EDGE-WIENER TYPE INVARIANTS OF SPLICES AND LINKS OF GRAPHS

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In this paper, we present explicit formulae for the first and second edge-Wiener type invariants of splices and links of graphs. As a consequence, the first and second edge-Wiener and edge hyper-Wiener indices of these classes of composite graphs will be determined.

Keywords: Distance, Edge-Wiener type invariants, Splice, Link.

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1. Introduction

In this paper, we are concerned with connected finite graphs without any loops or multiple edges. Let $G$ be such a graph with the vertex set $V(G)$ and the edge set $E(G)$. For $u \in V(G)$ and $e \in E(G)$, we denote by $\deg_G(u)$, the degree of $u$ in $G$ and by $V(e)$, the set of two end vertices of $e$. A topological index $\Top(G)$ of $G$ is a real number with the property that for every graph $H$ isomorphic to $G$, $\Top(H) = \Top(G)$. Vertex version of the Wiener index is the first reported distance-based topological index which was introduced in 1947 by Wiener [1], who used it for modeling the shape of organic molecules and for calculating several of their physico-chemical properties. The Wiener index $W(G)$ of $G$ is defined as:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v),$$

where $d(u,v|G)$ denotes the distance between the vertices $u$ and $v$ of $G$ which is defined as the length of any shortest path in $G$ connecting $u$ and $v$. Details on the Wiener index can be found in [2-4].

The hyper-Wiener index of acyclic graphs was introduced by Milan Randić in 1993. Then Klein et al. generalized Randić’s definition for all connected graphs in 1995 [5]. The vertex version of hyper-Wiener index of $G$ is defined as:

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\[ WW(G) = \frac{1}{2} [W(G) + \sum_{\{u,v\} \subseteq V(G)} d(u,v)^2]. \]

We encourage the reader to consult [6-7], for the mathematical properties of hyper-Wiener index and its applications in chemistry.

Edge versions of the Wiener index based on distance between all pairs of edges in a graph \( G \) were introduced in 2009 [8]. Two possible distances between the edges \( e = uv \) and \( f = zt \) of the graph \( G \) can be considered. The first distance is denoted by \( d_0(e,f|G) \) and defined as:

\[
d_0(e,f|G) = \begin{cases} 
  d_1(e,f|G) + 1 & e \neq f \\
  0 & e = f 
\end{cases},
\]

where \( d_1(e,f|G) = \min \{d(u,z|G), d(u,t|G), d(v,z|G), d(v,t|G)\} \). It is easy to see that \( d_0(e,f|G) = d(e,f|L(G)) \), where \( L(G) \) is the line graph of \( G \).

The second distance is denoted by \( d_4(e,f|G) \) and defined as:

\[
d_4(e,f|G) = \begin{cases} 
  d_2(e,f|G) & e \neq f \\
  0 & e = f 
\end{cases},
\]

where \( d_2(e,f|G) = \max \{d(u,z|G), d(u,t|G), d(v,z|G), d(v,t|G)\} \).

Related to the above distances, two edge versions of the Wiener index can be defined. The first and second edge-Wiener indices of \( G \) are denoted by \( W_{e_0}(G) \) and \( W_{e_4}(G) \), respectively and defined as [8]:

\[
W_{e_i}(G) = \sum_{\{e,f\} \subseteq E(G)} d_i(e,f|G), \quad i \in \{0,4\}.
\]

Obviously, \( W_{e_0}(G) = W(L(G)) \). For more information on the edge-Wiener indices see [9-14].

Edge version of hyper-Wiener index are defined based on the distances \( d_0 \) and \( d_4 \), as follows [15]:

\[
WW_{e_i}(G) = \frac{1}{2} [W_{e_i}(G) + \sum_{\{e,f\} \subseteq E(G)} d_i(e,f|G)^2], \quad i \in \{0,4\}.
\]

The definitions of the edge-Wiener and edge hyper-Wiener indices can be generalized by the following definition:

\[
W_{e_i}(\lambda)(G) = \sum_{\{e,f\} \subseteq E(G)} d_i(e,f|G)^\lambda, \quad i \in \{0,4\},
\]
where $\lambda$ is an arbitrary real number. The indices $W_{e_0}^{(\lambda)}(G)$ and $W_{e_4}^{(\lambda)}(G)$ are called the first and second edge-Wiener type invariants of $G$, respectively. Obviously for $i \in \{0, 4\},$

$$W_{e_i}^{(0)}(G) = \binom{|E(G)|}{2}, \quad W_{e_i}^{(1)}(G) = W_{e_i}(G) \text{ and } \frac{1}{2}[W_{e_i}^{(1)}(G) + W_{e_i}^{(2)}(G)] = WW_{e_i}(G).$$

In this paper, we present explicit formulae for the first and second edge-Wiener type invariants of splices and links of graphs. Then, we apply our results to compute the first and second edge-Wiener and edge hyper-Wiener indices of these classes of composite graphs. Readers interested in more information on computing topological indices of composite graphs can be referred to [4, 9-12, 16-19].

2. Discussion and results

In this section, we compute the first and second edge-Wiener type invariants of splices and links of graphs. We start by introducing some useful notations.

Let $G$ be a simple connected graph. Two possible distances between a vertex $u$ and an edge $e=ab$ of the graph $G$ can be considered [20]. The first distance is denoted by $D_1(u, e|G)$ and defined as:

$$D_1(u, e|G) = \min \{d(u, a|G), d(u, b|G)\},$$

and the second one is denoted by $D_2(u, e|G)$ and defined as:

$$D_2(u, e|G) = \max \{d(u, a|G), d(u, b|G)\}.$$

Note that, $D_1(u, e|G)$ is a nonnegative integer and $D_1(u, e|G) = 0$ if and only if $u \in V(e)$. Also, $D_2(u, e|G)$ is a positive integer and $D_2(u, e|G) = 1$ if and only if $u \in V(e)$ or $u, a$ and $b$ form a triangle in $G$.

Let $\lambda$ be a real number and let $u \in V(G)$. We define:

$$D_1^{(\lambda)}(u|G) = \sum_{e \in E(G); u \notin V(e)} D_1(u, e|G)^{\lambda}, \quad D_2^{(\lambda)}(u|G) = \sum_{e \in E(G)} D_2(u, e|G)^{\lambda}.$$

Note that, if $\lambda$ is a positive number then $D_1^{(\lambda)}(u|G) = \sum_{e \in E(G)} D_1(u, e|G)^{\lambda}$. In particular for $\lambda = 0$, $D_1^{(0)}(u|G) = |E(G)| - \deg_G(u)$, $D_2^{(0)}(u|G) = |E(G)|$. 
2.1 Splice

Let \( G_1 \) and \( G_2 \) be two simple connected graphs with the vertex sets \( V(G_1) \) and \( V(G_2) \) and the edge sets \( E(G_1) \) and \( E(G_2) \), respectively. For given vertices \( y \in V(G_1) \) and \( z \in V(G_2) \), a splice of \( G_1 \) and \( G_2 \) by vertices \( y \) and \( z \) is denoted by \((G_1,G_2)(y,z)\) and defined by identifying the vertices \( y \) and \( z \) in the union of \( G_1 \) and \( G_2 \) as shown in Fig. 1 [21]. We denote by \( n_i \) and \( e_i \) the order and size of the graph \( G_i \), respectively. It is easy to see that \( |V((G_1,G_2)(y,z))| = n_1 + n_2 - 1 \) and \( |E((G_1,G_2)(y,z))| = e_1 + e_2 \).

![Fig. 1. A splice of \( G_1 \) and \( G_2 \) by vertices \( y \) and \( z \).](image)

In the following Lemma, the distance between vertices of \((G_1,G_2)(y,z)\) is computed. The proof can be easily obtained from the definition of splice of graphs, so is omitted.

**Lemma 2.1** Let \( u, v \in V((G_1,G_2)(y,z)) \). Then

\[
d(u,v)((G_1,G_2)(y,z)) = \begin{cases} 
  d(u,v)(G_1) & u,v \in V(G_1) \\
  d(u,v)(G_2) & u,v \in V(G_2) \\
  d(u,v)(G_1) + d(z,v)(G_2) & u \in V(G_1), v \in V(G_2)
\end{cases}
\]

**Theorem 2.2** Let \( \lambda \) be a positive integer. The first and second edge-Wiener type invariants of \((G_1,G_2)(y,z)\) are given by:

(i) \( W_{e_0}^{(\lambda)}(G) = W_{e_0}^{(\lambda)}(G_1) + W_{e_0}^{(\lambda)}(G_2) + \deg_{G_1}(y)\deg_{G_2}(z) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} \deg_{G_2}(z)D_1^{(i)}(y|G_1) + \deg_{G_1}(y)D_1^{(i)}(z|G_2) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_1^{(i)}(y|G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} D_1^{(j)}(z|G_2) \)

(ii) \( W_{e_4}^{(\lambda)}(G) = W_{e_4}^{(\lambda)}(G_1) + W_{e_4}^{(\lambda)}(G_2) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_2^{(i)}(y|G_1)D_2^{(\lambda-i)}(z|G_2) \)

**Proof.** (i) By definition of the first edge-Wiener type invariant,
Now, we partition the above sum into three sums as follows:

The first sum $S_1$ consists of contributions to $W_{e_0}^{(\lambda)}(G)$ of pairs of edges from $G_1$. For edges $e, f \in E(G_1)$, $d_0(e, f|G) = d_0(e, f|G_1)$. So,

$$S_1 = \sum_{\{e, f\} \subseteq E(G)} d_0(e, f|G_1)^\lambda = W_{e_0}^{(\lambda)}(G_1).$$

The second sum $S_2$ consists of contributions to $W_{e_0}^{(\lambda)}(G)$ of pairs of edges from $G_2$. For edges $e, f \in E(G_2)$, $d_0(e, f|G) = d_0(e, f|G_2)$. So,

$$S_2 = \sum_{\{e, f\} \subseteq E(G_2)} d_0(e, f|G_2)^\lambda = W_{e_0}^{(\lambda)}(G_2).$$

The third sum $S_3$ is taken over all pairs of edges $e, f \in E(G)$ such that $e \in E(G_1)$ and $f \in E(G_2)$. It is easy to see that, $d_0(e, f|G) = 1 + D_t(y, e|G_1) + D_t(z, f|G_2)$. So,

$$S_3 = \sum_{e \in E(G_1), f \in E(G_2)} [1 + D_t(y, e|G_1) + D_t(z, f|G_2)]^\lambda.$$

In order to compute the sum $S_3$, we partition it into four sums $S_{31}$, $S_{32}$, $S_{33}$ and $S_{34}$ as follows:

The sum $S_{31}$ is equal to:

$$S_{31} = \sum_{e \in E(G_1), y \not\in V(e)} \sum_{f \in E(G_2), z \not\in V(f)} [1 + D_t(y, e|G_1) + D_t(z, f|G_2)]^\lambda \sum_{i=0}^{\lambda-1} \binom{\lambda}{i} D_t(y, e|G_1)^i \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} D_t(z, f|G_2)^j$$

$$= \sum_{e \in E(G_1), y \not\in V(e)} \sum_{f \in E(G_2), z \not\in V(f)} D_t(y, e|G_1)^\lambda \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} D_t(z, f|G_2)^j \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_t(y, e|G_1)^i$$

$$= \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_t(y, e|G_1)^i \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} D_t(z, f|G_2)^j \sum_{e \in E(G_1), y \not\in V(e)} \sum_{f \in E(G_2), z \not\in V(f)} D_t(y, e|G_1)^i \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} D_t(z, f|G_2)^j.$$

The sum $S_{32}$ is equal to:

$$S_{32} = \sum_{e \in E(G_1), y \not\in V(e)} \sum_{f \in E(G_2), z \not\in V(f)} [1 + D_t(y, e|G_1)]^\lambda.$$
\[
S_{33} = \sum_{e \in E(G_1); y \in V(e) \cap V(G_2)} \sum_{f \in E(G_2)} [1 + D_1(z, f|G_2)]^j
= \sum_{e \in E(G_1); y \in V(e) \cap V(G_2)} \sum_{f \in E(G_2)} \sum_{i=0}^{j} \binom{j}{i} D_1(z, f|G_2)^i
= \deg_{G_1}(y) \sum_{i=0}^{j} \binom{j}{i} D_1^{(i)}(z|G_2).
\]

The sum \( S_{33} \) is equal to:

\[
S_{34} = \sum_{e \in E(G_1); y \in V(e) \cap V(G_2)} \sum_{f \in E(G_2)} \sum_{i=0}^{j} l^i
= \deg_{G_1}(y) \deg_{G_2}(z).
\]

By adding the quantities \( S_{31}, S_{32}, S_{33} \) and \( S_{34} \), we obtain:

\[
S_3 = \deg_{G_1}(y) \deg_{G_2}(z) + \sum_{i=0}^{j} \binom{j}{i} \deg_{G_2}(z) D_1^{(i)}(y|G_1) + \deg_{G_1}(y) D_1^{(i)}(z|G_2) + \sum_{i=0}^{j} \binom{j}{i} D_1^{(i)}(y|G_1) \sum_{j=0}^{i} \binom{i}{j} D_1^{(j)}(z|G_2).
\]

The formula of \( W_{e_4}(\lambda)(G) \) is obtained by adding the quantities \( S_1, S_2, S_3 \).

(ii) Using a similar method as in the proof of part (i), we have:

\[
W_{e_4}(\lambda)(G) = \sum_{\{e, f\} \subseteq E(G_1)} d_4(e, f|G_1)^\lambda + \sum_{\{e, f\} \subseteq E(G_2)} d_4(e, f|G_2)^\lambda
+ \sum_{e \in E(G_1), f \in E(G_2)} [D_1(y, e|G_1) + D_2(z, f|G_2)]^\lambda = W_{e_4}(\lambda)(G_1) + W_{e_4}(\lambda)(G_2)
+ \sum_{e \in E(G_1), f \in E(G_2)} \sum_{i=0}^{j} \binom{j}{i} D_2(y, e|G_1)^i D_2(z, f|G_2)^{\lambda-i}
= W_{e_4}(\lambda)(G_1) + W_{e_4}(\lambda)(G_2) + \sum_{i=0}^{j} \binom{j}{i} D_2^{(i)}(y|G_1) D_2^{(\lambda-i)}(z|G_2).
\]

This completes the proof. \( \square \)
Using Theorem 2.2, we can get the formulae for the edge-Wiener and edge hyper-Wiener indices of \((G_1, G_2)(y, z)\).

**Corollary 2.3** The first and second edge-Wiener indices of \(G = (G_1, G_2)(y, z)\) are given by:

(i) \(W_{e_0}(G) = W_{e_0}(G_1) + W_{e_0}(G_2) + e_2 D_1^{(1)}(y|G_1) + e_1 D_1^{(1)}(z|G_2) + e_1 e_2\),

(ii) \(W_{e_4}(G) = W_{e_4}(G_1) + W_{e_4}(G_2) + e_2 D_2^{(1)}(y|G_1) + e_1 D_2^{(1)}(z|G_2)\).

**Corollary 2.4** The first and second edge hyper-Wiener indices of \(G = (G_1, G_2)(y, z)\) are given by:

(i) \(WW_{e_0}(G) = WW_{e_0}(G_1) + WW_{e_0}(G_2) + \frac{3}{2} e_2 D_1^{(1)}(y|G_1) + \frac{3}{2} e_1 D_1^{(1)}(z|G_2) + \frac{1}{2} e_2 D_1^{(2)}(y|G_1) + \frac{1}{2} e_1 D_1^{(2)}(z|G_2) + e_1 e_2\),

(ii) \(WW_{e_4}(G) = WW_{e_4}(G_1) + WW_{e_4}(G_2) + \frac{1}{2} e_2 [D_2^{(1)}(y|G_1) + D_2^{(2)}(y|G_1)] + \frac{1}{2} e_1 [D_2^{(1)}(z|G_2) + D_2^{(2)}(z|G_2)] + D_2^{(1)}(y|G_1) D_2^{(1)}(z|G_2)\).

### 2.2 Link

Let \(G_1\) and \(G_2\) be two simple connected graphs with the vertex sets \(V(G_1)\) and \(V(G_2)\) and the edge sets \(E(G_1)\) and \(E(G_2)\), respectively. For vertices \(y \in V(G_1)\) and \(z \in V(G_2)\), a link of \(G_1\) and \(G_2\) by vertices \(y\) and \(z\) is denoted by \((G_1 \sim G_2)(y, z)\) and obtained by joining \(y\) and \(z\) by an edge in the union of these graphs, as shown in Fig. 2 [21].

![Fig. 2. A link of \(G_1\) and \(G_2\) by vertices \(y\) and \(z\).](image)

We denote by \(n_i\) and \(e_i\) the order and size of the graph \(G_i\), respectively. It is easy to see that \(|V((G_1 \sim G_2)(y, z))| = n_1 + n_2\) and \(|E((G_1 \sim G_2)(y, z))| = e_1 + e_2 + 1\).

In the following Lemma, the distance between vertices of \((G_1 \sim G_2)(y, z)\) is computed. The proof is easy, so we omit it.

**Lemma 2.5** Let \(u, v \in V((G_1 \sim G_2)(y, z))\). Then
\[
d(d(u,v)|G_1 \sim G_2)(y,z)) = \begin{cases} 
   d(u,v|G_1) & u,v \in V(G_1) \\
   d(u,v|G_2) & u,v \in V(G_2) \\
   d(u,y|G_1) + d(z,v|G_2) + 1 & u \in V(G_1), v \in V(G_2)
\end{cases}
\]

**Theorem 2.6** Let \( \lambda \) be a positive integer. The first and second edge-Wiener type invariants of \( G = (G_1 \sim G_2)(y,z) \) are given by:

(i) \( W_{e_0}(\lambda)(G) = W_{e_0}(\lambda)(G_1) + W_{e_0}(\lambda)(G_2) + 2^2 \deg_{G_1}(y) \deg_{G_2}(z) \)

\[+ \deg_{G_1}(y) + \deg_{G_2}(z) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} \left[ D_1^{(i)}(y|G_1) + D_1^{(i)}(z|G_2) \right] \]

\[+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_1^{(i)}(y|G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} 2^{\lambda-i-j} D_1^{(j)}(z|G_2), \]

(ii) \( W_{e_4}(\lambda)(G) = W_{e_4}(\lambda)(G_1) + W_{e_4}(\lambda)(G_2) + 2^2 \deg_{G_1}(y) \deg_{G_2}(z) \)

\[+ \deg_{G_1}(y) + \deg_{G_2}(z) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} \left[ D_2^{(i)}(y|G_1) + D_2^{(i)}(z|G_2) \right] \]

\[+ \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_2^{(i)}(y|G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} D_2^{(j)}(z|G_2). \]

**Proof.** (i) By definition of the first edge-Wiener type invariant,

\[ W_{e_0}(\lambda)(G) = \sum_{\{e,f\} \subseteq E(G)} d_0(e,f|G)^{\lambda}. \]

Now, we partition the above sum into fifth sums as follows:

The first sum \( S_1 \) consists of contributions to \( W_{e_0}(\lambda)(G) \) of pairs of edges from \( G_1 \). For edges \( e,f \in E(G_1) \), \( d_0(e,f|G) = d_0(e,f|G_1) \). So,

\[ S_1 = \sum_{\{e,f\} \subseteq E(G_1)} d_0(e,f|G_1)^{\lambda} = W_{e_0}(\lambda)(G_1). \]

The second sum \( S_2 \) consists of contributions to \( W_{e_0}(\lambda)(G) \) of pairs of edges from \( G_2 \). For edges \( e,f \in E(G_2) \), \( d_0(e,f|G) = d_0(e,f|G_2) \). So,

\[ S_2 = \sum_{\{e,f\} \subseteq E(G_2)} d_0(e,f|G_2)^{\lambda} = W_{e_0}(\lambda)(G_2). \]

The third sum \( S_3 \) is taken over all pairs of edges \( e,f \in E(G) \) such that \( e \in E(G_1) \) and \( f = yz \). It is easy to see that, \( d_0(e,f|G) = 1 + D_1(y,e|G_1) \). So,
\[ S_3 = \sum_{e \in E(G_1)} [1 + D_1(y,e|G_1)]^\lambda = \sum_{e \in E(G_1); y \notin V(e)} [1 + D_1(y,e|G_1)]^\lambda + \sum_{e \in E(G_1); y \in V(e)} 1^\lambda \\
= \sum_{e \in E(G_1); y \notin V(e)} \sum_{i=0}^\lambda \binom{\lambda}{i} D_1(y,e|G_1)^i \text{deg}_{G_1}(y) \\
= \sum_{i=0}^\lambda \binom{\lambda}{i} D_1(i)(y|G_2) + \text{deg}_{G_2}(y). \]

The forth sum \( S_4 \) is taken over all pairs of edges \( e,f \in E(G) \) such that \( e \in E(G_2) \) and \( f = yz \). It is easy to see that, \( d_0(e,f|G) = 1 + D_t(z,e|G_2) \). So by a similar argument as used in the computation of \( S_3 \), we have:

\[ S_4 = \sum_{i=0}^\lambda \binom{\lambda}{i} D_1(i)(z|G_2) + \text{deg}_{G_2}(z). \]

The last sum \( S_5 \) is taken over all pairs of edges \( e,f \in E(G) \) such that \( e \in E(G_1) \) and \( f \in E(G_2) \). In this case,

\[ d_0(e,f|G) = 2 + D_1(y,e|G_1) + D_1(z,f|G_2). \]

So,

\[ S_5 = \sum_{e \in E(G_1), f \in E(G_2)} [2 + D_1(y,e|G_1) + D_1(z,f|G_2)]^\lambda. \]

Now, we partition the sum \( S_5 \) into four sums \( S_{51}, S_{52}, S_{53} \) and \( S_{54} \) as follows:

The sum \( S_{51} \) is equal to:

\[ S_{51} = \sum_{e \in E(G_1); y \notin V(e) f \in E(G_2); z \notin V(f)} [2 + D_1(y,e|G_1) + D_1(z,f|G_2)]^\lambda \\
= \sum_{e \in E(G_1); y \notin V(e) f \in E(G_2); z \notin V(f)} \sum_{i=0}^\lambda \binom{\lambda}{i} D_1(y,e|G_1)^i \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} 2^{\lambda-i-j} D_1(z,f|G_2)^j \\
= \sum_{i=0}^\lambda \binom{\lambda}{i} D_1(i)(y|G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} 2^{\lambda-i-j} D_1(j)(z|G_2). \]

The sum \( S_{52} \) is equal to:

\[ S_{52} = \sum_{e \in E(G_1); y \notin V(e) f \in E(G_2); z \notin V(f)} [2 + D_1(y,e|G_1)]^\lambda \\
= \sum_{e \in E(G_1); y \notin V(e) f \in E(G_2); z \notin V(f)} \sum_{i=0}^\lambda \binom{\lambda}{i} D_1(y,e|G_1)^i \]
\[ = \deg_{G_2}(z) \sum_{i=0}^{\lambda} \binom{\lambda}{i} 2^{\lambda-i} D_1^{(i)}(y|G_1). \]

The sum \( S_{53} \) is equal to:

\[
S_{53} = \sum_{e \in E(G_1), y \in V(e)} \sum_{f \in E(G_2), z \in V(f)} \sum_{i=0}^{\lambda} \binom{\lambda}{i} 2^{\lambda-i} D_1(z, f|G_2)^i.
\]

\[ = \deg_{G_1}(y) \sum_{i=0}^{\lambda} \binom{\lambda}{i} 2^{\lambda-i} D_1^{(i)}(z|G_2). \]

The sum \( S_{54} \) is equal to:

\[
S_{54} = \sum_{e \in E(G_1), y \in V(e)} \sum_{f \in E(G_2), z \in V(f)} 2^{\lambda} = 2^{\lambda} \deg_{G_1}(y) \deg_{G_2}(z).
\]

By adding the quantities \( S_{51}, S_{52}, S_{53} \) and \( S_{54} \), we obtain:

\[
S_5 = \sum_{i=0}^{\lambda} \binom{\lambda}{i} 2^{\lambda-i} [\deg_{G_2}(z)D_1^{(i)}(y|G_1) + \deg_{G_1}(y)D_1^{(i)}(z|G_2)]
\]

\[ + 2^{\lambda} \deg_{G_1}(y) \deg_{G_2}(z) + \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_1^{(i)}(y|G_1) \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} 2^{\lambda-i-j} D_1^{(j)}(z|G_2). \]

Now, the formula of \( W_{e_0}(\lambda)(G) \) is obtained by adding the quantities \( S_1, S_2, S_3, S_4 \) and \( S_5 \).

**ii** Using a similar method as in the proof of part (i), we have:

\[
W_{e_4}(\lambda)(G) = \sum_{\{e, f\} \subseteq E(G_1)} d_4(e, f|G_1)^{\lambda} + \sum_{\{e, f\} \subseteq E(G_2)} d_4(e, f|G_2)^{\lambda} + \sum_{e \in E(G_1)} [1 + D_2(y, e|G_1)]^{\lambda}
\]

\[ + \sum_{e \in E(G_2)} [1 + D_2(z, e|G_2)]^{\lambda} + \sum_{e \in E(G_1), f \in E(G_2)} [1 + D_2(y, e|G_1) + D_2(z, f|G_2)]^{\lambda}
\]

\[ = W_{e_4}(\lambda)(G_1) + W_{e_4}(\lambda)(G_2)
\]

\[ + \sum_{e \in E(G_1)} \sum_{i=0}^{\lambda} \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_2(y, e|G_1)^i + \sum_{e \in E(G_2)} \sum_{i=0}^{\lambda} \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_2(z, f|G_2)^i
\]

\[ + \sum_{e \in E(G_1), f \in E(G_2)} \sum_{i=0}^{\lambda} \sum_{i=0}^{\lambda} \binom{\lambda}{i} D_2(y, e|G_1)^i \sum_{j=0}^{\lambda-i} \binom{\lambda-i}{j} D_2(z, f|G_2)^j. \]
\[ W_{e_4}(G_1) + W_{e_4}(G_2) + \sum_{i=0}^{\lambda} \left( \frac{\lambda}{i} \right) D_2^{(i)}(y|G_1) + D_2^{(i)}(z|G_2) \]

\[ + \sum_{j=0}^{\lambda-i} \left( \frac{\lambda-i}{j} \right) D_2^{(j)}(z|G_2) \]

Using Theorem 2.6, we can get the formulae for the edge-Wiener and edge hyper-Wiener indices of \((G_1 \sim G_2)(y, z)\).

**Corollary 2.7** The first and second edge-Wiener indices of \((G_1 \sim G_2)(y, z)\) are given by:

(i) \[ W_{e_0}(G) = W_{e_0}(G_1) + W_{e_0}(G_2) + (e_2 + 1)D_1^{(1)}(y|G_1) + (e_1 + 1)D_1^{(1)}(z|G_2) \]

\[ + e_1 + e_2 + 2e_1e_2, \]

(ii) \[ W_{e_4}(G) = W_{e_4}(G_1) + W_{e_4}(G_2) + (e_2 + 1)D_2^{(1)}(y|G_1) + (e_1 + 1)D_2^{(1)}(z|G_2) \]

\[ + e_1 + e_2 + e_1e_2. \]

**Corollary 2.8** The first and second edge hyper-Wiener indices of \((G_1 \sim G_2)(y, z)\) are given by:

(i) \[ WW_{e_0}(G) = WW_{e_0}(G_1) + WW_{e_0}(G_2) + \frac{1}{2}(5e_2 + 3)D_1^{(1)}(y|G_1) + \frac{1}{2}(5e_1 + 3)D_1^{(1)}(z|G_2) \]

\[ + \frac{1}{2}(e_2 + 1)D_1^{(2)}(y|G_1) + \frac{1}{2}(e_1 + 1)D_1^{(2)}(z|G_2) + D_1^{(1)}(y|G_1)D_1^{(1)}(z|G_2) \]

\[ + e_1 + e_2 + 3e_1e_2, \]

(ii) \[ WW_{e_4}(G) = WW_{e_4}(G_1) + WW_{e_4}(G_2) + \frac{3}{2}(e_2 + 1)D_2^{(1)}(y|G_1) + \frac{3}{2}(e_1 + 1)D_2^{(1)}(z|G_2) \]

\[ + \frac{1}{2}(e_2 + 1)D_2^{(2)}(y|G_1) + \frac{1}{2}(e_1 + 1)D_2^{(2)}(z|G_2) \]

\[ + D_2^{(1)}(y|G_1)D_2^{(1)}(z|G_2) + e_1 + e_2 + e_1e_2. \]

### 3. Conclusions

In this paper, we studied the behavior of the edge-Wiener type invariants under the splices and links of graphs. Results were applied to compute the edge-Wiener and edge hyper-Wiener indices of these classes of composite graphs. It is also interesting to find explicit formulae for the edge-Wiener type invariants of other classes of composite graphs such as bridge and chain graphs. In order to achieve that goal, further research into mathematical properties of the edge-Wiener type invariants will be necessary.
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