP)dA.$$

Proof: Let $A(a,c), B(b,c), C(b,d)$ and $D(a,d)$ be the vertices of Ω in (36). Thus, the boundary $\partial\Omega$ of the rectangular domain Ω are the sides AB, BC, CD and DA.

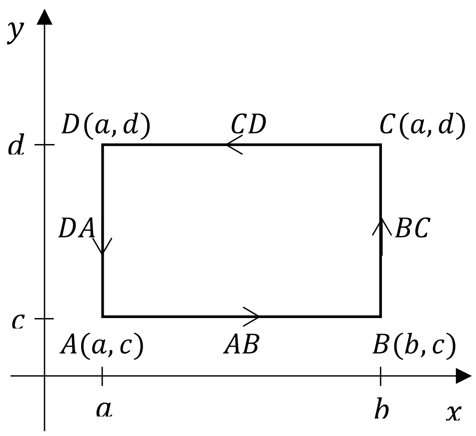

Fig 1: The rectangular domain Ω.

Before we proceed, it should be noted that

$$I_{\Omega}^{\alpha,\rho}[x,y] = I_{\Omega}^{\alpha,\rho}[x] I_{\Omega}^{\alpha,\rho}[y] = {}_aI_b^{\alpha,\rho}[x]\, {}_cI_d^{\alpha,\rho}[y].$$

Then,

$$\begin{aligned} &I_{\partial\Omega}^{\alpha,\rho}[x]P(x,y) + I_{\partial\Omega}^{\alpha,\rho}[y]Q(x,y) \\ &= I_{AB}^{\alpha,\rho}[x]P(x,c) + I_{CD}^{\alpha,\rho}[x]P(x,d) + I_{BC}^{\alpha,\rho}[y]Q(b,y) + I_{DA}^{\alpha,\rho}[y]Q(a,y) \\ &= [x]P(x,c) - {}_aI_b^{\alpha,\rho}[x]P(x,d) + {}_cI_d^{\alpha,\rho}[y]Q(b,y) - {}_cI_d^{\alpha,\rho}[y]Q(a,y) \\ &= {}_aI_b^{\alpha,\rho}[x]\big(P(x,c) - P(x,d)\big) + {}_cI_d^{\alpha,\rho}[y]\big(Q(b,y) - Q(a,y)\big). \end{aligned}$$

Applying (35) of FTFC we obtain

$$P(x,c) - P(x,d) = -{}_cI_d^{\alpha,\rho}[y]\, {}_c^CD_y^{\alpha,\rho}[t]P(x,t) \tag{38}$$

$$Q(b,y) - Q(a,y) = {}_aI_b^{\alpha,\rho}[x]\, {}_a^CD_x^{\alpha,\rho}[s]Q(s,y). \tag{39}$$

Hence,

$$\begin{aligned} &I_{\partial\Omega}^{\alpha,\rho}[x]P(x,y) + I_{\partial\Omega}^{\alpha,\rho}[y]Q(x,y) \\ &= {}_aI_b^{\alpha,\rho}[x]\left(-{}_cI_d^{\alpha,\rho}[y]\, {}_c^CD_y^{\alpha,\rho}[t]P(x,t)\right) + {}_cI_d^{\alpha,\rho}[y]\left({}_aI_b^{\alpha,\rho}[x]\, {}_a^CD_x^{\alpha,\rho}[s]Q(s,y)\right) \\ &= {}_aI_b^{\alpha,\rho}[x]\, {}_cI_d^{\alpha,\rho}[y]\left({}_a^CD_x^{\alpha,\rho}[s]Q(s,y) - {}_c^CD_y^{\alpha,\rho}[t]P(x,t)\right) \\ &= I_{\Omega}^{\alpha,\rho}[x,y]\left({}_a^CD_x^{\alpha,\rho}[s]Q(s,y) - {}_c^CD_y^{\alpha,\rho}[t]P(x,t)\right). \end{aligned}$$

3.2 Fractional Gauss' Theorem

The Gauss' theorem, also known as the divergence theorem, claims that the outward flux of a vector field through a closed surface is the same as the volume integral of the divergence over the region inside the surface.

Theorem 6 (Fractional Gauss' Theorem)
For the parallelpiped $\Omega = \{(x, y, z) \in \mathbb{R}^3 : a \leq x \leq b, c \leq y \leq d, g \leq z \leq h\}$, if $F_1(x, y, z)$, $F_2(x, y, z)$ and $F_3(x, y, z)$ are continuously differentiable real-valued function in Ω bounded by the closed surface $\partial\Omega$, then

$$\left(\boldsymbol{I}_{\partial\Omega}^{\alpha,\rho}, \boldsymbol{F}\right) = I_{\Omega}^{\alpha,\rho} \mathrm{Div}_{\Omega}^{\alpha,\rho} \boldsymbol{F}$$

Proof: The volume integral $I_{\Omega}^{\alpha,\rho}[x, y, z]$ can be written as

$$I_{\Omega}^{\alpha,\rho}[x, y, z] = {}_a I_b^{\alpha,\rho}[x] {}_c I_d^{\alpha,\rho}[y] {}_g I_h^{\alpha,\rho}[z].$$

Moreover,

$I_{\partial\Omega}^{\alpha,\rho}[y, z] = {}_c I_d^{\alpha,\rho}[y] {}_g I_h^{\alpha,\rho}[z], \qquad I_{\partial\Omega}^{\alpha,\rho}[x, z] = {}_a I_b^{\alpha,\rho}[x] {}_g I_h^{\alpha,\rho}[z],$
$I_{\partial\Omega}^{\alpha,\rho}[x, y] = {}_a I_b^{\alpha,\rho}[x] {}_c I_d^{\alpha,\rho}[y].$

Thus,

$$I_{\partial\Omega}^{\alpha,\rho}[y, z] F_1 + I_{\partial\Omega}^{\alpha,\rho}[x, z] F_2 + I_{\partial\Omega}^{\alpha,\rho}[x, y] F_3$$
$$= {}_c I_d^{\alpha,\rho}[y] {}_g I_h^{\alpha,\rho}[z] \big(F_1(b, y, z) - F_1(a, y, z)\big) + {}_a I_b^{\alpha,\rho}[x] {}_g I_h^{\alpha,\rho}[z] \big(F_2(x, d, z) - F_2(x, c, z)\big) + {}_a I_b^{\alpha,\rho}[x] {}_c I_d^{\alpha,\rho}[y] \big(F_3(x, y, g) - F_3(x, y, h)\big) =$$
$${}_a I_b^{\alpha,\rho}[x] {}_c I_d^{\alpha,\rho}[y] {}_g I_h^{\alpha,\rho}[z] \left({}_a^C D_x^{\alpha,\rho}[p] F_1(p, y, z) + {}_c^C D_y^{\alpha,\rho}[q] F_2(x, q, z) + {}_g^C D_z^{\alpha,\rho}[r] F_3(x, y, r) \right) = I_{\Omega}^{\alpha,\rho} \left({}^C\boldsymbol{D}_{\Omega}^{\alpha,\rho}, \boldsymbol{F} \right) = I_{\Omega}^{\alpha,\rho} \mathrm{Div}_{\Omega}^{\alpha,\rho} \boldsymbol{F}.$$

4. Conclusion

The generality of the generalized Caputo fractional derivative can be observed from the limiting case as $\rho \to 0$ leading to the Caputo-Hadamard fractional derivative since $\lim_{\rho \to 0} \frac{x^\rho - a^\rho}{\rho} = \log\left(\frac{x}{a}\right)$. Moreover, $\rho = 1$ gives the Caputo fractional derivatives. Thus, as the generalized Caputo fractional derivative is used in formulating the FTFC and the consequent theorems in this paper, the results obtained are also general. For instance, when $\rho = 1$, (27) gives the definition of Caputo fractional differential operator and its application in integral theorems of vector calculus given in [8]. In fact, we formulated a generalized fractional vector calculus and defined a generalized fractional differential vector operations. Moreover, formulations and proofs of some fractional integral theorems (Green's and Gauss' theorems) as an application of the generalized FTFC are given.

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