

## EXTENDED $b$ -METRIC SPACE, EXTENDED $b$ -COMPARISON FUNCTION AND NONLINEAR CONTRACTIONS

Maria Samreen<sup>1</sup>, Tayyab Kamran<sup>2</sup>, Mihai Postolache<sup>3</sup>

*Fagin et al. used some kind of relaxed triangle inequality for the distance measure in pattern matching and called this measure as non-linear elastic matching. Intrigued by such idea, recently, Kamran et al. introduced a new intuitive concept of distance measure to extend the notion of  $b$ -metric space. We present some fixed point theorems when the underlying ambient space is an extended  $b$ -metric space and the contraction condition involve a new class of comparison functions. To substantiate the validity of our results we obtain many well-known pre-existing results in literature. Moreover, we have applied our result to study the solution to an integral equation.*

**Keywords:**  $b$ -metric space, extended  $b$ -metric, comparison function, fixed point.

**MSC2010:** 47H10, 54H25.

### 1. Introduction

The celebrated Banach contraction principle is known as a very fundamental tool for providing the existence and solutions for many mathematical problems involving differential equations, integro-differential equations and integral equations. Efforts have been made to investigate and generalize this remarkable theorem to make it more viable in many other ambient spaces [3, 8, 9, 28]. In this context Bakhtin [2], Bourbaki [7], Czerwik [11] and Heinonen [15] generalized the structure of metric space by weakening the triangle inequality and called it the  $b$ -metric space. Since then several articles have appeared which dealt with fixed point theorems for single-valued and multi-valued functions in  $b$ -metric space: Ali *et al.* [1], Berinde [5], Păcurar [21], Samreen *et al.* [26], Shatanawi *et al.* [27]. Afterwards, Fagin *et al.* [14] utilized this kind of relaxation in the triangular inequality to incorporate with pattern matching. The same kind of concept was also used for trade measure, Cortelazzo *et al.* [10], and measure ice floes, McConnell *et al.* [20]. In this context, Kamran *et al.* [17] introduced the concept of extended  $b$ -metric space by further weakening the triangle inequality.

In this paper we introduce a new class of comparison functions and present some fixed point theorems for the mappings satisfying a contraction condition involving a new class of comparison functions when the underlying space is endowed with an extended  $b$ -metric. The results obtained for such settings become more viable in different avenues of applications.

### 2. Preliminaries

The purpose of this section is to recollect some notions and results from literature, for completeness.

<sup>1</sup> Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.

<sup>2</sup> Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan & School of Natural Sciences, National University of Science and Technology, H-12, Islamabad, Pakistan.

<sup>3</sup> (CORRESPONDING AUTHOR) China Medical University, Taichung 40402, Taiwan; Gh. Mihoc–C. Iacob Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Bucharest 050711, Romania; University "Politehnica" of Bucharest, Bucharest 060042, Romania, Email: emscolar@yahoo.com

Let  $X$  be a nonempty set and  $s \geq 1$  be a given real number.

A function  $d: X \times X \rightarrow \mathbb{R}^+$  is said to be a  $b$ -metric [2, 11] on  $X$  and the pair  $(X, d)$  is called a  $b$ -metric space if for all  $x, y, z \in X$ ,

1.  $d(x, y) = 0$  if and only if  $x = y$ ;
2.  $d(x, y) = d(y, x)$ ;
3.  $d(x, z) \leq s[d(x, y) + d(y, z)]$ .

Let  $X := l_p(\mathbb{R})$  with  $0 < p < 1$ , where  $l_p(\mathbb{R}) := \{x = \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ . Then  $d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p}$  is a  $b$ -metric on  $X$  with  $s = 2^{1/p}$ . Note that  $(X, d)$  is not a metric space.

A function  $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which is increasing, and satisfies the property that  $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$  for all  $t \geq 0$  is said to be a comparison function (Matkowski [19]).

A function  $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which is increasing, and satisfies the property that  $\sum_{n=0}^{\infty} \varphi^n(t)$  converges for all  $t > 0$  is called a  $c$ -comparison function.

Note that any  $c$ -comparison function is a comparison function but the converse may not be true, see, for example, Rus [23], Berinde [4].

Berinde [6, 23] introduced the class of  $b$ -comparison functions.

**Definition 2.1.** Let  $(X, d)$  be a  $b$ -metric space with  $s \geq 1$ . A function  $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is known as a  $b$ -comparison function if which is increasing, and satisfies the property that  $\sum_{n=0}^{\infty} s^n \varphi^n(t)$  converges for all  $t \in \mathbb{R}^+$ .

Let  $(X, d)$  be a  $b$ -metric space with coefficient  $s \geq 1$ . Then  $\varphi(t) = \kappa t ; t \in \mathbb{R}^+$  with  $0 < \kappa < \frac{1}{s}$  is a  $b$ -comparison function. The definition of the  $b$ -comparison function reduces to that of a  $c$ -comparison function when  $s = 1$ .

### 3. Main results

In this section we recall the notion of extended  $b$ -comparison function and prove some fixed point theorems involving such functions in the setting of extended  $b$ -metric spaces. In the following we recollect the definition and some properties of the extended  $b$ -metric space [17].

**Definition 3.1** ([17], lema). Let  $X$  be a non empty set and  $\theta: X \times X \rightarrow [1, \infty)$ . A function  $d_\theta: X \times X \rightarrow [1, \infty)$  is called an extended  $b$ -metric if for all  $x, y, z \in X$  it satisfies:

- $d_\theta 1.$   $d_\theta(x, y) = 0$  if and only if  $x = y$ ;
- $d_\theta 2.$   $d_\theta(x, y) = d_\theta(y, x)$ ;
- $d_\theta 3.$   $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$ .

The pair  $(X, d_\theta)$  is called extended  $b$ -metric space.

Remark that a  $b$ -metric is a special case of the extended  $b$ -metric when  $\theta(x, y) = s$  for  $s \geq 1$ .

To show the concreteness of the idea of extended  $b$ -metric space we include the following nontrivial example of extended  $b$ -metric space.

**Example 3.1.** Let  $X = \{1, 2, 3, \dots\}$ . Define  $\theta: X \times X \rightarrow [1, \infty)$  and  $d_\theta: X \times X \rightarrow \mathbb{R}^+$  as:

$$\theta(x, y) = \begin{cases} |x - y|^3 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases} \quad \text{and } d_\theta(x, y) = (x - y)^4$$

Then  $(X, d_\theta)$  is an extended  $b$ -metric space.

*Proof.* Note that  $(d_\theta 1)$  and  $(d_\theta 2)$  trivially hold.

For the real numbers  $a, b$  with  $a = 0$  or  $|a| \geq 1$  and  $b = 0$ , or  $|b| \geq 1$  we establish the following relation

$$(a + b)^4 \leq |a + b|^3 [a^4 + b^4]. \quad (1)$$

**Case i.** Inequality (1) is trivially satisfied if either  $a = 0$  or  $b = 0$ .

**Case ii.** For  $|a| \geq 1, |b| \geq 1$  we get

$$\begin{aligned} (a+b)^4 &= \frac{(a+b)^4}{(a^4+b^4)}(a^4+b^4) \\ &\leq \frac{(a+b)^4}{|a+b|}(a^4+b^4) \\ &= |a+b|^3(a^4+b^4). \end{aligned}$$

Finally by setting  $a = x - y, b = y - z$  where  $x, y, z \in X$  then  $a = 0$  or  $|a| \geq 1$  and  $b = 0$  or  $|b| \geq 1$ . Thus inequality (1) yields  $(d_\theta 3)$ .

$$(x-z)^4 \leq |x-z|^3[(x-y)^4 + (y-z)^4]$$

Note that  $\sup\{\theta(x, y) | x, y \in X\} = \infty$ .  $\square$

In the following, the concepts of convergence, Cauchy sequence and completeness are introduced in extended  $b$ -metric space.

**Definition 3.2** ([17]). Let  $(X, d_\theta)$  be an extended  $b$ -metric space. Then a sequence  $(x_n)$  in  $X$  is said to be:

(i) convergent if and only if there exists  $x$  in  $X$  such that  $d_\theta(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ , we write  $\lim_{n \rightarrow \infty} x_n = x$ ,

(ii) Cauchy if and only if  $d_\theta(x_n, x_m) \rightarrow 0$  as  $n, m \rightarrow \infty$ .

The extended  $b$ -metric space  $(X, d_\theta)$  is complete if every Cauchy sequence converges in  $X$ . We note that the extended  $b$ -metric  $d_\theta$  is not a continuous function in general. The following lemma is trivial.

**Lemma 3.2.** Let  $(X, d_\theta)$  be an extended  $b$ -metric space. Then every convergent sequence has a unique limit.

*Proof.* It is trivially followed from Definition 3.1  $(d_\theta 3)$ .  $\square$

We define the concept of  $f$ -orbital lower semi-continuity (in case of extended  $b$ -metric space) as used in [16].

**Definition 3.3.** Let  $f: D \subset X \rightarrow X$  and there exist some  $x_0 \in D$  such that the set  $\mathcal{O}(x_0) = \{x_0, fx_0, f^2x_0, \dots\} \subset D$ . The set  $\mathcal{O}(x_0)$  is known as an orbit of  $x_0 \in D$ . A function  $G$  from  $D$  into the set of real numbers is said to be  $f$ -orbitally lower semi-continuous at  $t \in D$  if  $\{x_n\} \subset \mathcal{O}(x_0)$  and  $x_n \rightarrow t$  implies  $G(t) \leq \liminf_{n \rightarrow \infty} G(x_n)$ .

For some technical reasons we introduce another class of comparison functions, as follows.

**Definition 3.4.** Let  $(X, d_\theta)$  be an extended  $b$ -metric space. We say that a function  $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an extended  $b$ -comparison function if it is increasing, and there exists a mapping  $f: D \subset X \rightarrow X$  such that for some  $x_0 \in D, \mathcal{O}(x_0) \subset D, \sum_{n=0}^{\infty} \varphi^n(t) \prod_{i=1}^n \theta(x_i, x_m)$  converges for all  $t \in \mathbb{R}^+$  and for every  $m \in \mathbb{N}$ . Here  $x_n = f^n x_0$  for  $n = 1, 2, \dots$ . We say that  $\varphi$  is an extended  $b$ -comparison function for  $f$  at  $x_0$ .

**Remark 3.3.** (I) It is easily seen that by taking  $\theta(x, y) = s \geq 1$  (constant) for any  $x, y \in X$ , Definition 3.4 coincides with Definition 2.1 for any arbitrary self-map  $f$  on  $X$ .

(II) Since  $\theta(x, y) \geq 1$  for every  $x, y \in X$ , then by setting  $s = \inf_{x, y \in X} \theta(x, y)$  we have

$$\sum_{n=0}^{\infty} s^n \varphi^n(t) \leq \sum_{n=0}^{\infty} \varphi^n(t) \prod_{i=1}^n \theta(x_i, x_m).$$

The above relation invokes that every extended  $b$ -comparison function is also a  $b$ -comparison function for some  $s \geq 1$ .

**Example 3.4.** Let  $(X, d_\theta)$  be an extended  $b$ -metric space,  $f$  a self-map on  $X$ , assume that for  $x_0 \in X$ ,  $\lim_{n,m \rightarrow \infty} \theta(x_n, x_m)$  exists. Define  $\varphi: [0, \infty) \rightarrow [0, \infty)$  as

$$\varphi(t) = \kappa t, \quad \text{such that } \lim_{n,m \rightarrow \infty} \theta(x_n, x_m) < \frac{1}{\kappa}.$$

Then by using ratio test one can easily see that the series  $\sum_{n=1}^{\infty} \varphi^n(t) \prod_{i=1}^n \theta(x_i, x_m)$  converges. Here  $x_n = f^n x_0$  for  $n = 1, 2, \dots$

Now we proceed to establish our main result.

**Theorem 3.5.** Let  $(X, d_\theta)$  be a complete extended  $b$ -metric space such that  $d_\theta$  is continuous. Let  $f: D \subset X \rightarrow X$  be such that  $\mathcal{O}(x_0) \subseteq D$ . Assume that

$$d_\theta(fx, f^2x) \leq \varphi(d_\theta(x, fx)), \quad \text{for each } x \in \mathcal{O}(x_0), \quad (2)$$

where  $\varphi$  is an extended  $b$ -comparison function for  $f$  at  $x_0$ . Then  $f^n x_0 \rightarrow \xi \in X$  (as  $n \rightarrow \infty$ ). Furthermore,  $\xi$  is a fixed point of  $f$  if and only if  $G(x) = d_\theta(x, fx)$  is  $f$ -orbitally lower semi-continuous at  $\xi$ .

*Proof.* Let  $x_0 \in X$  be arbitrary, define the iterative sequence  $\{x_n\}$  by

$$x_0, \quad fx_0 = x_1, \quad x_2 = fx_1 = f(fx_0) = f^2(x_0) \dots, \quad x_n = f^n x_0 \dots$$

Then by successively applying inequality (2), we obtain

$$d_\theta(x_n, x_{n+1}) \leq \varphi^n(d_\theta(x_0, x_1)) \quad (3)$$

By the triangle inequality and (3), for  $m > n$  we have

$$\begin{aligned} & d_\theta(x_n, x_m) \\ & \leq \theta(x_n, x_m) \varphi^n(d_\theta(x_0, x_1)) + \theta(x_n, x_m) \theta(x_{n+1}, x_m) \varphi^{n+1}(d_\theta(x_0, x_1)) \\ & \quad + \dots + \theta(x_n, x_m) \theta(x_{n+1}, x_m) \theta(x_{n+2}, x_m) \dots \theta(x_{m-2}, x_m) \theta(x_{m-1}, x_m) \varphi^{m-1}(d_\theta(x_0, x_1)) \\ & \leq \theta(x_1, x_m) \theta(x_2, x_m) \dots \theta(x_{n-1}, x_m) \theta(x_n, x_m) \varphi^n(d_\theta(x_0, x_1)) \\ & \quad + \theta(x_1, x_m) \theta(x_2, x_m) \dots \theta(x_n, x_m) \theta(x_{n+1}, x_m) \varphi^{n+1}(d_\theta(x_0, x_1)) + \dots + \\ & \quad \theta(x_1, x_m) \theta(x_2, x_m) \dots \theta(x_n, x_m) \theta(x_{n+1}, x_m) \dots \theta(x_{m-2}, x_m) \theta(x_{m-1}, x_m) \varphi^{m-1}(d_\theta(x_0, x_1)). \end{aligned}$$

The series  $\sum_{n=1}^{\infty} \varphi^n(d_\theta(x_0, x_1)) \prod_{i=1}^n \theta(x_i, x_m)$  converges for each  $m \in \mathbb{N}$ . Let

$$S = \sum_{n=1}^{\infty} \varphi^n(d_\theta(x_0, x_1)) \prod_{i=1}^n \theta(x_i, x_m), \quad S_n = \sum_{j=1}^n \varphi^j(d_\theta(x_0, x_1)) \prod_{i=1}^j \theta(x_i, x_m).$$

Thus for  $m > n$ , the above inequality implies

$$d_\theta(x_n, x_m) \leq [S_{m-1} - S_n].$$

Letting  $n \rightarrow \infty$ , we conclude that  $\{x_n\}$  is a Cauchy sequence. Since  $X$  is complete then  $x_n = f^n x_0 \rightarrow \xi \in X$ . Assuming that  $G$  is orbitally lower semi-continuous at  $\xi \in X$ , we obtain

$$\begin{aligned} d_\theta(\xi, f\xi) & \leq \liminf_{n \rightarrow \infty} d_\theta(f^n x_0, f^{n+1} x_0) \\ & \leq \liminf_{n \rightarrow \infty} \varphi^n(d_\theta(x_0, x_1)) = 0. \end{aligned}$$

Conversely, let  $\xi = f\xi$  and  $x_n \in \mathcal{O}(x)$  with  $x_n \rightarrow \xi$ . Then,

$$G(\xi) = d_\theta(\xi, f\xi) = 0 \leq \liminf_{n \rightarrow \infty} G(x_n) = d_\theta(f^n x_0, f^{n+1} x_0),$$

and this completes the proof.  $\square$

**Remark 3.6.** The assertion of Theorem 3.5 holds if we replace  $f$ -orbital continuity of  $G$  with continuity of  $f$ .

#### 4. Consequences and Applications

To substantiate the degree of validity of our result, in this section we include many results from literature obtained as special cases. Theorem 3.5 includes main results by Czerwik [12, Theorem 1] and Samreen *et al.* [25, Theorem 3.10 (5)] as special cases when underlying space is a  $b$ -metric space. It also yields some of the results by Proinov [22] and Hicks and Rhoades [16] in case of metric space.

In the following we include an analogue of the theorem of Hicks and Rhoades [16], in the setting of extended  $b$ -metric space.

**Theorem 4.1.** Let  $(X, d_\theta)$  be a complete extended  $b$ -metric space such that  $d_\theta$  is continuous,  $D \subseteq X$ , and  $x_0 \in D$ . Let  $f: D \subset X \rightarrow X$  be such that  $\mathcal{O}(x_0) \subseteq D$ . Suppose that  $\lim_{n,m \rightarrow \infty} \theta(x_n, x_m)$  exists and  $\kappa$  is a constant so that  $\lim_{n,m \rightarrow \infty} \theta(x_n, x_m) < \frac{1}{\kappa}$  for all  $x_n, x_m \in \mathcal{O}(x_0)$ . Assume that

$$d_\theta(fy, f^2y) \leq \kappa d_\theta(y, fy) \quad \text{for each } y \in \mathcal{O}(x_0).$$

Then  $f^n x_0 \rightarrow \xi \in X$  (as  $n \rightarrow \infty$ ). Furthermore  $\xi$  is a fixed point of  $f$  if and only if  $G(x) = d_\theta(x, fx)$  is  $f$ -orbitally lower semi-continuous at  $\xi$ .

*Proof.* Define  $\varphi: [0, \infty) \rightarrow [0, \infty)$ ,  $\varphi(t) = \kappa t$ . Then Example 3.4 invokes that  $\varphi$  is an extended  $b$ -comparison function for  $f$  at  $x_0$ . Hence the result follows from Theorem 4.1.  $\square$

Remark that Theorem 4.1 extends/generalizes [16, Theorem 1] to the case of extended  $b$ -metric spaces.

**Definition 4.1** ([24]). Let  $\alpha: X \times X \rightarrow [0, \infty)$ . A self-map  $f$  on  $X$  is said to be  $\alpha$ -admissible if

$$\alpha(x, y) \geq 1 \implies \alpha(fx, fy) \geq 1 \quad x, y \in X.$$

**Theorem 4.2.** Let  $(X, d_\theta)$  be a complete generalized  $b$ -metric space such that  $d_\theta$  is a continuous function. Let  $f: X \rightarrow X$  satisfy

$$\alpha(x, y) d_\theta(fx, fy) \leq \varphi(d_\theta(x, y)), \quad \text{for every } x, y \in X, \quad (4)$$

where  $\varphi$  is an extended  $b$ -comparison function for the self-map  $f$  at  $x_0 \in X$ .

Assume that:

- (i).  $f$  is  $\alpha$ -admissible.
- (ii).  $\alpha(x_0, fx_0) \geq 1$  for  $x_0 \in X$ .

Then  $f^n x_0 \rightarrow \xi \in X$  (as  $n \rightarrow \infty$ ).

Furthermore  $\xi$  is a fixed point of  $f$  if and only if  $G(x) = d_\theta(x, fx)$  is  $f$ -orbitally lower semi-continuous at  $\xi$ .

*Proof.* By using condition (i) and (ii) we obtain

$$\alpha(f^n x_0, f^{n+1} x_0) \geq 1, \quad n = 1, 2, \dots$$

For  $n = 1, 2, \dots$ , the contraction condition (4) implies

$$d_\theta(f^n x_0, f^{n+1} x_0) \leq \alpha(f^n x_0, f^{n+1} x_0) d_\theta(f^n x_0, f^{n+1} x_0) \leq \varphi(d_\theta(f^{n-1} x_0, f^n x_0)). \quad (5)$$

The above inequality (5) is equivalent to (2). Thus all the conditions of Theorem 3.5 are satisfied. Hence the result follows.  $\square$

**Remark 4.3.** 1. Given that  $d_\theta$  is a continuous function, we note that if the self-map  $f$  is continuous at some  $\xi \in X$  then the function  $G(x) = d_\theta(x, fx)$  is also continuous at  $\xi$ . Thus, the orbital continuity of  $G$  in Theorem 4.2 can be replaced by the continuity of  $f$ .

2. For  $\theta(x, y) = 1$ , Theorem 4.2 reduces to [24, Theorem 2.1].

Before moving towards another application, we state the following lemma which is an extension of the idea from Example 3.4.

**Lemma 4.4.** Let  $(X, d_\theta)$  be an extended  $b$ -metric space,  $f$  a self-map on  $X$ ,  $x_0 \in X$  and  $\lim_{n,m \rightarrow \infty} \theta(x_n, x_m) < \frac{1}{\kappa}$ , where  $\kappa \in (0, 1)$ . Assume that  $\psi$  is a comparison function then  $\varphi(t) = \kappa\psi$  is an extended  $b$ -comparison function for  $f$  at  $x_0$ .

It is well-known that nonlinear integral equations play a fundamental role in modeling and solving many nonlinear problems, see [13]. Inspired by [18], an application of our result, we shall establish the existence of a solution to the following type of integral equation.

$$x(t) = p(t) + \int_0^t S(t, u)g(u, x(u))du, \quad t \in [0, 1], \quad (6)$$

where  $g: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $p: [0, 1] \rightarrow \mathbb{R}$  are two bounded continuous functions and  $S: [0, 1] \times [0, 1] \rightarrow [0, \infty)$  is a function such that  $S(t, \cdot) \in L^1([0, 1])$  for all  $t \in [0, 1]$ .

Consider the operator

$$f: X \rightarrow X, \quad f(x)(t) = p(t) + \int_0^t S(t, u)g(u, x(u))du. \quad (7)$$

Observe that each fixed point of  $f$  is a solution of integral equation (6). Also,  $f$  is well-defined since  $g$  and  $p$  are two bounded continuous functions. Thus, we state and prove the following result for the existence of a fixed point for (7), which in turn reduces to the result for the existence of a solution to (6).

**Theorem 4.5.** Let  $f: X \rightarrow X$  be the integral operator given by (7). Suppose that the following conditions hold:

(i) for  $x \in X$  and for every  $u \in [0, 1]$ , we have

$$0 \leq g(u, x(u)) - g(u, f(x)(u)) \leq \frac{1}{4} \sqrt{\ln(1 + |x(u) - f(x)(u)|^2)};$$

(ii) for every  $u \in [0, 1]$ , we have

$$\left\| \int_0^1 S(t, u)du \right\|_\infty < 1.$$

Then  $f$  has a fixed point.

*Proof.* Consider the space  $X = C([0, 1], \mathbb{R})$  of all continuous real valued functions defined on  $[0, 1]$ . Note that  $X$  is a complete  $b$ -metric space with respect to  $d_\theta(x, y) = \|x - y\|_\infty = \sup_{t \in [a, b]} |x(t) - y(t)|^2$ , where  $\theta: X \times X \rightarrow [1, \infty)$  is defined as  $\theta(x, y) = 2$ . By construction, it is evident that  $f$  is continuous. Also, by condition (ii), for  $x \in X$ , we write

$$\begin{aligned} |f(x)(t) - f^2(x)(t)|^2 &= \left| \int_0^t S(t, u)[g(u, x(u)) - g(u, f(x)(u))]du \right|^2 \\ &\leq \int_0^t |S(t, u)|^2 |g(u, x(u)) - g(u, f(x)(u))|^2 du \\ &\leq \int_0^t |S(t, u)|^2 \frac{\ln(1 + |x(u) - f(x)(u)|^2)}{16} du \\ &\leq \frac{1}{16} \ln(1 + \|x - f(x)\|_\infty). \end{aligned}$$

Then we have

$$\|f(x) - f^2(x)\|_\infty \leq \frac{1}{16} \ln(1 + \|x - f(x)\|_\infty).$$

Hence, we deduce that

$$d(f(x), f^2(x)) \leq \kappa\varphi(d(x, f(x))),$$

where  $\kappa = \frac{1}{8}$  and  $\varphi(r) = \frac{1}{2} \ln(1+r)$  is a comparison function. It follows from Lemma 4.4 that  $\kappa\varphi$  is an extended  $b$ -comparison function. Further, For  $x_0 \in X$ ,  $\lim_{m,n \rightarrow \infty} \theta(x_n, x_m) = 2$ . Thus, all the conditions of Theorem 3.5 are immediately satisfied and hence the operator  $f$  has a fixed point, which is a solution of the integral equation (6).  $\square$

As a special case of Theorem 4.5, we give the following result for a fractional-order integral equation.

**Corollary 4.6.** Let  $f: X \rightarrow X$  be the integral operator given by

$$f(x)(t) = p(t) + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} g(u, x(u)) du, \quad t \in [0, 1], \quad \alpha \in (0, 1),$$

where  $\Gamma$  is the Euler gamma function given by  $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ . Suppose that the following conditions hold:

(i) for  $x \in X$ , we have

$$0 \leq g(u, x(u)) - g(u, f(x)(u)) \leq \frac{\Gamma(\alpha+1)}{8} \sqrt{\ln(1+|x(u) - f(x)(u)|^2)}$$

for every  $u \in [0, 1]$ .

Then  $f$  has a fixed point.

#### Acknowledgements

The first and the second authors are thankful to Higher Education Commission of Pakistan for providing financial support under the project no. 21-928/SRGP/R&D/HEC/2016.

#### REFERENCES

- [1] M. U. Ali, T. Kamran, M. Postolache, Solution of Volterra integral inclusion in  $b$ -metric spaces via new fixed point theorem, *Nonlinear Anal. Modelling Control*, **22** (2017), No. 1, 17-30.
- [2] I. A. Bakhtin, The contraction mapping principle in almost metric spaces. *Funct. Anal.*, **30** (1989), 26-37.
- [3] G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal.*, **65** (2006), No. 7, 1379-1393.
- [4] V. Berinde, *Contractii Generalizate si Aplicatii*. Cub Press, Baia Mare, 1997.
- [5] V. Berinde, Sequences of operators and fixed points in quasimetric spaces. *Studia Univ. Babeş-Bolyai, Math.*, **16** (1996), 23-27.
- [6] V. Berinde, Generalized contractions in quasimetric spaces, *Seminar on Fixed Point Theory (Preprint)*. Babeş-Bolyai University of Cluj-Napoca, **3** (1993), 3-9.
- [7] N. Bourbaki, *Topologie Generale*. Herman, Paris (1974).
- [8] B. S. Choudhury, N. Metiya, M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, *Fixed Point Theory Appl.* **2013**, Art. No. 152 (2013).
- [9] L. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **45** (1974), 267-273.
- [10] G. Cortelazzo, G. Mian, G. Vezzi, P. Zamperoni, Trademark shapes description by string matching techniques. *Pattern Recognit.*, **27** (1994), 1005-1018.
- [11] S. Czerwik, Nonlinear set-valued contraction mappings in  $b$ -metric spaces. *Atti Sem. Mat. Univ. Modena*, **46** (1998), 263-276.
- [12] S. Czerwik, Contraction mappings in  $b$ -metric spaces. *Acta Math. Inform. Univ. Ostraviensis*, **1** (1993), 5-11.
- [13] K. Deimling, *Nonlinear Functional Analysis*. Springer-Verlag, Berlin, 1985.
- [14] R. Fagin, L. Stockmeyer, Relaxing the triangle inequality in pattern matching. *Int. J. Comput. Vis.*, **30** (1998), 219-231.
- [15] J. Heinonen, *Lectures on Analysis on Metric Spaces*. Springer, Berlin (2001).
- [16] T. L. Hicks, B. E. Rhoades, A Banach type fixed point theorem. *Math. Japonica*, **24** (1979), 327-330.
- [17] T. Kamran, M. Samreen. Q. Ul. Ain, A generalization of  $b$ -metric space and some fixed point theorems, *Mathematics*, **5** (2017), doi:10.3390/math5020019.
- [18] T. Kamran, C. Vetro, M. U. Ali, M. A. Waheed, Fixed point theorem for G-monotone multivalued mapping with application to nonlinear integral equations. *Filomat*, **31** (2017), 2045-2052.
- [19] J. Matkowski, Integrable solutions of functional equations. *Dissertationes Math.*, **127** (1975), 1-68.

- [20] R. McConnell, R. Kwok, J. Curlander, W. Kober, S. Pang,  $\Psi$ -S correlation and dynamic time warping: Two methods for tracking ice floes. *IEEE Trans. Geosci. Remote Sens.*, **29** (1991), 1004–1012.
- [21] M. Păcurar, A fixed point result for  $\varphi$ -contractions on  $b$ -metric spaces without the boundedness assumption. Preprint.
- [22] P. D. Proinov, A generalization of the Banach contraction principle with high order of convergence of successive approximations. *Nonlinear Anal.*, **67** (2007), 2361–2369.
- [23] I. A. Rus, *Generalized Contractions and Applications*. Cluj Univ. Press, 2001.
- [24] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for  $\alpha$ - $\psi$ -contractive type mappings. *Nonlinear Anal.*, **75** (2012), 2154–2165.
- [25] M. Samreen, Q. Kiran, T. Kamran, Fixed point theorems for  $\varphi$ -contractions. *J. Inequal. Appl.*, **2014**, Art. No. 266 (2014).
- [26] M. Samreen, T. Kamran, N. Shahzad, Some fixed point theorems in  $b$ -metric space endowed with graph. *Abstr. Appl. Anal.*, **2013**, ID 967132.
- [27] W. Shatanawi, A. Pitea, R. Lazovic, Contraction conditions using comparison functions on  $b$ -metric spaces. *Fixed Point Theory Appl.*, **2014**, Art. No. 135 (2014).
- [28] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, *Proc. Amer. Math. Soc.*, **136** (2008), No. 5, 1861–1869.