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ANNULI FOR THE ZEROS OF A POLYNOMIAL

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To Octavian Stănăşilă on his 75th birthday

In this paper, ring shaped regions containing all the zeros of a polynomial with complex coefficients involving binomial coefficients and Fibonacci–Pell numbers are given. Furthermore, bounds for strictly positive polynomials involving their derivatives are also presented. Finally, using MAPLE, some examples illustrating the bounds proposed are computed and compared with other existing explicit bounds for the zeros.

Keywords: inequalities, polynomials location of the zeros, Fibonacci and Pell numbers, positive polynomials, MAPLE

1. Introduction

Problems involving polynomials in general, and location of their zeros in particular, have a long history. But, recently come under reexamination because of their importance in many areas of applied mathematics such as control theory, signal processing and electrical networks, coding theory, cryptography, combinatorics, number theory, and engineering among others [16, 17, 12, 13]. Specially, the zeros of polynomials play an important role to solve control engineering problems [2], digital audio signal processing problems [18], eigenvalue problems in mathematical physics [10], and in mathematical biology, where polynomials with strictly positive coefficients have efficiently used [3]. A lot of methods to approximate the actual value of the zeros of polynomials with real or complex coefficients, such as Sturm sequence method start with an estimate of an upper bound for the moduli of the zeros. If we have and accurate estimation for the bound, then the amount of work needed to search the range of possible values to begin with can be considerably reduced in comparison with the classical starting searchers such as the bisection methods. Therefore, it should be important to have a set of available bounds to choose what is most accurate to begin the computation. It is well-known that one of the first explicit bounds for the zeros is a classical result due to Cauchy (1829) [5] and

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since then a lot of papers devoted to the subject can be found in the literature. Some of them that will be used in the computations performed to obtain the numerical results presented at the end of this paper, and for ease of reference, are summarized in the following three theorems.

Theorem A (Upper bounds for the zeros) *Let $A(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ be a nonconstant polynomial with complex coefficients. Then, all its zeros lie in the disc $\mathcal{C} = \{z \in \mathbb{C} : |z| \leq r\}$, where*

(i): $r < 1 + \max_{0 \leq k \leq n-1} \left\{ |a_k| \right\}$ *(Cauchy [5])*

(ii): $r = \max \left\{ 1, \sum_{k=0}^{n-1} |a_k| \right\}$ *(Tikoo [14])*

(iii): $r < 1 + \delta_3$, *where δ_3 is the unique positive root of the equation*

$$Q_3(x) = x^3 + (2 - |a_{n-1}|)x^2 + (1 - |a_{n-1}| - |a_{n-2}|)x - A = 0$$

Here $A = \max_{0 \leq k \leq n-1} |a_k|$ *(Sun et al. [11])*

(iv): $r < \sqrt{1 + A}$, *where $A = \max_{0 \leq k \leq n-1} \left\{ |a_k^2 + 2(-1)^k (B - C)| \right\}$ and*

$$B = \sum_{\substack{0 \leq i < j \leq [n/2] \\ i+j=k}} a_{2i} a_{2j} \quad \text{and} \quad C = \sum_{\substack{0 \leq i < j \leq [(n-1)/2] \\ i+j=k-1}} a_{2i+1} a_{2j+1}$$

(Zilovic [17])

(v): $r = \sum_{k=0}^{n-1} |a_k|^{1/n-k}$ *(Walsh [15])*

(vi): $r = \sqrt{1 + \sum_{k=0}^{n-1} |a_k|^2}$ *(Carmichael et. al. [4])*

Another refinement of the explicit bound of Cauchy was given by Díaz-Barrero [6], by proving the following

Theorem B *Let $A(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ be a nonconstant polynomial with complex coefficients. Then, all its zeros lie in the annulus $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, where*

$$r_1 = \frac{3}{2} \min_{1 \leq k \leq n} \left\{ \frac{2^n F_k C(n,k)}{F_{4n}} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}, \text{ and } r_2 = \frac{2}{3} \max_{1 \leq k \leq n} \left\{ \frac{F_{4n}}{2^n F_k C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}$$

Here F_k are Fibonacci's numbers, namely, $F_0 = 0$, $F_1 = 1$, and for $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$. Furthermore, $C(n,k)$ are the binomial coefficients.

Recently Affane-Aji et al. [1] have generalized the results obtained in [11], by given the following

Theorem C *Let $A(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ be a nonconstant polynomial with complex coefficients. Then, all its zeros lie in the annulus $\mathcal{C} = \{z \in$

$\mathbb{C} : r_1 \le |z| \le r_2\}$, *where*

$$r_1 = \min_{1\le k\le n}\left\{\frac{kC(n,k)}{n2^{n-1}}\left|\frac{a_0}{a_k}\right|\right\}^{1/k}, \text{ and } r_2 = 1+\delta_k.$$

Here, δ_k is the unique positive root of the equation

$$Q_k(z) = z^k + \sum_{\nu=2}^{k}\left[C(k-1,k-\nu) - \sum_{j=1}^{\nu-1} C(k-j-1,k-\nu)|a_{n-j}|\right] z^{k+1-\nu} - A = 0,$$

where $A = \max_{0\le k\le n-1}|a_k|$, $a_k = 0$ *if* $k < 0$.

Our goal in this paper is to present some ring-shaped regions in the complex plane containing all the zeros of a polynomial involving the elements of second order recurrences such as the Fibonacci and Pell numbers and binomial coefficients. Moreover, bounds polynomials with strictly positive coefficients are also obtained. To illustrate the results given, numerical computations using a MAPLE code are performed and the results obtained are exhibited in the last section where they are compared with other bounds appeared previously.

2. Main results

In the following some theorems on the location of the zeros are given. We begin with

Theorem 2.1. *Let $A(z) = \sum_{k=0}^{n} a_k z^k$, ($a_k \ne 0$) be a nonconstant polynomial with complex coefficients. Then, all its zeros lie in the ring shaped region $\mathcal{C} = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$, where*

$$r_1 = \min_{1\le k\le n}\left\{2^{3(1-n)}\binom{2n}{n+k}\mathcal{P}_k^2\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \tag{1}$$

and

$$r_2 = \max_{1\le k\le n}\left\{2^{3(n-1)}\binom{2n}{n+k}^{-1}\mathcal{P}_k^{-2}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k} \tag{2}$$

Here $\mathcal{P}_k$ stands for the k^{th} Pell number defined by $\mathcal{P}_0 = 0$, $\mathcal{P}_1 = 1$, and for $n \ge 2$, $\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2}$.

Proof. To prove the preceding result, we will use the identity

$$\sum_{k=0}^{n}\binom{2n}{n+k}(z^k + z^{-k}) = \binom{2n}{n} + z^{-n}(z+1)^{2n}, \tag{3}$$

valid for all nonnegative integer n and for all nonzero complex number z. In fact, developing the LHS of the preceding expression, we have

$$\begin{aligned} z^n\sum_{k=0}^{n}\binom{2n}{n+k}(z^k+z^{-k}) &= 2\binom{2n}{n}z^n + \binom{2n}{n+1}(z^{n+1}+z^{n-1}) + \ldots + \binom{2n}{2n}z^n(z^n+z^{-n}) \\ &= 2\binom{2n}{n}z^n + \binom{2n}{n+1}z^{n+1} + \ldots + \binom{2n}{2n}z^{2n} + \binom{2n}{n-1}z^{n-1} + \ldots + \binom{2n}{0}z^0 \end{aligned}$$

$$= \binom{2n}{n} z^n + \sum_{k=0}^{2n} \binom{2n}{k} z^k = \binom{2n}{n} z^n + (z+1)^{2n}$$

and (3) is proven after dividing by z^n. Now, using the above result we can obtain

$$\sum_{k=0}^{n} \binom{2n}{n+k} \mathcal{P}_k^2 = 2^{3(n-1)} \tag{4}$$

Indeed, putting $z = -1$ in (3) the identity becomes $\sum_{k=0}^{n}(-1)^k \binom{2n}{n+k} = \frac{1}{2}\binom{2n}{n}$. Putting $z = \alpha = 1 + \sqrt{2}$, a characteristic root of the second order recurrence $x^2 - 2x - 1 = 0$ that defines Pell's numbers in (3), we get $\sum_{k=0}^{n} \binom{2n}{n+k} \left(\alpha^{2k} + \alpha^{-2k}\right) = \frac{1}{\alpha^{2n}} \left(\alpha^2 + 1\right)^{2n} + \binom{2n}{n}$. On account of Binet's formulae, we have $\mathcal{P}_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta} = \dfrac{\alpha^n - \beta^n}{2\sqrt{2}}$, where $\alpha = 1 + \sqrt{2}$, and $\beta = 1 - \sqrt{2}$. Squaring the last expression, and taking into account that $\alpha\beta = -1$, yields

$$\begin{aligned} \mathcal{P}_n^2 &= \frac{1}{2^3}\left(\alpha^n - \beta^n\right)^2 = \frac{1}{2^3}\Big(\alpha^{2n} + \beta^{2n} - 2(\alpha\beta)^n\Big) \\ &= \frac{1}{2^3}\left(\alpha^{2n} + \left(\frac{-1}{\alpha}\right)^{2n} - 2(-1)^n\right) = \frac{1}{2^3}\Big(\alpha^{2n} + \alpha^{-2n} - 2(-1)^n\Big) \end{aligned}$$

from which follows

$$\begin{aligned} \sum_{k=0}^{n} \binom{2n}{n+k} \mathcal{P}_k^2 &= \sum_{k=0}^{n} \frac{1}{2^3} \binom{2n}{n+k} \Big(\alpha^{2k} + \alpha^{-2k} - 2(-1)^k\Big) \\ &= \frac{1}{2^3} \sum_{k=0}^{n} \binom{2n}{n+k} \left(\alpha^{2k} + \alpha^{-2k}\right) - \frac{1}{2^2} \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} \end{aligned}$$

Finally,

$$\sum_{k=0}^{n} \binom{2n}{n+k} \mathcal{P}_k^2 = \frac{1}{2^3}\left[\frac{1}{\alpha^{2n}}(\alpha^2+1)^{2n} + \binom{2n}{n}\right] - \frac{1}{2^3}\binom{2n}{n} = \frac{1}{2^3}\left(\frac{1}{\alpha} + \alpha\right)^{2n} = 2^{3(n-1)}$$

on account of the preceding and the fact that $\alpha + \alpha^{-1} = 2\sqrt{2}$, as can be easily checked.

From (1), it follows for $1 \le k \le n$,

$$r_1^k \le 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2 \left|\frac{a_0}{a_k}\right| \tag{5}$$

Suppose that $|z| < r_1$, then

$$|A(z)| = \left|\sum_{k=0}^{n} a_k z^k\right| \ge |a_0| - \sum_{k=1}^{n} |a_k||z|^k > |a_0| - \sum_{k=1}^{n} |a_k| r_1^k$$

$$= |a_0| \left(1 - \sum_{k=1}^{n} \left|\frac{a_k}{a_0}\right| r_1^k\right) \geq |a_0| \left(1 - \sum_{k=1}^{n} 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2\right) = 0$$

on account of (4) and (5). Consequently, $A(z)$ does not have zeros in $\{z \in \mathbb{C} : |z| < r_1\}$.

It is known ([5], [7]) that all the zeros of $A(z)$ have modulus less or equal than the unique positive root of the equation

$$G(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - \cdots - |a_1|z - |a_0|$$

Hence, the second part of our statement will be proved if we show that $G(r_2) \geq 0$. From (2), it follows for $1 \leq k \leq n$,

$$\left|\frac{a_{n-k}}{a_n}\right| \leq 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2 r_2^k \tag{6}$$

Then,

$$G(r_2) = |a_n| \left[r_2^n - \sum_{k=1}^{n} \left|\frac{a_{n-k}}{a_n}\right| r_2^{n-k}\right] \geq |a_n| \left[r_2^n - \left(\sum_{k=1}^{n} 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2 r_2^k\right) r_2^{n-k}\right]$$

$$= |a_n| r_2^n \left(1 - \sum_{k=1}^{n} 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2\right) = 0,$$

and the proof is complete. □

Using Fibonacci numbers instead of Pell's numbers, we state and prove the following

Theorem 2.2. *Let* $A(z) = \sum_{k=0}^{n} a_k z^k$, $(a_k \neq 0)$ *be a nonconstant polynomial with complex coefficients. Then, all its zeros lie in the ring shaped region* $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, *where*

$$r_1 = \min_{1 \leq k \leq n} \left\{5^{1-n} \binom{2n}{n+k} F_k^2 \left|\frac{a_0}{a_k}\right|\right\}^{1/k}$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{5^{n-1} \binom{2n}{n+k}^{-1} F_k^{-2} \left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}$$

Here F_k *stands for the* k^{th} *Fibonacci number defined by* $F_0 = 0, F_1 = 1$, *and for all* $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

Proof. To prove the preceding result we will use the same technique as in the proof of Theorem 1. So, we begin claiming that

$$\sum_{k=0}^{n} \binom{2n}{n+k} F_k^2 = 5^{n-1} \tag{7}$$

Indeed, since the roots of the characteristic equation of the second order recurrence for Fibonacci numbers, $x^2 - x - 1 = 0$ are $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$, then we obtain $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta} = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$, as it is well-known. From the preceding immediately follows $F_n^2 = \frac{1}{5}\Big(\alpha^n - \beta^n\Big)^2 = \frac{1}{5}\Big(\alpha^{2n} + \alpha^{-2n} - 2(-1)^n\Big)$ and

$$\begin{aligned}\sum_{k=0}^{n}\binom{2n}{n+k}F_k^2 &= \sum_{k=0}^{n}\binom{2n}{n+k}\frac{1}{5}\left(\alpha^{2k} + \alpha^{-2k} - 2(-1)^k\right)\\ &= \frac{1}{5}\sum_{k=0}^{n}\binom{2n}{n+k}\left(\alpha^{2k} + \alpha^{-2k}\right) - \frac{2}{5}\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k}\end{aligned}$$

Putting $z = -1$ and $z = \alpha = \dfrac{1+\sqrt{5}}{2}$ in (3) and substituting the identities obtained in the preceding expression, immediately follows

$$\sum_{k=0}^{n}\binom{2n}{n+k}F_k^2 = \frac{1}{5}\left[\frac{1}{\alpha^{2n}}(\alpha^2+1)^{2n} + \binom{2n}{n}\right] - \frac{1}{5}\binom{2n}{n} = \frac{1}{5}\left(\frac{1}{\alpha} + \alpha\right)^{2n} = 5^{n-1}$$

because $\alpha + \alpha^{-1} = \sqrt{5}$, as can be easily checked.

Now using (7) instead of (4), and carrying out step by step the proof of Theorem 1 the proof of Theorem 2 can be easily completed. We omit the details. □

Hereafter, we state and prove two results involving explicit bounds for the zeros of polynomials with strictly positive coefficients. We begin with a bound involving the derivative of the polynomial. It is stated in the following

Theorem 2.3. *Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a nonconstant polynomial with strictly positive coefficients. Then, all its zeros lie in the ring shaped region $\mathcal{C} = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$, where*

$$r_1 = \min_{1\le k\le n}\left\{kp^{k-1}\left(\frac{a_0}{A'(p)}\right)\right\}^{1/k} \tag{8}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{q^{k-n}}{n-k+1}\left(\frac{B'(q)}{a_n}\right)\right\}^{1/k} \tag{9}$$

Here $p > 0, q > 0$ are positive real numbers and $B(z) = z\Big(A(z) - a_n z^n\Big)$.

Proof. We will argue as in the proof of Theorem 1. So, we begin assuming that $|z| < r_1$, and we have

$$|A(z)| = \left|\sum_{k=0}^{n} a_k z^k\right| \ge |a_0| - \sum_{k=1}^{n}|a_k||z|^k > |a_0| - \sum_{k=1}^{n}|a_k| r_1^k$$

$$= |a_0| \left(1 - \sum_{k=1}^{n} \left|\frac{a_k}{a_0}\right| r_1^k\right) = a_0 \left(1 - \sum_{k=1}^{n} \left(\frac{a_k}{a_0}\right) r_1^k\right)$$

From (8), we have for $1 \leq k \leq n$, $\left(\frac{a_k}{a_0}\right) r_1^k \leq kp^{k-1} \left(\frac{a_k}{A'(p)}\right)$. Substituting in the preceding, yields

$$|A(z)| > a_0 \left(1 - \sum_{k=1}^{n} \left(\frac{a_k}{a_0}\right) r_1^k\right) \geq a_0 \left(1 - \sum_{k=1}^{n} kp^{k-1} \left(\frac{a_k}{A'(p)}\right)\right) = 0$$

Therefore, $A(z)$ does not have zeros in $\{z \in \mathbb{C} : |z| < r_1\}$ and we are done with the lower bound. For the upper bound we have to see that all the zeros of $A(z)$ have modulus less than or equal to the unique positive root of the equation $G(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - \ldots - |a_1|z - |a_0| = 0$. So, it will be suffice to prove that $G(r_2) \geq 0$. Indeed, from (9) we get for $1 \leq k \leq n$, $\frac{a_{n-k}}{a_n} \leq (n-k+1)q^{n-k} \left(\frac{a_{n-k}}{B'(q)}\right) r_2^k$. Since $G(r_2) = a_n \left(r_2^n - \sum_{k=1}^{n} \frac{a_{n-k}}{a_n} r_2^{n-k}\right)$, then immediately follows

$$\begin{aligned} G(r_2) &\geq a_n \left(r_2^n - \sum_{k=1}^{n} \left(\frac{a_{n-k}(n-k+1)q^{n-k} r_2^k}{B'(q)}\right) r_2^{n-k}\right) \\ &= a_n r_2^n \left(1 - \sum_{k=1}^{n} \frac{a_{n-k}(n-k+1)q^{n-k}}{B'(q)}\right) = 0 \end{aligned}$$

on account that $B'(z) = \sum_{k=1}^{n} (n-k+1)a_{n-k} z^{n-k}$. This completes the proof.

□

Finally, we close this section stating a proving a result involving higher order derivatives of the polynomial.

Theorem 2.4. *Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a nonconstant polynomial with strictly positive coefficients. Then, all its zeros lie in the ring shaped region $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$, where*

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{(p-q)^k A^{(k)}(q)}{k!\,[A(p) - A(q)]} \left(\frac{a_0}{a_k}\right) \right\}^{1/k} \tag{10}$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ \frac{k!\,[A(r) - A(s)]}{A^{(k)}(s)\,(r-s)^k} \left(\frac{a_{n-k}}{a_n}\right) \right\}^{1/k} \tag{11}$$

Here $p > q > 0$ and $r > s > 0$ are strictly positive real numbers.

Proof. Arguing out as in the proof of Theorem 1, if we suppose that $|z| < r_1$, then we have

$$|A(z)| = \left|\sum_{k=0}^{n} a_k z^k\right| \geq a_0 - \sum_{k=1}^{n} a_k |z|^k > a_0 - \sum_{k=1}^{n} a_k r_1^k = a_0 \left(1 - \sum_{k=1}^{n} \frac{a_k}{a_0}\, r_1^k\right)$$

From (10), we get for $1 \leq k \leq n$, $\left(\frac{a_k}{a_0}\right) r_1^k \leq \frac{A^{(k)}(q)\,(p-q)^k}{k!\,[A(p)-A(q)]}$. Substituting in the preceding expression, we obtain

$$|A(z)| > a_0 \left(1 - \sum_{k=1}^{n} \frac{a_k}{a_0}\, r_1^k\right) \geq a_0 \left(1 - \sum_{k=1}^{n} \frac{A^{(k)}(q)\,(p-q)^k}{k!\,[A(p)-A(q)]}\right) = 0,$$

because $A(p)-A(q) = A'(q)(p-q) + \frac{A''(q)}{2!}(p-q)^2 + \ldots + \frac{A^{(n)}(q)}{n!}(p-q)^n$ on account of Taylor's formulae applied to $A(p)$ at point q. Consequently, $A(z)$ does not have zeros in $\{z \in \mathbb{C} : |z| < r_1\}$. For the upper bound we have to find the unique positive root of the equation $G(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - \ldots - |a_1|z - |a_0| = 0$ as it is well-known ([7], [8], [9]). Hence, we have to prove that $G(r_2) \geq 0$ and we are done. From (11), we get for $1 \leq k \leq n$, $\frac{a_{n-k}}{a_n} \leq \frac{A^{(k)}(s)(r-s)^k}{k!\,[A(r)-A(s)]}\, r_2^k$. Since $G(r_2) = a_n \left(r_2^n - \sum_{k=1}^{n} \frac{a_{n-k}}{a_n}\, r_2^{n-k}\right)$, then

$$\begin{aligned} G(r_2) &\geq a_n \left[r_2^n - \sum_{k=1}^{n} \left(\frac{A^{(k)}(s)(r-s)^k}{k!\,[A(r)-A(s)]}\, r_2^k\right) r_2^{n-k}\right] \\ &= a_n r_2^n \left(1 - \sum_{k=1}^{n} \frac{A^{(k)}(s)(r-s)^k}{k!\,[A(r)-A(s)]}\right) = 0, \end{aligned}$$

because $A(r) - A(s) = A'(s)(r-s) + \frac{A''(s)}{2!}(r-s)^2 + ... + \frac{A^{(n)}(s)}{n!}(r-s)^n$ on account of Taylor's formulae applied to $A(r)$ at point s. This completes the proof.

□

3. Applications using MAPLE

In this section, we use a MAPLE code to compute the inner and outer radius of a ring shaped region containing all the zeros for the polynomials $A_1(z) = z^3 + 0.1z^2 + 0.3z + 0.7$ and $A_2(z) = z^3 + 0.1z^2 + 0.1z + 0.04$, obtainable by the results presented in this paper. Also, the annuli obtained from the known results given in Theorem A, Theorem B and Theorem C are computed and compared with our results. These numerical results are shown in Table 1. Furthermore, by using MAPLE code, we also computed the actual zeros of both polynomials. We found that for $A_1(z)$ they all lie in the annulus $\mathcal{C} = \{z \in \mathbb{C} : 0.80579 \leq |z| \leq 0.93205\}$ which has area equal to 0.68933. For $A_2(z)$ the annulus containing all the zeros is $\mathcal{C} = \{z \in \mathbb{C} : 0.27228 \leq |z| \leq 0.38328\}$

with area equal to 0.22862. These values has been used to express in terms of percentage error the area of the annuli obtained. They can be found in columns fourth and seventh of Table 1.

Table 1. Inner radius r_1, outer radius r_2 and Error for polynomials $A_1(z)$ and $A_2(z)$.

	$A_1(z)$			$A_2(z)$		
Bounds	r_1	r_2	Error	r_1	r_2	Error
Cauchy [5]	0.41176	1.7	1139.82	0.03846	1.1	1560.69
Tikoo [14]	0.49999	1.1	337.51	0.03333	1.0	1272.63
Sun et al. [11]	0.58693	1.43399	680.16	0.21858	1.1	1497.07
Zilovic [17]	0.57346	1.22065	429.18	0.03996	1.00499	1285.70
Walsh [15]	0.51739	1.53563	825.71	0.14275	0.75822	662.00
Carmichael et al. [4]	0.55514	1.26095	484.18	0.03957	1.01074	1301.68
Díaz-Barrero [6]	0.58333	1.23127	435.85	0.1	0.51639	252.70
Affane-Aji et al. [1]	0.55934	1.17510	386.73	0.1	1.05039	1402.40
Theorem 2.1	0.54687	1.21463	436.07	0.09375	0.51639	254.36
Theorem 2.2	0.48202	1.63553	1013.21	0.18566	0.64549	425.20
Theorem 2.3	0.74413	1.05601	155.87	0.21782	0.52254	210.01
Theorem 2.4	0.74665	1.03928	138.18	0.15370	0.53742	264.43

In the computation of the values presented in Table 1 for $A_1(z)$ using Theorem 2.3 we have taken $p = 0.43$, $q = 0.61$, and using Theorem 2.4 we have taken $p = 0.75$, $q = 0.0001$, $r = 2.07$ and $s = 0.24$. For the polynomial $A_2(z)$ the values used for Theorem 2.3 were $p = 0.13$, $q = 0.29$, and for Theorem 2.4 we have taken $p = 0.24$, $q = 0.016$, $r = 3.00$ and $s = 0.90$.

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REFERENCES

[1] *C. Affane-Aji, S. Biaz, N.K. Govil*, On anulli containing all the zeros of a polynomial, Math. Comput. Modelling, **52** (2010) 1532-1537.

[2] *C. Bissel*, Control Engineering, 2nd Edition, CRC Press, 1996.

[3] *W.E. Briggs*, Zeros and factors of polynomials with positive coefficients and protein-ligand binding, Rocky Mountain J. Math. **15** (1985) 75-89.

[4] *D.R. Carmichael and T. E. Mason*, Note on roots of algebraic equations, Bull. Amer. Math. Soc. **21** (1914), 14–22.

[5] *A. L. Cauchy*, Exercices de Mathématique, in Oeuvres **9(2)** (1829), 122.

[6] *J. L. Díaz-Barrero*, An annulus for the zeros of polynomials, J. Math. Anal. Appl. **237** (2002), 349–352.

[7] *M. Marden*, The Geomeetry of the Zeros of a Polynomial in a Complex Variable, American Mathematical Society, 1966.
[8] *M. Mignote, D. Stefanescu*, Polynomials - An Algorithmic Approach, Springer-Verlag, 1999.
[9] *M. Parodi*, La Localisation des Valeurs Caracteristiques de Matrices et ses Applications, Gauthier-Villars, Paris 1959.
[10] *H. Sagan*, Boundary and Eigenvalue Problems in Mathematical Physics, Dover Publications, 1989.
[11] *Y.J. Sun and J.G. Hsieh*, A note on circular bound of polynomial zeros, IEEE Trans. Circuits Syst. I **43** (1996), 476–478.
[12] *Y.J. Sun*, New result for the annular bounds of complex-coefficient polynomials, IEEE Trans. Automat. Contr. **49** (2004), 813-814.
[13] *Y.J. Sun*, A new annular bound for the roots of characteristic equations of uncertaint discrete systems, Journal of the Chinese Institute of Engineers **28** (2005), 563-566.
[14] *M.L. Tikoo*, Location of the zeros of a polynomial, Am. Math. Mon. **74** (1967), 688-690.
[15] *J.L. Walsh*, An inequality for the roots of an algebraic equation, Ann. Math. **2** (1924), 285-286.
[16] *E. Zeheb*, On the largest modulus of polynomial zero, IEEE Trans. Circuits Syst. **38** (1991), 333-337.
[17] *M.S. Zilovic, L.M. Roytman, P.L. Combettes and M.N. Swamy*, A bound for the zeros of polynomials, IEEE Trans. Circuits Syst. I **39** (1992), 476-478.
[18] *U. Zölzer*, Digital Audio Signal Processing, John Wiley & Sons, 1997.