ALGORITHMIC ANALYSIS OF TSENG'S ITERATES FOR PSEUDO-MONOTONE VARIATIONAL INEQUALITIES

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The iterative methods for solving the variational inequality problem are studied and reviewed. We reconsider the convergence of the Tseng's algorithm for solving the pseudo-monotone variational inequalities. Under some mild assumptions, we show that the Tseng's algorithm converges weakly to a solution of the pseudo-monotone variational inequality.

Keywords: pseudo-monotone variational inequalities, Tseng's method, pseudo-monotone operator, weak convergence.

MSC2010: 47J20, 49J40, 49M30.

1. Introduction

Let H be a real Hilbert space with its inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed and convex subset of H.

Definition 1.1. *An operator $A : C \to H$ is said to be monotone if*

$$\langle Ax - Ay, x - y \rangle \geq 0, \ \forall x, y \in C.$$

An operator $A : C \to H$ is said to be pseudo-monotone if

$$\langle Ay, x - y \rangle \geq 0 \ \textit{implies} \ \langle Ax, x - y \rangle \geq 0, \ \ \forall x, y \in C.$$

Definition 1.2. *An operator $A : C \to H$ is called L-Lipschitz continuous if there exists a positive constant L such that*

$$\|Ax - Ay\| \leq L\|x - y\|, \ \forall x, y \in C.$$

Definition 1.3. *Recall that an operator $A : C \to H$ is said to be*

(*i*) *strongly monotone if (for some positive constant γ)*

$$\langle Ax - Ay, x - y \rangle \geq \gamma \|x - y\|^2, \ \forall x, y \in C.$$

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(*ii*) *inverse strongly monotone if (for some positive constant* γ*)*

$$\langle Ax - Ay, x - y\rangle \geq \gamma\|Ax - Ay\|^2,\ \forall x, y \in C.$$

(*iii*) *sequently weakly continuous if for each sequence* $\{x^n\}$*, we have:* $\{x^n\}$ *converges weakly to* $\tilde{x}$ *implies* $\{Ax^n\}$ *converges weakly to* $A\tilde{x}$*.*

In this work, we are interested in the investigation on the classical variational inequality problem which is formulated as

$$\text{find } \tilde{x} \in C \text{ such that } \langle A\tilde{x}, x - \tilde{x}\rangle \geq 0,\ \forall x \in C. \tag{1}$$

We denote the solution set of (1) by $Sol(C, A)$.

Variational inequalities as a useful computational framework has been applied widely to physics, engineering, economics, optimization and control problems, traffic network problems, equilibrium problems, etc., see [5, 12, 14, 15]. The theory of approximation and iterative algorithms, as an active topic of variational inequalities, has attracted so much attention to explore and analyze relevant convergent results and provide possible error analysis, see [16]. Among them, the more popular method is the following projected-type method ([1, 6, 10, 11, 13]): for $x^0 \in C$, calculate iteratively the sequence $\{x^n\}$ through

$$x^{n+1} = P_C[x^n - \tau Ax^n], \quad n \geq 0,$$

where P_C is the metric projection and $\tau > 0$ is the step-size.

The projected-type algorithm is an effective method for solving variational inequalities. However, the involved operator should be strongly monotone or inverse strongly monotone. In order to overcome this flaw, in [9], Korpelevich suggested an extragradient method for solving variational inequalities. For given current iteration x^n, calculate the next iteration x^{n+1} by the form

$$\begin{cases} y^n = P_C[x^n - \tau Ax^n], \\ x^{n+1} = P_C[x^n - \tau Ay^n],\ n \geq 0 \end{cases}$$

where the step-size $\tau \in (0, 1/L)$ with L being the Lipschitz constant of A.

Korpelevich's algorithm has received so much attention by a range of scholars, who improved it in several ways; see, e.g., [2, 7, 8, 17, 21]. Very recently, Vuong [20] proved that the extragradient method can be successfully applied for solving pseudo-monotone variational inequality (1). However, the operator A has been imposed an additional assumption with A being sequently weakly continuous on C.

Among drawbacks of Korpelevich's extragradient algorithm is the necessity of two projections onto the admissible set C (that are resource-consuming when the structure of the set C is complicated) to pass to the next iteration. We restrict our attention to an alternative of the extragradient method. It was proposed in [19] by Tseng with the following remarkable form

$$\begin{cases} y^n = P_C[x^n - \tau_n Ax^n], \\ x^{n+1} = y^n + \tau_n(Ax^n - Ay^n). \end{cases}$$

Consequently, Tseng's algorithm was studied extensively by many authors, see [3, 18].

The purpose of this paper is to reconsider the convergence of the Tseng's algorithm for solving pseudo-monotone variational inequalities. By using new analysis technique, we show that the Tseng's algorithm converges weakly to a solution of pseudo-monotone variational inequalities. Our result improves some existing results in the literature.

2. **Preliminaries**

Let C be a nonempty closed convex subset of a real Hilbert space H. For fixed $z \in H$, there exists a unique $z^\dagger \in C$ satisfying

$$\|z - z^\dagger\| = \inf\{\|z - \tilde{z}\| : \tilde{z} \in C\}.$$

Denote $z^\dagger$ by $P_C[z]$.

The following inequality is an important property of projection P_C([**?**]): for given $x \in H$,

$$\langle x - P_C[x], y - P_C[x]\rangle \leq 0, \ \forall y \in C. \tag{2}$$

There holds the relation

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \ \forall x, y \in C. \tag{3}$$

In what follows, we shall use the following expressions:

- $u^n \rightharpoonup z^\dagger$ denotes the weak convergence of u^n to $z^\dagger$.
- $w_\omega(u^n) = \{u^\dagger : \exists \{u^{n_i}\} \subset \{u^n\} \textit{ such that } u^{n_i} \rightharpoonup u^\dagger (i \to \infty)\}$.

Lemma 2.1 ([4]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous and pseudo-monotone operator. Then $x^\dagger \in Sol(C, A)$ iff $x^\dagger$ solves the following dual variational inequality*

$$\langle Au^\dagger, u^\dagger - x^\dagger\rangle \geq 0, \ \forall u^\dagger \in C.$$

3. **Main results**

In this section, we show the convergence of Tseng's method for solving the pseudo-monotone variational inequality (1).

Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the following conditions are satisfied

(C1): A is pseudo-monotone on H;

(C2): A is L-Lipschitz continuous on C;

(C3): A possesses the property (P): Let $\{x^n\} \subset C$ be a sequence. If $x^n \rightharpoonup u \in C$ and $\liminf_{n\to\infty} \|Ax^n\| = 0$, then $Au = 0$.

Remark 3.1. If A is demi-closed at zero (recall that an operator A is demi-closed at zero if $x^n \rightharpoonup u$ and $Ax^n \to 0$ imply that $Au = 0$), then A satisfies property (P).

Next we introduce our iterative steps of the Tseng's method. Let $\{\tau_n\}_{n=0}^{\infty}$ be a sequence in $(0, \infty)$.

Step 1. Let $x^0 \in C$ be an initiation and set $n = 0$.

Step 2. Let x^n be the iteration of the n-th step which has been calculated. If $x^n = P_C(x^n - \tau_n Ax^n)$, then stop. Otherwise, calculate the next iteration x^{n+1} by the following form

$$\begin{cases} y^n = P_C[x^n - \tau_n Ax^n], \\ x^{n+1} = y^n + \tau_n(Ax^n - Ay^n). \end{cases} \tag{4}$$

Step 3. Set $n := n + 1$ and return to step 2.

Remark 3.2. According to (2), we know that $\hat{x}$ solves VI (1) if and only if $\hat{x} = P_C[\hat{x} - \mu\hat{x}]$ for any $\mu > 0$. Hence, if $x^n = P_C[x^n - \tau_n Ax^n]$, then x^n is a solution of VI (1).

Remark 3.3. Assume that the above iterate does not terminate in a finite steps. In this case, it is obvious that $Ax^n \neq 0$ for all $n \geq 0$, otherwise $x^n = P_C[x^n - \tau_n Ax^n]$ and the iterate stops voluntarily.

Next, we prove the convergence of the infinite iterations $\{x^n\}$ generated by the above algorithm.

Theorem 3.1. *Suppose that $Sol(C, A) \neq \emptyset$. Then the sequence $\{x^n\}$ generated by (4) converges weakly to $x^\dagger \in Sol(C, A)$ provided* $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < 1/L$.

Proof. Pick any $\hat{x} \in Sol(C, A)$. From (2) and (4), we have

$$\langle y^n + \tau_n Ax^n - x^n, y^n - \hat{x}\rangle \leq 0. \tag{5}$$

Since $y^n \in C$ and $\hat{x} \in Sol(C, A)$, we have

$$\langle A\hat{x}, y^n - \hat{x}\rangle \geq 0.$$

By the pseudo-monotonicity of A, we deduce

$$\langle Ay^n, y^n - \hat{x}\rangle \geq 0. \tag{6}$$

From (4), (5) and (6), we obtain

$$\begin{aligned} \langle x^{n+1} - x^n, y^n - \hat{x}\rangle &= \langle y^n + \tau_n(Ax^n - Ay^n) - x^n, y^n - \hat{x}\rangle \\ &= \langle y^n + \tau_n Ax^n - x^n, y^n - \hat{x}\rangle - \tau_n\langle Ay^n, y^n - \hat{x}\rangle \\ &\leq 0. \end{aligned}$$

It follows that

$$\begin{aligned} \langle x^{n+1} - x^n, x^n - \hat{x}\rangle &= \langle x^{n+1} - x^n, x^n - y^n\rangle + \langle x^{n+1} - x^n, y^n - \hat{x}\rangle \\ &\leq \langle x^{n+1} - x^n, x^n - y^n\rangle \\ &= \langle y^n + \tau_n(Ax^n - Ay^n) - x^n, x^n - y^n\rangle \\ &= -\|y^n - x^n\|^2 + \tau_n\langle Ax^n - Ay^n, x^n - y^n\rangle. \end{aligned} \tag{7}$$

By (4), we get

$$\begin{aligned}\|x^{n+1}-\hat{x}\|^2 &= \|y^n - x^n + x^n - \hat{x} + \tau_n(Ax^n - Ay^n)\|^2\\ &= \|y^n - x^n\|^2 + \|x^n - \hat{x}\|^2 + \tau_n^2\|Ax^n - Ay^n\|^2\\ &\quad + 2\langle y^n - x^n, x^n - \hat{x}\rangle + 2\tau_n\langle y^n - x^n, Ax^n - Ay^n\rangle\\ &\quad + 2\tau_n\langle x^n - \hat{x}, Ax^n - Ay^n\rangle\\ &= \|y^n - x^n\|^2 + \|x^n - \hat{x}\|^2 + \tau_n^2\|Ax^n - Ay^n\|^2\\ &\quad + 2\tau_n\langle y^n - x^n, Ax^n - Ay^n\rangle + 2\langle x^{n+1} - x^n, x^n - \hat{x}\rangle.\end{aligned} \tag{8}$$

Combining (7) with (8), we have

$$\|x^{n+1}-\hat{x}\|^2 \le \|x^n - \hat{x}\|^2 - \|y^n - x^n\|^2 + \tau_n^2\|Ax^n - Ay^n\|^2. \tag{9}$$

Noting that A is L-Lipschitz, we have

$$\|Ax^n - Ay^n\| \le L\|x^n - y^n\|.$$

This together with (9) implies that

$$\begin{aligned}\|x^{n+1}-\hat{x}\|^2 &\le \|x^n - \hat{x}\|^2 - \|y^n - x^n\|^2 + L^2\tau_n^2\|x^n - y^n\|^2\\ &= \|x^n - \hat{x}\|^2 - (1 - L^2\tau_n^2)\|x^n - y^n\|^2\end{aligned} \tag{10}$$

which implies that the sequence $\{\|x^n - \hat{x}\|\}$ is monotone descending. Thus, the limit $\lim_{n\to\infty}\|x^n - \hat{x}\|$ exists and the sequence $\{x^n\}$ is bounded. Hence, there exists a subsequence $\{x^{n_i}\} \subset \{x^n\}$ such that $x^{n_i} \rightharpoonup x^\dagger \in C$.

Next, we show that $x^\dagger \in Sol(C, A)$. In fact, by the assumption $0 < \liminf_{n\to\infty}\tau_n \le \limsup_{n\to\infty}\tau_n < 1/L$, there exists a, b and N_0 such that

$$0 < a \le \tau_n \le b < 1/L$$

when $n \ge N_0$.

By terms of (10), we obtain

$$(1 - aL)\|x^n - y^n\|^2 \le \|x^n - \hat{x}\|^2 - \|x^{n+1} - \hat{x}\|^2 \to 0, n \ge N_0.$$

It follows that

$$\lim_{n\to\infty}\|x^n - y^n\| = 0. \tag{11}$$

From (2) and (4), we deduce

$$\langle x^{n_i} - \tau_{n_i}Ax^{n_i} - y^{n_i}, u - y^{n_i}\rangle \le 0,\ \forall u \in C.$$

It follows that

$$\frac{1}{\tau_{n_i}}\langle x^{n_i} - y^{n_i}, u - y^{n_i}\rangle + \langle Ax^{n_i}, y^{n_i} - x^{n_i}\rangle \le \langle Ax^{n_i}, u - x^{n_i}\rangle,\ \forall u \in C. \tag{12}$$

According to (11) and (12), we deduce

$$\liminf_{i\to\infty}\langle Ax^{n_i}, u - x^{n_i}\rangle \ge 0. \tag{13}$$

Now, we consider two possible cases.

Case 1: $\liminf_{i\to\infty}\|Ax^{n_i}\| = 0$. In this case, noting that $x^{n_i} \rightharpoonup x^\dagger$ and A satisfies property (P), we deduce that $Ax^\dagger = 0$. Consequently, $x^\dagger \in Sol(C, A)$.

Case 2: $\liminf_{i\to\infty}\|Ax^{n_i}\| > 0$. First, we note that $Ax^{n_i} \neq 0$ for each $i \geq 0$, otherwise, x^{n_i} is a solution of VI(1). In terms of (13), we obtain

$$\liminf_{i\to\infty}\left\langle\frac{Ax^{n_i}}{\|Ax^{n_i}\|}, u - x^{n_i}\right\rangle \geq 0. \tag{14}$$

By (14), we can choose a positive real numbers sequence $\{\theta_i\}$ verifying $\theta_i \to 0$ as $i \to \infty$. For each θ_i, there exists the smallest positive integer N_i such that

$$\left\langle\frac{Ax^{n_i}}{\|Ax^{n_i}\|}, u - x^{n_i}\right\rangle + \theta_i \geq 0,\ \forall i \geq N_i.$$

It follows that

$$\langle Ax^{n_i}, u - x^{n_i}\rangle + \theta_i\|Ax^{n_i}\| \geq 0,\ \forall i \geq N_i. \tag{15}$$

Set $u^{n_i} = \dfrac{Ax^{n_i}}{\|Ax^{n_i}\|^2}$. Thus, we have

$$\langle Ax^{n_i}, u^{n_i}\rangle = 1$$

for each i.

From (15), we deduce

$$\langle Ax^{n_i}, u + \theta_i\|Ax^{n_i}\|u^{n_i} - x^{n_i}\rangle \geq 0,\ \forall i \geq N_i. \tag{16}$$

Since A is pseudo-monotone, it follows from (16) that

$$\langle A(u + \theta_i\|Ax^{n_i}\|u^{n_i}), u + \theta_i\|Ax^{n_i}\|u^{n_i} - x^{n_i}\rangle \geq 0,\ \forall i \geq N_i. \tag{17}$$

Since $\lim_{i\to\infty}\theta_i\|Ax^{n_i}\|\|u^{n_i}\| = \lim_{i\to\infty}\theta_i = 0$ and noting that A is L-Lipschitz continuous and $x^{n_i} \rightharpoonup x^\dagger$, by taking the limit as $i \to \infty$ in (17), we obtain

$$\langle Au, u - x^\dagger\rangle \geq 0. \tag{18}$$

Applying Lemma 2.1 to (18), we conclude that $x^\dagger \in Sol(C, A)$.

Finally, we show that the entire sequence $\{x^n\}$ converges weakly to $x^\dagger$. It is enough to show that $w_\omega(x^n)$ is singleton. Let $\tilde{x} \in w_\omega(x^n)$. Hence, there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ such that $x^{n_j} \rightharpoonup \tilde{x}$. It is obvious that $\tilde{x} \in VI(C, A)$ and $\lim_{n\to\infty}\|x^n - \tilde{x}\|$ exists. In turn, since

$$\|x^n - x^\dagger\|^2 - \|x^n - \tilde{x}\|^2 + \|\tilde{x}\|^2 - \|x^\dagger\|^2 = 2\langle x^n, \tilde{x} - x^\dagger\rangle,$$

the limit $\lim_{n\to\infty}\langle x^n, \tilde{x} - x^\dagger\rangle$ exists. Hence,

$$\langle x^\dagger, \tilde{x} - x^\dagger\rangle = \lim_{i\to\infty}\langle x^{n_i}, \tilde{x} - x^\dagger\rangle = \lim_{j\to\infty}\langle x^{n_j}, \tilde{x} - x^\dagger\rangle = \langle \tilde{x}, \tilde{x} - x^\dagger\rangle.$$

Therefore, $\tilde{x} = x^\dagger$ and the sequence x^n weakly converges to $x^\dagger \in Sol(C, A)$. This completes the proof. □

It is obvious that monotonicity implies pseudo-monotonicity. Thus, we obtain the following corollary.

Corollary 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the operator $A : H \to H$ be monotone and L-Lipschitz continuous. Assume that A possesses the property (P). Suppose that $Sol(C, A) \neq \emptyset$. Then the sequence $\{x^n\}$ generated by (4) converges weakly to $x^\dagger \in Sol(C, A)$ provided $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < 1/L$.*

Remark 3.4. It is obvious that if A is sequentially weakly continuous, then A satisfies the above property (P).

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the operator $A : H \to H$ be monotone and L-Lipschitz continuous. Assume that A is sequentially weakly continuous on C. Suppose that $Sol(C, A) \neq \emptyset$. Then the sequence $\{x^n\}$ generated by (4) converges weakly to $x^\dagger \in Sol(C, A)$ provided $0 < \liminf_{n\to\infty} \tau_n \leq \limsup_{n\to\infty} \tau_n < 1/L$.*

4. Conclusions

In this paper, we investigate the pseudomonotone variational inequality problem in Hilbert spaces. Several existing algorithms for solving the pseudomonotone variational inequality are reviewed. We reconsider the Tseng's algorithm and demonstrate its convergence by using new technique for solving the pseudomonotone variational inequalities. Noting that in Tseng's iterative sequence (4), the involved operator A requires an extra assumption ($C3$). A natural problem arises: how to weaken this assumption?

5. Acknowledgements

Yonghong Yao was partially supported by the grant TD13-5033. Li-Jun Zhu was supported by the National Natural Science Foundation of China (61362033) and the Natural Science Foundation of Ningxia province(NZ17015).

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