ON (2-CLOSED) REGULAR HYPERGROUPS

Mohammad HAMIDI[1] and Violeta LEOREANU-FOTEA[2]

In this paper we construct regular hypergroups on all nonempty set. Fundamental group is obtained via a fundamental relation on a regular hypergroup and it is shown that any group is a fundamental group (especially free group). Moreover, closed hypergroups are investigated.

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1. Introduction

The hyperstructure theory was introduced by F. Marty at the 8th congress of Scandinavian Mathematicians in 1934 [14]. Marty introduced the concept of hypergroups as a generalization of groups and used it in different contexts like algebraic functions, rational fractions and non commutative groups. Fundamental relations are a main tool which connects algebraic hyperstructures theory with algebraic structures theory. The fundamental relation β^* is one of the most important relations that was introduced on hypergroups by Koskas [12] and studied mainly by Corsini [5]. Up to now, many researchers have studied hyperstructures theory and its applications. S. Sh. Mousavi et. al showed that semihypergroup category and hypergroup theory have not free objects and defined and investigated the notion of weak free (semi) hypergroups [17]. M. Hamidi et. al defined a new fundamental relation τ^* on divisible hypergroups and obtained divisible groups via this relation and computed automorphism groups of very thin H_v–groups [9, 10]. Extensive hypergroups were introduced by Chvalina [4] and studied also by C. Massouros [15, 16], who called them closed hypergroups. Further materials regarding groups and hypergroups are available in the literature, see for instance [1, 2, 3, 6, 7, 8, 11, 13, 18]. In this paper, first we construct regular hypergroups on a nonempty set and investigate properties of this class of regular subhypergoups. We analyse closed hypergroups, in the context of regular hypergroups. Then we define and investigate generated subhypergroups in regular hypergroups. Regular subhypergroups are analysed, using the

[1]Professor, Department of Mathematics, Faculty of Mathematical Sciences, University "PAYAME NOOR" of Tehran, Iran, E-mail: m.hamidi@pnu.ac.ir

[2] Professor, Department of Mathematics, Faculty of Mathematical Sciences, University "AL. I. CUZA" of Iasi, Romania, E-mail: violeta.fotea@uaic.ro

fundamental relation β; moreover, we show that all (free) group is a fundamental group. The results of this paper are accompanied by various examples.

2. **Preliminaries**

In this section, we mention some definitions and results from [19], which we need in what follows.

Let H be a nonempty set and $P^*(H)$ be the family of all nonempty subsets of H. A function $\circ : H \times H \longrightarrow P^*(H)$ is called a *hyperoperation.* For all x, y of H, $x \circ y$ is called the *hyperproduct* of x and y. An algebraic system $(H, \circ_1, \circ_2, \ldots, \circ_n)$ which has at least one a hyperoperation $\circ_i$ is called a *hyperstructure* and a binary structure $(H, \circ)$ endowed with one hyperoperation $\circ$ is called a *hypergroupoid.* For all two nonempty subsets A and B of $H, A \circ B$ means $\bigcup\limits_{a\in A, b\in B} a \circ b$. A *hypergroupoid* $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H, (x \circ y) \circ z = x \circ (y \circ z)$ and a semihypergroup $(H, \circ)$ is called a *hypergroup* if satisfies in the *reproduction axiom*, i.e. for all $x \in H, x \circ H = H \circ x = H$. A hypergroup $(H, \circ)$ is called a *regular hypergroup*, if it has at least an identity element and all element of H has at least an inverse. In other words, there exists $e \in H$, such that for all $x \in H$, we have $x \in (x \circ e) \cap (e \circ x)$ and for all $x \in H$ there exists $x' \in H$ such that $e \in (x \circ x') \cap (x' \circ x)$. A regular hypergroup $(H, \circ)$ is called *commutative,* if for all $x, y \in H, x \circ y = y \circ x$. A commutative hypergroup $(H, \circ)$ is called a *join space* if for all elements $a, b, c, d \in H$, $a/b \cap c/d \neq \emptyset$ implies that $(a \circ d) \cap (b \circ c) \neq \emptyset$, where a/b denotes the set $\{x \in H \mid a \in x \circ b\}$. A join space $(H, \circ)$ is called *geometric*, if there exists a $x \in H$ such that $x \circ x = \{x\} = x/x$. A nonempty subset K of a hypergroup $(H, \circ)$ is a subhypergroup if for all a of K we have $a \circ K = K \circ a = K$ and subhypergroup K of H is called a *closed* subhypergroup, if for all $k_1, k_2 \in K$ and $x \in H$, $k_1 \in (x \circ k_2) \cap (k_2 \circ x)$ implies that $x \in K$. Let H_1 and H_2 be hypergroups. A map $f : H_1 \to H_2$ is called an *homomorphism* if for all $x, y \in H_1, f(x \cdot y) \subseteq f(x) \cdot f(y)$ and is called a good homomorphism if for all $x, y \in H_1$, we have $f(x \cdot y) = f(x) \cdot f(y)$.

A hypergroup $(H, \circ)$ is called *flat*, if for all subhypergroup K of G, we have $w_K = w_H \cap K$, where $w_H = \{x \in H \mid \varphi(x) = 1\}$ is the *heart* of H. For all homomorphism $f : H_1 \longrightarrow H_2,\ Ker(f) = \{x \in H_1 \mid f(x) \in w_{H_2}\}$. Let $(H, \circ)$ be a semihypergroup and A be a nonempty subset of G. We say that A is a *complete part* of G if for all nonzero natural number n and for all $a_1, ..., a_n$ of H, $A \cap \prod\limits_{i=1}^{n} a_i \neq \emptyset$ implies that $\prod\limits_{i=1}^{n} a_i \subseteq A$.

3. **On closed hypergroups**

In this section, we analyse regular hypergroups, mainly closed hypergroups, and highlight identity and inverse elements. Generated subhypergroups are studied in the context of regular hypergroups.

From now on, for a regular hypergroup H we denote by $Id(H)$ the set of all identity elements of H and for all $x \in H, x_e^{-1}$ denotes an inverse of x with respect to the identity element $e \in Id(H)$. Denote by $In_e(x)$ the set of all inverse elements of x with respect to e.

Theorem 3.1. *Let H be a nonempty set. Then there exists a binary hyperoperation "$\circ$" on H such that $(H, \circ)$ is a commutative regular hypergroup.*

Proof. Let $|H| \geq 1$. For all $x, y \in H$ define a hyperoperation "$\circ$" on H by $x \circ y = \begin{cases} \{x\}, & \text{if } x = y, \\ \{x, y\}, & \text{otherwise.} \end{cases}$. Clearly, $Id(H) = H$.

□

Theorem 3.2. *Let $(G, \circ)$, $(H, \circ')$ be regular hypergroups, $f : (G, \circ) \to (H, \circ')$ be a homomorphism and $x, y \in G$. Then*

(i) if $e \in Id(G)$, then $e \in In_e(e)$;
(ii) if $e \in Id(G)$, then $f(e) \in Id(H)$;
(iii) if $e \in Id(G)$, then we have $f(x_e^{-1}) \in In_{f(e)}(f(x))$;
(iv) $x\beta^* y$ implies that $f(x)\beta^* f(y)$;
(v) if f is a bijection, then for all $x \in G$, $f(\beta^*(x)) = \beta^*(f(x))$.
(vi) if $(G, \circ) \cong (H, \circ')$, then $G/\beta^* \cong H/\beta^*$.

Example 3.1. *Let $H = \{0, 1, 2\}$. Then $(H, \circ)$ is a hypergroup, where $\circ$ is defined as follows:*

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0, 1, 2\}$
1	$\{1\}$	$\{1\}$	$\{0, 1, 2\}$
2	$\{2\}$	$\{2\}$	$\{0, 1, 2\}$

Clearly $K_1 = \{0\}$ and $K_2 = \{1\}$ are subhypergroups of H but $K_1 \cap K_2$ is not a subhypergroup of H.

Remark 3.1. *If $(G, \circ)$ is a regular hypergroup and $\{H_i \mid i \in I\}$ is a family of its regular subhypergroups, then $\bigcap_{i \in I} H_i$ is not necessarily a regular subhypergroup.*

Example 3.2. *Let $G = \{a, b, c, d\}$. Define a hyperoperation "$\circ$" on G as follows:*

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a, b, d\}$
b	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a, b, d\}$
c	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{c, d\}$
d	$\{a, b, d\}$	$\{a, b, d\}$	$\{c, d\}$	$\{a, b, d\}$

Clearly $(G, \circ)$ is a regular hypergroup. It is easy to see that $Id(G) = \{c, d\}$, $In_c(a) = \{c\} = In_c(b)$, $In_c(c) = G$, $In_d(a) = \{d\} = In_d(b)$ and $In_d(d) = G$. Set $H_1 = \{a, b, c\}$ and $H_2 = \{a, b, d\}$. Clearly H_1 and H_2 are regular subhypergroups of G, but $H_1 \bigcap H_2 = \{a, b\}$ is not a regular subhypergroup.

Remark 3.2. *Let* $(G, \circ)$ *be the regular hypergroup, defined in Example 3.2. We have* $In_c(a \circ b) = In_c(a) = \{c\}$*, while* $In_c(b) \circ In_c(a) = c \circ c = \{a, b, c\}$*. Hence in general* $In_c(a \circ b) \neq In_c(b) \circ In_c(a)$*.*

Definition 3.1. *Let* $(G, \circ)$ *be a hypergroup.* G *is called a closed hypergroup, if for all* $x, y \in G, \{x, y\} \subseteq x \circ y$*. The hypergroup* $(G, \circ)$ *where for all* $x, y \in G, x \circ y = \{x, y\}$ *is called* 2*–closed hypergroup.*

The following result can be easily checked for the 2–closed hypergroup $(G, \circ)$, which is commutative.

Theorem 3.3. *Let* $x, y \in G$*. We have*

(i) $x/x = G$.
(ii) $x/y = \{x\}$ *for* $x \neq y$.
(iii) $Id(G) = G$.
(iv) $In_x(x) = G$ *and* $In_x(y) = \{x\}$ *for* $x \neq y$.
(v) *Every subhypergroup of* G *is a closed subhypergroup.*

Theorem 3.4. *Let* $(G, \circ)$ *be the* 2*–closed hypergroup and* $x, y, z, t \in G$*. Then*

(i) $x/y \cap z/t \neq \emptyset$ *if and only if* $x \in \{y, z\}$ *or* $z = t$;
(ii) $(G, \circ)$ *is a join space.*
(iii) $(G, \circ)$ *is geometric if and only if* $|G| = 1$.

Theorem 3.5. *Let* $(G, \circ)$ *be a closed hypergroup and* $\emptyset \neq H \subseteq G$*. Then*

(i) $(G, \circ)$ *is a regular hypergroup;*
(ii) $Id(G) = G$ *and for all* $x, y \in G, x \in In_x(y)$ *and* $In_x(x) = G$;
(iii) *if* $(G, \circ)$ *is* 2*–closed, then for all* $x, y, z \in G$ *we have* $In_x(y \circ z) = In_x(y) \circ In_x(z)$;
(iv) *if* $|G| = n$ *and* $(G, \circ)$ *is* 2*–closed, then* $(H, \circ)$ *is a regular subhypergroup of* $(G, \circ)$;
(v) *if* $\{H_i\}_{i \in I}$ *is a family of regular subhypergroups of* G *and* $\bigcap_{i \in I} H_i \neq \emptyset$*, then* $\bigcap_{i \in I} H_i$ *is a regular subhypergroup of* G.

Proof. $(i), (ii)$ Let $x, y \in G$. Then $\{x, y\} \subseteq x \circ y$, so $(G, \circ)$ is a regular hypergroup and $Id(G) = G$. Moreover, $x \in In_x(y)$ and $In_x(x) = G$.

(iii) We have that $x \circ y = \{x, y\}$ and so $In_x(y) = \{x\}$ for all $x, y \in G$. Hence

$$In_x(y \circ z) = In_x\{y, z\} = \{x\} = x \circ x = In_x(y) \circ In_x(z).$$

(iv) $H = \{x_1, x_2, \ldots, x_k\}$, where $1 \leq k \leq n$. So $(H, \circ)$ is a regular hypergroup.

(v) It is obvious. □

Corollary 3.1. *Let* $(G, \circ)$ *be a* 2*–closed hypergroup. If* $|G| = n$*, then* $|\mathcal{S}_h| = 2^n - 1$*, where* $\mathcal{S}_h = \{H \mid H$ *is a subhypergroup of* $G\}$*.*

Definition 3.2. *A regular hypergroup $(G, \circ)$ is called a quasi complete regular hypergroup, if for all $a \in G$ and $e \in Id(G)$, $In_e(a)$ is a complete part, that it means, if for all nonzero natural number n and for all $a_1, \ldots, a_n \in G$, the following implication holds:*

$$In_e(a) \cap (a_1 \circ a_2 \circ \ldots \circ a_n) \neq \emptyset \Longrightarrow (a_1 \circ a_2 \circ \ldots \circ a_n) \subseteq In_e(a).$$

Proposition 3.1. *Let $(G, \circ)$ be a quasi complete regular hypergroup and $a, b \in G$. Then for all $e \in Id(G)$ we have $In_e(a \circ b) \subseteq In_e(b) \circ In_e(a)$.*

Proof. Let $t \in In_e(a \circ b)$, then $e \in t \circ (a \circ b) = (t \circ a) \circ b$. It implies that $(t \circ a) \cap In_e(b) \neq \emptyset$. Since $(G, \circ)$ is a quasi complete regular hypergroup, we obtain that $(t \circ a) \subseteq In_e(b)$ and so $t \in (t \circ e) \subseteq t \circ (a \circ In_e(a)) = (t \circ a) \circ In_e(a) \subseteq In_e(b) \circ In_e(a)$. It follows that $In_e(a \circ b) \subseteq In_e(b) \circ In_e(a)$. □

Example 3.3. *Consider the regular hypergroup $(G, \circ)$ in Example 3.2. Clearly $In_c(a) \cap (b \circ c) \neq \emptyset$, but $(b \circ c) \nsubseteq In_c(a)$. So the converse of Proposition 3.1 is not necessarily true.*

Notation 3.1. *Let $(G, \circ)$ be a regular hypergroup and $A \subseteq G$. Then for all $x \in G$ and all $n \in \mathbb{Z}$, we denote*

$$x^{\uparrow n} = \begin{cases} \underbrace{x \circ x \circ \ldots \circ x,}_{n \ times} & \text{if } n > 0 \\ Id(G), & \text{if } n = 0 \\ \bigcup\limits_{e \in Id(G)} In_e(x), & \text{if } n < 0 \end{cases} .$$

Denote by $\mathcal{L}(A)$ the set of all finite products of elements A with respect to $\circ$.

If $A = \{a_1, a_2, \ldots a_n\}$, denote $\bigodot\limits_{i=1}^{n} a_i = a_1 \circ a_2 \circ \ldots a_n$.

Proposition 3.2. *Let $(G, \circ)$ be a 2–closed hypergroup. Then*

(*i*) *for all $x \in G, x^{\uparrow n} = \begin{cases} \{x\} & n > 0 \\ G & n \leq 0 \end{cases}$;*

(*ii*) *for all $A \subseteq G$, we have $\mathcal{L}(A)$ is the set of finite parts of A.*

Proof. (*i*) For $n \leq 0, x^{\uparrow n} = G$, since $In_e(x) = \{e\}$ and for all $n > 0$, $x^{\uparrow n} = \{x\}$.

(*ii*) For all $1 \leq k \leq n, \bigodot\limits_{i=1}^{k} a_i = a_1 \circ a_2 \circ \ldots a_k = \{a_1, a_2, \ldots, a_k\}$. It follows that $\mathcal{L}(A) = \{\bigodot\limits_{i=1}^{k} a_i \mid k \in \mathbb{N}^*\} = \{\{a_1, a_2, \ldots, a_k\} \mid k \in \mathbb{N}^*\}$. □

Definition 3.3. *Let $(G, \circ)$ be a regular hypergroup, A be a subset of G and $\{H_i \mid i \in I\}$ be the family of all subhypergroups of G which contain A, such that $\bigcap\limits_{i \in I} H_i$ is a* *subhypergroup of G. The subhypergroup $< A >$ generated by A is $\bigcap_{i\in I} H_i$. If $A = \{a_1, a_2, \ldots, a_n\}$, then the subhypergroup $< A >$ is denoted by $< a_1, a_2, \ldots, a_n >$.*

Example 3.4. *Let $G = \{0, 1, 2, 3\}$. Then $(G, \circ)$ is a regular hypergroup, defined as follows:*

$\circ$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,2\}$	$\{3\}$	$\{0,2\}$
2	$\{2\}$	$\{3\}$	$\{0\}$	$\{1\}$
3	$\{3\}$	$\{0,2\}$	$\{1\}$	$\{0,2\}$.

Then $< 0 >= \{0\}, < 2 >= \{0, 2\}, < 1 >= G$ and $< 3 >= G$. Moreover, $< 0, 2 >= \{0, 2\}$ and for others $x, y \in G, < x, y >= G$.

Theorem 3.6. *Let $(G, \circ)$ be a 2–closed hypergroup and $A = \{a_1, a_2, \ldots, a_n\}$ be a subset of G.*

(i) $B = \{t \mid t \in a_1^{\uparrow k_1} \circ a_2^{\uparrow k_2} \ldots \circ a_n^{\uparrow k_n}, k_i \in \mathbb{N}\}$ is a regular subhypergroup of $(G, \circ)$.
(ii) $< A >= A$.
(iii) If $a \in G$, then $< a >= \{t \mid t \in a^{\uparrow n}, n \in \mathbb{N}\}$.

Proof. (i) By Proposition 3.2, we have

$$B = \{t \mid t \in a_1^{\uparrow k_1} \circ a_2^{\uparrow k_2} \ldots \circ a_n^{\uparrow k_n}, k_i \in \mathbb{N}\} = \{t \mid t \in a_1 \circ a_2 \circ \ldots \circ a_n\} = \{a_1, a_2, \ldots, a_n\}.$$

By Theorem 3.5, B is a regular subhypergroup of G.

(ii) By (i), since A is the smallest regular subhypergroup of G which contains A, it follows that $< A >= A$.

(iii) Let $a \in G$. By (ii), we have $< a >= \{a\}$. □

Example 3.5. *Let $G = \{a, b, c, d, e\}$. Define a hyperoperation "$\circ$" on G as follows:*

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{a,e\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,d\}$	$\{b,e\}$
c	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$	$\{c,e\}$
d	$\{a,d\}$	$\{b,d\}$	$\{c,d\}$	$\{d\}$	$\{d,e\}$
e	$\{a,e\}$	$\{b,e\}$	$\{c,e\}$	$\{d,e\}$	$\{e\}$

Clearly $(G, \circ)$ is 2–closed hypergroup. It is easy to see that for all $x \in G$, $< x >= \{x\}$ and for all $x, y \in G, < x, y >= \{x, y\}$ is a regular subhypergroup of $(G, \circ)$ and for all $x, y, z \in G, < x, y, z >= \{x, y, z\}$.

Theorem 3.7. *Let $(G, \circ)$ be a quasi complete regular hypergroup and $A = \{a_1, \ldots, a_n\}$ be a nonempty subset of G. Then*

(i) $B = \{t \mid t \in a_1^{\uparrow k_1} \circ a_2^{\uparrow k_2} \ldots \circ a_n^{\uparrow k_n}, a_i \in A, k_i \in \mathbb{Z}\}$ is a regular subhypergroup of $(G, \circ)$;
(ii) if $a \in G$, then $< a >= \{t \mid t \in a^{\uparrow n}, n \in \mathbb{Z}\}$.

Proof. (i) Since $Id(G) \cap B \neq \emptyset$, it follows that $B \neq \emptyset$. Clearly $(B, \circ)$ is a semihypergroup. For all $x \in B$, there exist $k_1, \ldots, k_n, j_1, \ldots j_m \in \mathbb{Z}$ such that $x \in a_1^{\uparrow k_1} \circ a_2^{\uparrow k_2} \ldots \circ a_n^{\uparrow k_n}$, so

$$x \circ B = \bigcup_{y \in B} x \circ y \subseteq \bigcup \{ \bigodot_{i=1}^{n} a_i^{\uparrow k_i} \circ \bigodot_{j=1}^{m} a_j'^{\uparrow l_j} \mid (l_1, \ldots, l_m) \in \mathbb{Z}^m \} \subseteq B,$$

where $a'_1, \ldots, a'_m \in A$. Moreover, for all $t \in B$, we have $t \in e \circ t \subseteq x \circ x^{-1} \circ t \subseteq x \circ B$, it concludes that $B \subseteq x \circ B$ and so $B = x \circ B = B \circ x$. Let $x \in B$. By Proposition 3.1, for all $e \in Id(G)$,

$$In_e(x) \subseteq In_e(a_1^{\uparrow k_1} \circ a_2^{\uparrow k_2} \ldots \circ a_n^{\uparrow k_n}) \subseteq In_e(a_n^{\uparrow k_n}) \circ \ldots \circ In_e(a_1^{\uparrow k_1}) \subseteq B$$

and so $\emptyset \neq In_e(x) \subseteq B$. Therefore, $(B, \circ)$ is a regular hypergroup.

(ii) It is immediate. □

Example 3.6. (i) *Let $G = \{1, 2, 3, 4\}$. Define a hyperoperation "$\circ$" on G as follows:*

$\circ$	1	2	3	4
1	$\{1\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$
2	$\{1,2\}$	$\{1,2\}$	$\{2,3\}$	$\{2,4\}$
3	$\{1,3\}$	$\{3,2\}$	$\{1,3\}$	$\{3,4\}$
4	$\{1,4\}$	$\{4,2\}$	$\{4,3\}$	$\{1,4\}$

Clearly $(G, \circ)$ is a closed hypergroup. It is easy to see that $< 1 >= \{1\}, < 2 >= \{1, 2\}, < 3 >= \{1, 3\}$ and $< 4 >= \{1, 4\}$. For all $x, y \in G, < x, y >= \{1, x, y\}$ is a regular subhypergroup of $(G, \circ)$ and for all $x, y, z \in G \setminus \{1\}, < x, y, z >= G$.

(ii) *Let $G = \{0, 2, 4, 6\}$. Define a hyperoperation "$\circ$" on G as follows:*

$\circ$	0	2	4	6
0	$\{0\}$	$\{2\}$	$\{4\}$	$\{6\}$
2	$\{2\}$	$\{4\}$	$\{6\}$	$\{0\}$
4	$\{4\}$	$\{6\}$	$\{0\}$	$\{2\}$
6	$\{6\}$	$\{0\}$	$\{2\}$	$\{4\}$

Clearly $(G, \circ)$ is a quasi complete regular hypergroup which is actually isomorphic to the group $(\mathbb{Z}_4, +)$. It is easy to see that $< 0 >= \{0\}, < 2 >= G, < 4 >= \{0, 4\}$ and $< 6 >= G$. Moreover $< 0, 4 >=< 4 >$ is a regular subhypergroup of $(G, \circ)$ and for all $x, y, z \in G, < x, y, z >= G$.

3.1. Construction of regular hypergroups

In this subsection, we construct regular hypergroups using the concept of reduced words on a set.

Definition 3.4. *Let A be a nonempty set and consider $A^{-1} = \{a^{-1} \mid a \in A\}$ a disjoint set with A, such that there is a bijection $A \longrightarrow A^{-1}$, which we denote by $a \longrightarrow a^{-1}$. Now, we choose an element $1 \notin A \bigcup A^{-1}$ and we call $A \bigcup A^{-1} \bigcup \{1\}$ the alphabet on A.*

Define a nonempty word w on A, as a formal expression of the form $w = a_1^{\delta_1}a_2^{\delta_2}a_3^{\delta_2}\ldots = \prod_i a_i^{\delta_i}$ *where* $\delta_i = \pm 1$, $a_i \in A\bigcup A^{-1}\bigcup\{1\}$ *and for some* $k \in \mathbb{N}, a_k = 1$ *for all* $k \geq n$ *.*

We say that two such expressions represent the same word if they have the same elements in the same positions . The sequence $w = 111\ldots$ is called the *empty word* and we denote it by 1. If $a \in A$, then a^1 will be denoted by a and a^0 will be denoted by 1.

A word $w = a_1^{\delta_1}a_2^{\delta_2}a_3^{\delta_2}\ldots$ on A is said to be a *reduced word* if either it is empty or for all $a_i \in A, a_i$ and a_i^{-1} are never adjacent and if there exists an index k such that $a_k^{\delta_k} = 1$, then for all $i \geq k, a_i^{\delta_i} = 1$. Every nonempty reduced word has the form $w = a_1^{\delta_1}a_2^{\delta_2}a_3^{\delta_n}\ldots a_n^{\delta_n}111\ldots$ and in what follows we shall denote this word by $w = a_1^{\delta_1}a_2^{\delta_2}a_3^{\delta_n}\ldots a_n^{\delta_n}$, where $n \in \mathbb{N}$.

Let $A^* = \{w \mid w$ is a reduced word on A$\}$, where $A \neq \emptyset$.

Notation 3.2. *Let A be a nonempty set and $A^* = \{w \mid w$ is a reduced word on $A\}$. Then for all $x \in G$ and all $n \in \mathbb{Z}$ we denote*

$$x^{\downarrow n} = \begin{cases} \underbrace{xx\ldots x}_{(n-times)} & n > 0 \\ 1 & n = 0 \\ \underbrace{x^{-1}x^{-1}\ldots x^{-1}}_{(n-times)} & n < 0 \end{cases},$$

For example $x^{\downarrow 4}x^{\downarrow 2}$ and $x^{\downarrow 6}$ are both abbreviations for xxxxxx and so $x^{\downarrow 4}x^{\downarrow 2} = x^{\downarrow 6}$.

Now, we define a binary hyperoperation " $\circ$ " on set $A^* \times A^*$, as follows:

$$(w_1, w_2) \circ (w_3, w_4) = \{(w_1w_3, y) \mid y \text{ contains of letters of } w_2, w_4\}. \tag{1}$$

Hence if $(w_1, w_2) = (\prod_{i=1}^{n} a_i^{\delta_i}, \prod_{i=1}^{m} b_i^{\epsilon_i})$ and $(w_3, w_4) = (\prod_{i=1}^{k} c_i^{\lambda_i}, \prod_{i=1}^{l} d_i^{\mu_i})$, then by juxtaposition (also called concatenation) we have $w_1w_3 = \prod_{i=1}^{n} a_i^{\delta_i}\prod_{i=1}^{k} c_i^{\lambda_i}, w1 = 1w$ and $ww^{-1} = w^{-1}w = 1$, where $w \in A^*$. For example,

$$\begin{aligned} & (a_1 1 a_2 a_3, 1b_1) \circ (a_3^{-1}c_1, d_1 1 d_2 111) \\ = \; & \{(a_1a_2c_1, b_1), (a_1a_2c_1, d_1), (a_1a_2c_1, d_2), (a_1a_2c_1, b_1d_1), (a_1a_2c_1, d_1b_1), \\ & (a_1a_2c_1, b_1d_2), (a_1a_2c_1, d_2b_1), (a_1a_2c_1, d_1d_2), (a_1a_2c_1, d_2d_1), (a_1a_2c_1, b_1d_1d_2), \\ & (a_1a_2c_1, b_1d_2d_1), (a_1a_2c_1, d_1d_2b_1), (a_1a_2c_1, d_1b_1d_2), (a_1a_2c_1, d_2d_1b_1), \\ & (a_1a_2c_1, d_2b_1d_1), \ldots\} \end{aligned}$$

Example 3.7. *Let $B = \{b\}$. Then $B^* = \{b\}^* = \{1, b, bb, bbb, b^{-1}, b^{-1}b^{-1}, \ldots\} = \{b^{\downarrow n} \mid n \in \mathbb{Z}\}$.*

Proposition 3.3. *Let A be a non-empty set. Then*
(i) $A \hookrightarrow A^*$;
(ii) $A \hookrightarrow A^* \times A^*$;
(iii) A^* *and* $A^* \times A^*$ *are infinite.*

Proof. Let $w \in A^*$. For all $a \in A$, $\sigma_1 : A \longrightarrow A^* \times A^*$ defined by $\sigma_1(a) = (a, w)$ and $\sigma_2 : A \longrightarrow A^*$ defined by $\sigma_2(a) = a$ are inclusion maps. □

Theorem 3.8. *Let A be a nonempty set. Then there exists a hyperoperation "$\circ$" on $A^* \times A^*$ such that $(A^* \times A^*, \circ)$ is a regular hypergroup.*

Proof. We consider the hyperoperation $\circ$ defined in definition (1) on $A^* \times A^*$.

It is easy to see that other axioms are satisfied and so $(A^* \times A^*, \circ)$ is a regular hypergroup. □

Theorem 3.9. *Let A be a nonempty set. Then there exists a hyperoperation "$\bullet$" on A^* such that $(A^*, \bullet)$ is a regular hypergroup.*

Proof. For all w_1 and w_2, define a hyperoperation $\circ$ on A^* as follows:

$$w_1 \bullet w_2 = \begin{cases} \{1, a_1, a_2, \ldots, a_n\}, & \text{if } w_1 = w_2 = \prod_{i=1}^{n} a_i^{\delta_i}, \\ \{a_1, \ldots, a_n, b_1, \ldots, b_m, w_1, w_2\ \}, & \text{if } w_1 = \prod_{i=1}^{n} a_i^{\delta_i}\ ,\ w_2 = \prod_{i=1}^{m} b_i^{\epsilon_i}. \end{cases}$$

Hence $(A^*, \bullet)$ is a regular hypergroup. □

Example 3.8. *Let $A = \{a\}$. Then*

(*i*) $(A^*, \bullet) = \{1, a, a^{\downarrow 2}, a^{\downarrow 3}, \ldots, a^{-1}, a^{\downarrow -2}, \ldots\}$ *is a regular hypergroup, for which* $Id(A^*) = \{1\}$ *and for all* $w \in A^*, \{1, w\} \subseteq In_1(w)$;

(*ii*) $(A^*, .)$ *is a commutative group, where for all* $a^{\downarrow n}, a^{\downarrow m} \in A^*, a^{\downarrow n}.a^{\downarrow m} = a^{\downarrow m+n}$;

(*iii*) $(A^* \times A^*, \circ) = \{(1,1), (1,a), (a,1), (a,a), (a^{\downarrow 2}, 1), \ldots, (a^{\downarrow 4}, a), \ldots\}$ *is a regular hypergroup such that* $|Id(A^* \times A^*)| \geq 1$.

4. Regular hypergroups and fundamental relations

In this section, we show that all groups are fundamental groups via the fundamental relation on regular hypergroups.

Definition 4.1. (*i*) *A group $(G, \cdot)$ is said to be a fundamental group if there exists a nontrivial regular hypergroup $(H, \circ)$ such that* $(\frac{(H,\circ)}{\beta}, *) \cong (G, \cdot)$.
(*ii*) *A fundamental group $(G, .)$ is said to be a self fundamental group if there exists a nontrivial hyperoperation $\circ$ on G such that* $(\frac{(G,\circ)}{\beta}, *) \cong (G, \cdot)$.

Lemma 4.1. *Let $(G, .)$ be a group. Then for all group $(H, \cdot)$, there exists a binary hyperoperation "$\circ$" on group $G \times H$ such that $(G \times H, \circ)$ is a regular hypergroup.*

Proof. Let $(H,\cdot)$ be a nonzero group. Define a hyperoperation " $\circ$ " on $G\times H$, by $(g,h)\circ(g',h')=\{(g\cdot g',h),(g\cdot g',h')\}$. Therefore $(G\times H,\circ)$ is a regular hypergroup. □

Remark 4.1. *If G and H are groups, then:*

(*i*) *The regular hypergroup $(G\times H,\circ)$ is called the associated regular hypergroup to G via H (or shortly, associated regular hypergroup) and denote it by $(G_H,\circ)$.*

(*ii*) *The mapping $\varphi: G\longrightarrow G_H$ defined by $\varphi(g)=(g,1)$ is an embedding.*

(*iii*) *For $H=\mathbb{Z}$, we denote G_H by $\overline{G}$.*

Theorem 4.1. *Let $(G_1,\cdot)$ and $(G_2,\cdot)$ be isomorphic groups. Then, for all group $(H,\cdot)$, G_{1_H} and G_{2_H} are isomorphic regular hypergroups.*

Theorem 4.2. *Every group is a fundamental group.*

Proof. Let $(G,\cdot)$ be a group. By Lemma 4.1, for all group $(H,\cdot)$, $(G\times H,\circ)$ is a hypergroup. For all $g\in G$ and $(h,h')\in H\times H$, we have $\{(g,h),(g,h')\}=(g,h)\circ(1,h')$, then $(g,h)\beta(g,h')$. Thus $\beta(g,h)=\{(g,x)\mid x\in H\}$. Define the mapping $\varphi: (\frac{(G\times H,\circ)}{\beta},*)\longrightarrow (G,.)$ by $\varphi(\beta(g,h))=g$. Obviously, φ is well-defined and one to one, since for all $(g,h),(g',h')\in G\times H$, $\beta((g,h))=\beta((g',h'))$ if and only if $g=g'$ if and only if $\varphi(\beta(g,h))=\varphi(\beta(g',h'))$. Let $(g,h),(g',h')\in G\times H$. Then,

$$\begin{aligned}\varphi(\beta(g,h)*\beta(g',h'))=\varphi(\beta(g.g',h)) &= \varphi(\beta(g\cdot g',h'))\\ &= g\cdot g'=\varphi(\beta(g,h))\cdot\varphi(\beta(g',h'))\end{aligned}$$

Thus, φ is a homomorphism. Now, for all $g\in G, \varphi(\beta(g,1))=g$, hence φ is onto. Therefore, φ is an isomorphism and $(\frac{(G\times H,\circ)}{\beta},*)\cong(G,.)$. □

Theorem 4.3. *Let $(G,\cdot)$ be a group. Then for all group $(H,\cdot)$,*

(*i*) *$(Id(G_H),\circ)$ is a regular hypergroup;*

(*ii*) *$Id(Id(G_H))=Id(G_H)$;*

(*iii*) *$(Id(Id(G_H)),\cdot)$ is a group;*

(*iv*) *for all $e:=(1_G,h)\in Id(G_H)$ we have $In_e(In_e(G_H))=G_H$;*

(*v*) *$(In_e(In_e(G_H)),\cdot)$ is a group;*

(*vi*) *$(In_e(In_e(G_H)),\circ)$ is a regular hypergroup.*

Proof. $(i),(ii)$ Since $Id(G_H)=\{1_G\}\times H$, so, for all $h,h'\in H$ we have $(1_G,h)\circ(1_G,h')=\{(1_G,h),(1_G,h')\}$. Thus, $(Id(G\times H),\circ)$ is a hypergroup. Clearly $Id(\{1_G\}\times H)=\{1_G\}\times H$ and for all $(1_G,h)\in Id(\{1_G\}\times H)$ and for all $(1_G,h')\in\{1_G\}\times H$ we have $In_{(1_G,h)}(1_G,h')=\{(1_G,h)\}$.

(iii) Define a map $\varphi:(Id(Id(G_H)),\cdot)\to(H,\cdot)$ by $\varphi(1_G,h)=h$. It is easy to see that φ is an isomorphism and so $(Id(Id(G_H)),\cdot)$ is a group.

$(iv),(v),(vi)$ Let $e:=(1_G,h)\in Id(G_H)$. Then we have $In_e(G_H)=\bigcup\limits_{g\in G,h'\in H}(g^{-1},h')=G\times H$ and so $In_e(In_e(G_H))=G_H$. □

In what follows, we apply the fundamental relation β to regular hypergroups of identities.

Theorem 4.4. *Let $(G, \cdot)$ be a group. Then for all group $(H, \cdot)$,*

(i) $(Id(G_H), \circ)/\beta$ is a subgroup of $(Id(Id(G_H)), \cdot)$,

(ii) for all $e := (1_G, h) \in Id(G_H)$ we have $(In_e(G_H), \circ)/\beta$ is a subgroup of $(In_e(In_e(G_H), \cdot)$.

Proof. (i) By Theorem 4.3, we have $Id(G_H) = \{1_G\} \times H$ so for all $(h, h') \in H \times H$, we have $\{(1_G, h), (1_G, h')\} = (1_G, h) \circ (1_G, h')$, whence $(1_G, h)\beta(1_G, h')$. Thus $\beta(1_G, h) = \{(1_G, x) \mid x \in H\}$ and $(Id(G_H), \circ)/\beta = \{\beta(1_G, h) \mid h \in H\}$. It follows that $|(Id(G_H), \circ)/\beta| = 1$.

(ii) By Theorem 4.3, we have $(In_e(G_H), \circ) = G_H$ so $(In_e(G_H), \circ)/\beta \cong G$. □

Theorem 4.5. *Let G and H be two sets such that $|G| = |H|$. If $(G, \circ)$ is a regular hypergroup, then there exist a hyperoperation "$\circ'$" on H , such that $(G, \circ)$ and $(H, \circ')$, are isomorphic regular hypergroups.*

Theorem 4.6. $(\mathbb{Z}, +)$ *is a self fundamental group.*

Proof. For $(\mathbb{Z}, +)$, consider the associated hypergroup $(\mathbb{Z} \times \mathbb{Z}_2, \circ)$. By Lemma 4.1, $(\mathbb{Z} \times \mathbb{Z}_2, \circ)$ is a regular hypergroup and by Theorem 4.2, $(\frac{(\mathbb{Z}\times\mathbb{Z}_2,\circ)}{\beta}, *) \cong (\mathbb{Z}, +)$. Since $\mathbb{Z}$ is an infinite set, it follows that $|\mathbb{Z}| = |\mathbb{Z} \times \mathbb{Z}_2|$ and by Theorem 4.5, there exists a hyperoperation "$\circ'$" on $\mathbb{Z}$, such that $(\mathbb{Z}, \circ')$ and $(\mathbb{Z} \times \mathbb{Z}_2, \circ)$, are isomorphic regular hypergroups. We have $(\mathbb{Z}, +) \cong (\frac{(\mathbb{Z}\times\mathbb{Z}_2,\circ)}{\beta}, *) \cong (\frac{(\mathbb{Z},\circ')}{\beta}, *)$. Therefore, $(\mathbb{Z}, +)$ is a self fundamental group. □

Theorem 4.7. *Let A be a nonempty set. Then for all arbitrary element b, the fundamental group $(A^* \times \{b\}^*/\beta, *)$ is isomorphic to group $(A^*, \cdot)$.*

Proof. Clearly by Theorem 3.8, $(A^* \times \{b\}^*, \circ)$ is a regular hypergroup. Moreover, $(A^* \times \{b\}^*)/\beta, *)$ and $(A^*, .)$ are groups and we define a map $\theta : (A^* \times \{b\}^*)/\beta \longrightarrow (A^*, \cdot)$ by $\theta(\beta^*(\prod_{i=1}^{n} a_i^{\delta_i}, b^{\downarrow k})) = \prod_{i=1}^{n} a_i^{\delta_i}$. It is easy to see that θ is an isomorphism and so $(A^* \times \{b\}^*)/\beta \cong (A^*, \cdot)$. □

Corollary 4.1. *Let A be a nonempty set. Then $(A^* \times A^*)/\beta \cong (< A >, \cdot)$.*

Example 4.1. *Let $A = \{a\}$. Consider the regular hypergroup $(\{a\}^* \times \{a\}^*, \circ)$ with basis $\{a\}$, defined in Example 3.8. Now for all $m, n \in \mathbb{Z}$, we have $\beta^*(a^{\downarrow n}, a^{\downarrow m}) = \{(a^{\downarrow n}, a^{\downarrow k}) \mid k \in \mathbb{Z}\}$. Define a map $\theta : (\{a\}^* \times \{a\}^*)/\beta, *) \longrightarrow (< a >, \cdot)$ by $\theta(\beta(a^{\downarrow n}, a^{\downarrow k})) = a^n$, where $(< a >, \cdot)$ is the infinite free group. Firstly we show that θ is well-defined and one to one. Let $(a^{\downarrow n}, a^{\downarrow k}), (a^{\downarrow n'}, a^{\downarrow k'}) \in \{a\}^* \times \{a\}^*$. Since $| < a > |$ is infinite, it follows that $\theta(\beta(a^{\downarrow n}, a^{\downarrow k})) = \theta(\beta(a^{\downarrow n'}, a^{\downarrow k'}))$ if and only if $a^n = a^{n'}$ if if and only if $n = n'$ if and only if $\beta(a^{\downarrow n}, a^{\downarrow k}) = \beta(a^{\downarrow n'}, a^{\downarrow k'})$. Moreover, for $p \in \mathbb{Z}$, $\theta(\beta(a^{\downarrow n}, a^{\downarrow k})) * \theta(\beta(a^{\downarrow n'}, a^{\downarrow k'})) = \theta(\beta(a^{\downarrow n+n'}, a^{\downarrow p})) = a^{n+n'} = \theta(\beta(a^{\downarrow n}, a^{\downarrow k})) \cdot \theta(\beta(a^{\downarrow n'}, a^{\downarrow k'}))$, so θ is a homomorphism. Clearly θ is an onto map so θ is an isomorphism and $(\{a\}^* \times \{a\}^*)/\beta, *) \cong (< a >, \cdot)$.*

The following results concerning 2-closed hypergroups can be checked.

Theorem 4.8. *Let $(G,\circ)$ be a regular hypergroup. Then*

(i) *if $e \in Id(G)$, then $\beta(e)$ is an identity element in the fundamental group $(\frac{(G,\circ)}{\beta},*)$;*
(ii) *$Id(G) \cap w_G = Id(G)$;*
(iii) *if $|w_G| \geq 2$, then there exists $x \in G$ such that $w_G = \beta(x)$;*
(iv) *if $|Id(G)| \geq 2$, then there exists $x \in G$ such that $Id(G) = \beta(x)$;*
(v) *$Id(G)/\beta = w_G/\beta$;*
(vi) *if $|Id(G)| \geq 2$, then canonical homomorphism φ is not necessarily one to one.*

Example 4.2. (i) *Let $G = \{a,b,c,d,e\}$. Consider the regular hypergroup $(G,\circ)$, defined in Example 3.5. Since $Id(G) = G$, for all $x \in G$ it follows that $\beta(x) = G$. So $G/\beta = \{\beta(x)\}$, where $x \in Id(G)$.*
(ii) *Consider the regular hypergroup defined in Example 3.4. Clearly $Id(G) = \{0\}$ and $w_G = \{0,2\}$, hence $Id(G) \neq w_G$.*

Corollary 4.2. *Let $(G,\circ)$ be a 2–closed hypergroup. Then*

(i) *$w_G = G$;*
(ii) *$(G,\circ)$ is a flat hypergroup.*

Theorem 4.9. *Let $f : (G,\circ) \to (G',\circ')$ be a regular hypergroup homomorphism. Then*

(i) *$Ker(f) \supseteq f^{-1}(Id(G'))$;*
(ii) *$Ker(f) \supseteq Id(G)$;*
(iii) *$f(Id(G)) \subseteq Id(G')$;*
(iv) *if $(G',\circ)$ is a 2–closed hypergroup , then $Ker(f) = G$;*

Theorem 4.10. *Let $(G,\circ)$ be 2–closed hypergroup, $A \subseteq G$ and $\{H_i \mid i \in I\}$ be a family of subhypergroups of G.*

(i) *$< A > /\beta = \beta(A)$,*
(ii) *if $\bigcap_{i\in I} H_i \neq \emptyset$, then $(\bigcap_{i\in I} H_i)/\beta = \bigcap_{i\in I}(H_i/\beta)$ and $(\bigcap_{i\in I} H_i)/\beta$ is a group,*
(iii) *$(\bigcup_{i\in I} H_i)/\beta = \bigcup_{i\in I}(H_i/\beta)$.*
(iv) *$(G,\circ)/\beta \cong 1$.*

Example 4.3. *Let $G = \{1,2,3,4,5,6,7\}$. Consider the hypergroup $(G,\circ)$, defined as follows:*

$\circ$	1	2	3	4	5	6	7
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6,7\}$	$\{6,7\}$
2	$\{2\}$	$\{1\}$	$\{5\}$	$\{6,7\}$	$\{3\}$	$\{4\}$	$\{4\}$
3	$\{3\}$	$\{6,7\}$	$\{1\}$	$\{5\}$	$\{4\}$	$\{2\}$	$\{2\}$
4	$\{4\}$	$\{5\}$	$\{6,7\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{3\}$
5	$\{5\}$	$\{4\}$	$\{2\}$	$\{3\}$	$\{6,7\}$	$\{1\}$	$\{1\}$
6	$\{6,7\}$	$\{3\}$	$\{4\}$	$\{2\}$	$\{1\}$	$\{5\}$	$\{5\}$
7	$\{6,7\}$	$\{3\}$	$\{4\}$	$\{2\}$	$\{1\}$	$\{5\}$	$\{5\}$.

Then $(G,\circ)$ *is a regular hypergroup,* $G/\beta \cong S_3, w_G = Id(G) = \{1\}$,

$$H_1 =< 1 >= \{1\}, H_2 =< 2 >= \{1,2\}, H_3 =< 3 >= \{1,3\}, H_4 =< 4 >= \{1,4\},$$
$$H_5 =< 5 >= H_6 =< 6 >= H_7 =< 7 >= \{1,5,6,7\},$$
$$H_1/\beta = \{\beta(1)\}, H_2/\beta = \{\beta(1),\beta(2)\},$$
$$H_3/\beta = \{\beta(1),\beta(3)\}, H_4/\beta = \{\beta(1),\beta(4)\} \text{ and}$$
$$H_5/\beta = H_6/\beta = H_7/\beta = \{\beta(1),\beta(5),\beta(6)\}, \text{ since } \beta(6) = \beta(7).$$

Clearly $(\bigcap_{i=1}^{7} H_i)/\beta = \bigcap_{i=1}^{7}(H_i/\beta) = \{\beta(1)\}$, *but* $(G,\circ)$ *is not* 2*–closed regular hypergroup, so the converse of Theorem 4.10, is not necessarily true.*

5. **Conclusions**

We analysed in this paper a class of regular hypergroups, that is closed hypergroups, in particular 2-closed hypergroups. Other results are dedicated to quasi complete regular hypergroups and to construction of regular hypergroups. Finally, this construction is used in order to obtain groups through fundamental relations. This topic can be extented to the study of other classes of hypergroups, using a similar construction.

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