LAPLACE TRANSFORM APPROACH FOR ONE-DIMENSIONAL FOKKER-PLANCK EQUATION

S. ZARRINKAMAR[1*] , H. PaNAHI[2] and F. HOSSEINI[2]

The Fokker-Planck equation is considered in (1+1)-dimensions. Using already established transformations, the time dependence is removed and the spatial part is written as a Schrödinger-like equation. The equation is then considered with quadratic, exponential and logarithmic drift terms and the analytical solutions are reported using Laplace transform approach.

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1. Introduction

The Fokker-Planck equation (FPE) was originally formulated to study Brownian motion [1]. Now a day, however, it is used to investigate the stochastic phenomena [2]. In a very efficient method, the FPE is transformed into a linear ordinary second order differential equation which resembles the nonrelativistic Schrödinger equation with an effective potential. Consequently, we expect that the common techniques of quantum mechanics are applied to the FPE. In particular, we recognize the idea of supersymmetry as a powerful tool in mathematical physics. In recent interesting papers, the supersymmetry structure of the FPE has been discussed with Morse and Hulthén terms [3-6].
On the other hand, the Laplace integral transform has already been applied to various wave equations of quantum mechanics including Schrödinger, Dirac and Klein-Gordon equations for different potentials including harmonic, Morse, Coulomb, etc [7-13]. Unlike many common analytical techniques of ordinary differential equations which cannot be generalized to nonlinear, partial or fractional cases, the Laplace transform has been successfully applied to such problems. In particular, the transformation is used to study nonlinear Klein-Gordon equation [14] and fractional Schrödinger equation [15]. In our short note, we are going to consider

[1] Department of Basic Sciences, Garmsar Branch, Islamic Azad University, Garmsar, Iran, email: zarrinkamar.s@gmail.com (corresponding author)
[2] Department of Physics , University of Guilan, Rasht 41635-1914, Iran

the FPE in one spatial equation with quadratic and exponential terms and solve the arising differential equations using the Laplace transform.

2. Fokker-Planck Equation in (1+1)-Dimensions

In order to preserve the brevity, we start from [3-6]

$$f_t(x,t)-\frac{1}{2}f_{xx}(x,t)-\left(U'(x)f(x,t)\right)_x=0, \qquad (1)$$

where $U(x)$ is called the drift term. Using the transformation [3-6]

$$f(x,t)=\frac{1}{2}\exp\left(-U(x)\right)\psi(x,t), \qquad (2)$$

we are left with

$$\psi_t(x,t)=\psi_{xx}(x,t)+\left(U''(x)-\left(U'(x)\right)^2\right)\psi(x,t). \qquad (3)$$

Now, using the expansion

$$\psi(x,t)=\sum_{k\in I}c_k\exp(-\lambda_k t)\psi_k(x), \qquad (4)$$

where λ_k denotes the eigenvalue, as well as introducing

$$f(x,t)=\exp\left(-U(x)-\frac{\lambda}{2}t\right)\psi(x,t), \qquad (5)$$

the time component is removed and we obtain the Schrödinger-like equation [3-6]

$$\psi''(x)+\left(\lambda+U''(x)-\left(U'(x)\right)^2\right)\psi(x)=0. \qquad (6)$$

3. Laplace Transform Approach

3.1. Quadratic Drift Potential

We first consider a quadratic drift potential of the form

$$U(x)=ax^2, \qquad (7)$$

which yields the differential equation

$$\frac{d^2\psi(x)}{dx^2}+\left(\lambda+(2a)-(2ax)^2\right)\psi(x)=0. \qquad (8)$$

Introducing $c=\left(\frac{1}{4a^2}\right)^{1/4}$ and $k=c^2(\lambda+2a)$, as well as applying the change of variable

$$x=c\xi, \qquad (9)$$

Eq. (8) appears as

$$\frac{d^2\psi}{d\xi^2}+\left(k-\xi^2\right)\psi=0. \qquad (10)$$

Since $\psi \approx e^{-\xi^2/2}$ as $\xi \to \pm\infty$,one can factor out the asymptotic solution $\psi = e^{-\xi^2/2}$ and focus on the function $\upsilon(\xi)$ defined by

$$\psi = e^{-\xi^2/2}\ \upsilon(\xi). \qquad (11)$$

Substituting Eq. (11) into (10), it becomes

$$\upsilon'' - 2\xi\upsilon' + (k-1)\upsilon = 0. \qquad (12)$$

Now, using

$$\begin{cases} L\left[\upsilon(\xi)\right]=V(s) \\ L\left[\upsilon^n(\xi)\right]=s^nV(s) \\ L\left[\upsilon''\right]=s^2V(s) \\ L\left[\xi^n\upsilon(\xi)\right]=(-1)^n\left\{L\left[\upsilon(\xi)\right]\right\}^{(n)}. \\ L\left[\xi\upsilon'\right]=-\left(L\left[\upsilon'\right]\right)'=-\left[sV(s)\right]'=-sV'(s)-V(s) \end{cases} \qquad (13)$$

the corresponding Laplace space equation appears as

$$2sV' + \left(s^2+k+1\right)V = 0. \qquad (14)$$

which possesses the solution

$$V = C\, s^{-(k+1)/2} e^{-s^2/4}. \qquad (15)$$

Applying the Laplace inverse transform

$$\upsilon = C\int_\Gamma \left(s-\frac{1}{2}\right)^{-\frac{3}{4}+\frac{k}{4}} \left(s+\frac{1}{2}\right)^{-\frac{3}{4}-\frac{k}{4}} e^{s\xi}ds \qquad (16)$$

Note that in the following we shall denote all the constants by C, although they may represent different values in different place. Only when $(k+1)/2 = n+1$ and $n = 0,1,2,3...$ the residue in Eq. (16) is nonzero and one has

$$\upsilon = CH_n(\xi), \qquad (17)$$

where

$$H_n(\xi) = C\int_\Gamma s^{-(n+1)} e^{-s^2/4} e^{s\xi}ds \qquad (18)$$

is the integral form of the Hermite function. One can also use
$e^{-s^2/4} = \sum_{j=0}^{\infty} \left(s^2/4 \right)^j / (j!)$ to obtain

$$H_n(\xi) = C \int_\Gamma \sum_{j=0}^{\infty} \frac{(-1/4)^j}{j!} \frac{e^{s\xi}}{s^{n+1-2j}} ds \qquad (19)$$

$$= C \sum_{j=0}^{[n/2]} \frac{(-1)^j \, 2^{-2j} \, \xi^{n-2j}}{j!(n-2j)!}. \qquad (20)$$

This is the series form of the Hermite function.The residue for the integral in Eq. (19) is nonzero only for the terms with $k \le [n/2]$. The solution of ψ, from Eqs. (11) and (17) ,appears as $\psi(\xi) = Ce^{-\xi^2/2} H_n(\xi).$ (21)

3.2. Exponential Drift Potential

In this section, we consider a drift term of exponential type

$$\psi(\xi) = Ce^{-\xi^2/2} H_n(\xi). \qquad (22)$$

Therefore, we have to deal with

$$\psi''(x) + \left(\lambda + \left(ab^2 e^{bx} - a^2 b^2 e^{2bx} \right) \right) \psi(x) = 0 \qquad (23)$$

Introducing $c = \frac{\sqrt{-\lambda}}{b}$ and applying the change of variable

$\xi = 2ae^{bx}$ (24)

we find

$$\left(\frac{d^2}{d\xi^2} + \frac{1}{\xi} \frac{d}{d\xi} + \left(-\frac{1}{4} + \frac{1}{2\xi} - \frac{c^2}{\xi^2} \right) \right) \psi(\xi) = 0 \qquad (25)$$

using the gauge transformation,

$$\psi(\xi) = \xi^c \upsilon(\xi) \qquad (26)$$

we have

$$\xi \upsilon'' + (2c+1)\upsilon' - \left(\frac{\xi}{4} - \frac{1}{2} \right) \upsilon = 0 \qquad (27)$$

At this stage, we use the boundary conditions and basic formulae

$$L\left[\upsilon(\xi)\right]=V(s)$$
$$L\left[\upsilon'\right]=sV(s),$$
$$L\left[\xi\upsilon\right]=-V'(s),$$
$$L\left[\xi\upsilon''\right]=-\left(L\left[\upsilon''\right]\right)'=-\left[s^2V(s)\right]'=-s^2V'(s)-2sV(s), \quad (28)$$

which gives the first-order Laplace space equation

$$\left(-s^2+\frac{1}{4}\right)V'(s)+\left[(2c-1)s+\frac{1}{2}\right]V(s)=0 \qquad (29)$$

Eq. (29) possesses the solution

$$V(s)=C\left(s-\frac{1}{2}\right)^{c}\left(s+\frac{1}{2}\right)^{c-1} \qquad (30)$$

Applying the Laplace inverse transform

$$\upsilon=C\int\left(s-\frac{1}{2}\right)^{c}\left(s+\frac{1}{2}\right)^{c-1} e^{s\xi}ds \qquad (31)$$

and using the change of variable $s+\frac{1}{2}=t$,one can see that

$$\upsilon=Ce^{-\frac{\xi}{2}}\,{}_1F_1\left(d,N;\xi\right) \qquad (32)$$

where

$${}_1F_1\left(d,N;\xi\right)=C\int_{\Gamma}t^{d-1}(1-t)^{N-d-1}e^{t\xi}dt \qquad (33)$$

is the integral form of the confluent hypergeometric function and

$d=c$, $\mathrm{N}=2c+1$ (34)

Recalling

$$(1-t)^{N-d-1}=\sum_{k=0}^{\infty}\binom{N-d-1}{k}(-t)^k, \qquad (35)$$

where

$$\binom{\alpha}{k}\equiv\frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}, \qquad (36)$$

Eq. (33) can be writen as

$${}_1F_1\left(d,N;\xi\right)=C\sum_{k=0}^{\infty}\binom{N-d-1}{k}\int_{\Gamma}(-t)^{k+d-1}e^{t\xi}dt. \qquad (37)$$

Only when $d=-n$ and $n=0,1,2,3,\ldots$, the residue is nonzero and one has

$$
{}_1F_1(-n,N;\xi)=C\sum_{k=0}^{n}\binom{N-d-1}{k}(-1)^{n-k+1}\frac{\xi^{n-k}}{(n-k)!}
$$
$$
=C\sum_{k=0}^{n}\binom{N-d-1}{n-k}(-1)^{k+1}\frac{\xi^{k}}{k!}, \qquad (38)
$$

The solution of ψ is then by Eq. (26) and (32)

$$
\psi(\xi)=C\xi^{c}e^{-\frac{\xi}{2}}\,{}_1F_1(-n,N;\xi) \qquad (39)
$$

3.3. Logarithmic Drift Potential

In this section, we consider

$$
U(x)=a\log\left(bx^{2}\right) \qquad (40)
$$

which corresponds to the equation

$$
\psi''(x)+\left(\lambda-\frac{2a}{x^{2}}-\frac{4a^{2}}{x^{2}}\right)\psi(x)=0 \qquad (41)
$$

A gauge transformation of the form

$\psi(x)=x\,f(x)$ (42)

brings Eq. (28) into the form

$$
\frac{d^{2}f(x)}{dx^{2}}+\frac{2}{x}\frac{df(x)}{dx}+\left(\lambda-\frac{2a}{x^{2}}-\frac{4a^{2}}{x^{2}}\right)f(x)=0 \qquad (43)
$$

Using the chance of variable

$$
\rho=\sqrt{\lambda}\,x \qquad (44)
$$

we have

$$
\left(\rho^{2}\frac{d^{2}}{d\rho^{2}}+2\rho\frac{d}{d\rho}+\left(\rho^{2}-2a(2a+1)\right)\right)f(\rho)=0 \qquad (45)
$$

which, via

$$
f(\rho)=z(\rho)\rho^{-\frac{1}{2}} \qquad (46)
$$

appears as

$$
\rho^{2}\frac{d^{2}z(\rho)}{d\rho^{2}}+\rho\frac{dz(\rho)}{d\rho}+\left(\rho^{2}-\left(2a+\frac{1}{2}\right)^{2}\right)z(\rho)=0 \qquad (47)
$$

$$
\frac{d^{2}z(\rho)}{d\rho^{2}}+\frac{1}{\rho}\frac{dz(\rho)}{d\rho}+\left(1-\frac{\left(2a+\frac{1}{2}\right)^{2}}{\rho^{2}}\right)z(\rho)=0 \qquad (48)
$$

Finally, using

$$z(\rho) = \rho^{\beta}\, \upsilon(\rho) \qquad (49)$$

We are left with

$$\frac{d^2\upsilon(\rho)}{d\rho^2} + \frac{(2\beta+1)}{\rho}\frac{d\upsilon(\rho)}{d\rho} + \left(1 + \frac{\beta^2 - \left(2a + \frac{1}{2}\right)^2}{\rho^2}\right)\upsilon(\rho) = 0 \qquad (50)$$

Considering

$$\beta^2 - \left(2a + \frac{1}{2}\right)^2 = 0 \qquad ;\ \beta = \pm\mu = \pm\left(2a + \frac{1}{2}\right) \qquad (51)$$

Eq. (46) takes the form

$$\rho\upsilon'' + (2\beta+1)\upsilon' + \rho\upsilon = 0 \qquad (52)$$

or, in Laplace space

$$-\left(s^2+1\right)V'(s) + (2\beta-1)sV(s) = 0 \qquad (53)$$

Eq. (50) possesses the solution

$$V = C\left(s^2+1\right)^{\frac{(2\beta-1)}{2}} \qquad (54)$$

Applying the Laplace inverse transform

$$\upsilon = C\int_{\Gamma}\left(s^2+1\right)^{\frac{(2\beta-1)}{2}} e^{s\rho}ds \qquad (55)$$

where

$$\beta = 1, 2, 3, \ldots \qquad ;\ \beta = -\left(2a + \frac{1}{2}\right) = -\mu \qquad (56)$$

$$\upsilon = C\int_{\Gamma} s^{-(2\mu+1)}\left(1 + \frac{1}{s^2}\right)^{-\frac{(2\mu+1)}{2}} e^{s\rho}ds = C\int_{\Gamma}\sum_{k=0}^{\infty}\begin{pmatrix} -\left(\frac{2\mu+1}{2}\right) \\ k \end{pmatrix} s^{-(2\mu+1+2k)} e^{s\rho}ds \qquad (57)$$

we obtain

$$\upsilon = C\sum_{k=0}^{\infty}\begin{pmatrix} -\left(\frac{2\mu+1}{2}\right) \\ k \end{pmatrix}\frac{\rho^{2k+2\mu}}{(2k+2\mu)!} = C'\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+\mu)!}\frac{\rho^{2k+2\mu}}{2^{2k+\mu}} \qquad (58)$$

with $C' = \dfrac{C}{(2\mu-1)!!}$. In summary, the final solution is written in terms of Bessel function as

$$z(\rho) = \rho^{-\mu}\upsilon = C'\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+\mu)!}\left(\frac{\rho}{2}\right)^{2k+\mu} = CJ_{\mu}(\rho) \qquad (59)$$

4. Conclusions

In this article, we considered the Fokker-Planck equation in (1+1)-dimensions. Bearing mind, the merits of analytical solution and the physical insight they provide, we used the Laplace transform in the calculations to solve the equation with quadratic and exponential drift terms. The calculations revealed that the arising equations respectively appeared as the nonrelativistic one-dimensional harmonic, Morse and free particle problems. The Laplace approach has the potential to be extended to more realistic three-dimensional case with time-dependence terms. In addition, the generalized exponential-type interactions have been successfully used in the modeling of tumor growth and we hope the present work motivates further studies in the field.

R E F E R E N C E S

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