NEW FIXED POINT RESULTS FOR MULTI-VALUED MAPS VIA MANAGEABLE FUNCTIONS AND AN APPLICATION ON A BOUNDARY VALUE PROBLEM

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In this paper, by using the concepts of α-admissible mappings and manageable functions, we establish some fixed point results for multi-valued-maps in the setting of metric-like spaces. Some examples and an application on a boundary value problem are presented making effective our results.

Keywords: Hausdorff metric-like, fixed point, manageable functions.

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1. Introduction and Preliminaries

In 1969, Nadler [19] was the first who generalized the Banach contraction principle for multi-valued mappings. Later, this theorem has been generalized and extended in many directions. The notion of metric-like spaces (also named as dislocated metric spaces) were considered by Hitzler and Seda [13] as a generalization of the notion of partial metric spaces [17]. Many authors proved some (common) fixed point results on (generalized) metric-like spaces. In 2008, Aage and Salunke [1] established some fixed point results in dislocated and dislocated quasi-metric spaces. Recently, Aydi and Karapinar [6] (see also [9]) studied the case of generalized $\alpha-\psi$-contractions. Later, some best proximity point theorems on metric-like spaces have been presented in [8]. Moreover, Karapinar and Salimi [15] gave some details on dislocated metric spaces to metric spaces. For other related results, see [16,23,25]. In what follows, we recall some definitions and results we will need in the sequel.

Definition 1.1. *[12, 13] Let X be a nonempty set. A function $\sigma : X \times X \to \mathbb{R}^+$ is said to be a metric-like*
(or a dislocated metric) on X if for any $x, y, z \in X$, the following conditions hold:
(P$_1$) $\sigma(x, y) = 0 \Longrightarrow x = y$;
(P$_2$) $\sigma(x, y) = \sigma(y, x)$;
(P$_3$) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.
The pair (X, σ) is then called a metric-like (dislocated metric) space.

Let (X, σ) be a metric-like space. A sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $\sigma(x, x) = \lim_{n\to\infty} \sigma(x, x_n)$. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite. (X, σ) is said to be complete if every Cauchy sequence

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$\{x_n\}$ in X converges to a point $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x, x_n) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_n, x_m).$$

We also have

$$\sigma(x, x) \leq 2\sigma(x, y) \quad \text{for all } x, y \in X. \tag{1}$$

Very recently, Aydi et al. [5] introduced the concept of a Hausdorff metric-like. Let $CB^\sigma(X)$ be the family of all nonempty, closed and bounded subsets of the metric-like space (X, σ), induced by the metric-like σ. For $A, B \in CB^\sigma(X)$ and $x \in X$, define

$$\begin{aligned} \sigma(x, A) &= \inf\{\sigma(x, a),\ a \in A\},\ \ \delta_\sigma(A, B) = \sup\{\sigma(a, B) : a \in A\} \quad \text{and} \\ \delta_\sigma(B, A) &= \sup\{\sigma(b, A) : b \in B\}. \end{aligned}$$

We have the the following useful lemmas.

Lemma 1.1. *[5, 7] Let (X, σ) be a metric-like space and A any nonempty set in (X, σ), then*

$$\sigma(a, A) = 0 \Longrightarrow a \in \bar{A},$$

where $\overline{A}$ denotes the closure of A with respect to the metric-like σ. Also, if $\{x_n\}$ is a sequence in (X, σ) that is τ_σ-convergent to $x \in X$, then

$$\lim_{n\to\infty} |\sigma(x_n, A) - \sigma(x, A)| = \sigma(x, x).$$

Let (X, σ) be a metric-like space. For $A, B \in CB^\sigma(X)$, define

$$H_\sigma(A, B) = \max\{\delta_\sigma(A, B), \delta_\sigma(B, A)\}.$$

We have also some properties of $H_\sigma : CB^\sigma(X) \times CB^\sigma(X) \to [0, \infty)$.

Proposition 1.1. *[5, 7] Let (X, σ) be a metric-like space. For any $A, B, C \in CB^\sigma(X)$, we have the following:*

$$\begin{aligned} (i) &\ :\ H_\sigma(A, A) = \delta_\sigma(A, A) = \sup\{\sigma(a, A) : a \in A\}; \\ (ii) &\ :\ H_\sigma(A, B) = H_\sigma(B, A); \\ (iii) &\ :\ H_\sigma(A, B) = 0 \text{ implies that } A = B; \\ (iv) &\ :\ H_\sigma(A, B) \leq H_\sigma(A, C) + H_\sigma(C, B). \end{aligned}$$

The mapping $H_\sigma : CB^\sigma(X) \times CB^\sigma(X) \to [0, \infty)$ is called a Hausdorff metric-like induced by σ.

The following definition we find it in [2, 18].

Definition 1.2. *Let X be a nonempty set and $T : X \to 2^X$, be a multi-valued mapping. We say that*

T is $\alpha-$admissible if, for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$.

We have the following useful lemma.

Lemma 1.2. *Let (X, σ) be a metric-like space, $B \in CB^\sigma(X)$ and $c > 0$. If $a \in X$ and $\sigma(a, B) < c$ then there exists $b = b(a) \in B$ such that $\sigma(a, b) < c$.*

In 2014, Du and Khojasteh [11], introduced a new class of mappings called manageable functions and they obtained some fixed point theorems. Very recently, Hussain et al. [14] established some fixed point theorems for manageable contractions in the setting of metric spaces.

Definition 1.3. *[11] A manageable function is a mapping $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:*

(η_1) *$\eta(t,s) < s - t$ for all $t, s > 0$;*
(η_2) *for any bounded sequence $\{t_n\}$ in $(0,\infty)$ and any non-increasing sequence $\{s_n\}$ in $(0,\infty)$, it holds that*

$$\limsup_{n\to\infty} \frac{t_n + \eta(t_n, s_n)}{s_n} < 1.$$

Let $\widehat{Man}(\mathbb{R})$ be the set of manageable functions. We provide the following two examples.

Example 1.1. *[11] Let $k \in [0,1)$. Then $\eta_k : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by*

$$\eta_k(t,s) = ks - t$$

is a manageable function.

Example 1.2. *Let $\eta : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the function defined by*

$$\eta(t,s) = \begin{cases} \psi(s) - \varphi(t) & \text{if } (t,s) \in [0,\infty)\times[0,\infty), \\ f(s,t) & \text{otherwise,} \end{cases}$$

where $f : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is any function and $\psi, \varphi : [0,\infty) \to \mathbb{R}$ are two functions such that $\psi(t) < t \le \varphi(t)$ for all $t > 0$ and $\limsup_{r\to t^+} \frac{\psi(r)}{r} < 1$ for all $t \in [0,\infty)$. Then, $\eta \in \widehat{Man}(\mathbb{R})$. Indeed, for any $s, t > 0$,

$$\eta(t,s) = \psi(s) - \varphi(t) < s - t,$$

so, (η_1) holds. Let $\{t_n\}$ in $(0,\infty)$ be a bounded sequence and $\{s_n\}$ in $(0,\infty)$ be a non-increasing sequence. Then $\lim_{n\to\infty} s_n$ exists in $[0,\infty)$. Hence $\limsup_{n\to\infty}\frac{\psi(s_n)}{s_n} = \limsup_{r\to t^+}\frac{\psi(r)}{r} < 1$. Thus, we get

$$\limsup_{n\to\infty}\frac{t_n + \eta(t_n,s_n)}{s_n} = \limsup_{n\to\infty}\frac{\psi(s_n) + t_n - \varphi(t_n)}{s_n} \le \limsup_{n\to\infty}\frac{\psi(s_n)}{s_n} < 1.$$

It follows that (η_2) holds.

In this paper, we present variant fixed point results for multivalued mappings involving manageable contractions via α-admissible mappings in the class of metric-like spaces. Some examples and an application on a boundary value problem are given illustrating the presented concepts and obtained results.

2. **Fixed points via manageable functions**

Now, we state and prove our first main result.

Theorem 2.1. *Let (X,σ) be a complete metric-like space and $T : X \to CB^\sigma(X)$ be a given multi-valued mapping. Suppose that there exist a manageable function $\eta \in \widehat{Man}(\mathbb{R})$ and $\alpha : X\times X \to [0,\infty)$ such that*

$$\eta(H_\sigma(Tx,Ty), M_\sigma(x,y)) \ge 0 \tag{2}$$

for all $x, y \in X$ satisfying $\alpha(x,y) \ge 1$, where

$$M_\sigma(x,y) = \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{1}{4}[\sigma(x,Ty) + \sigma(Tx,y)]\}.$$

Assume that

(*i*) *T is α-admissible mapping;*
(*ii*) *there exist elements $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \ge 1$;*

(*iii*) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x$ *in* (X, σ) *as* $n \to \infty$*, then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k.

Then T *has a fixed point.*

Proof. By assumption (*ii*), there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Clearly, if $x_1 = x_0$ or $x_1 \in Tx_1$, we conclude that x_1 is a fixed point of T and so the proof is finished. Now, we assume that $x_1 \neq x_0$ and $x_1 \notin Tx_1$. So, $\sigma(x_0, x_1) > 0$ and $\sigma(x_1, Tx_1) > 0$.

Since $\alpha(x_0, x_1) \geq 1$, by (2), we have

$$\eta(H_\sigma(Tx_0, Tx_1), M_\sigma(x_0, x_1)) \geq 0, \tag{3}$$

where

$$\begin{aligned} M_\sigma(x_0, x_1) &= \max\{\sigma(x_0, x_1), \sigma(x_0, Tx_0), \sigma(x_1, Tx_1), \frac{1}{4}[\sigma(x_0, Tx_1) + \sigma(x_1, Tx_0)]\} \\ &= \max\{\sigma(x_0, x_1), \sigma(x_1, Tx_1), \frac{1}{4}[\sigma(x_0, Tx_1) + \sigma(x_1, x_1)]\}. \end{aligned}$$

Note that

$$\frac{1}{4}[\sigma(x_0, Tx_1) + \sigma(x_1, x_1)] \leq \frac{1}{4}[\sigma(x_1, Tx_1) + 3\sigma(x_0, x_1)]\} \leq \max\{\sigma(x_0, x_1), \sigma(x_1, Tx_1)\}.$$

Therefore

$$M_\sigma(x_0, x_1) = \max\{\sigma(x_0, x_1), \sigma(x_1, Tx_1)\}).$$

Define the function $\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\lambda(t, s) = \begin{cases} \frac{t+\eta(t,s)}{s} & \text{if } t, s > 0, \\ 0 & \text{otherwise.} \end{cases}$$

By (η_1), we have

$$0 < \lambda(t, s) < 1 \quad \text{for all } \ t, s > 0. \tag{4}$$

Also, if $\eta(t, s) \geq 0$, then

$$0 < t \leq s\lambda(t, s) \quad \text{for all}\, t, s > 0. \tag{5}$$

From (3) and (4), we get

$$0 < \lambda(H_\sigma(Tx_0, Tx_1), M_\sigma(x_0, x_1)) < 1. \tag{6}$$

Since $\sigma(x_1, Tx_1) > 0$, by using (6), we have

$$\sigma(x_1, Tx_1) < \frac{1}{\sqrt{\lambda(H_\sigma(Tx_0, Tx_1), M_\sigma(x_0, x_1))}}\sigma(x_1, Tx_1).$$

Lemma 1.2 implies the existence of a point $x_2 \in Tx_1$ such that

$$\sigma(x_1, x_2) < \frac{1}{\sqrt{\lambda(H_\sigma(Tx_0, Tx_1), M_\sigma(x_0, x_1))}}\sigma(x_1, Tx_1). \tag{7}$$

From (5), we have

$$H_\sigma(Tx_0, Tx_1) \leq M_\sigma(x_0, x_1)\lambda(H_\sigma(Tx_0, Tx_1), M_\sigma(x_0, x_1)) < M_\sigma(x_0, x_1).$$

Then

$$\sigma(x_1, Tx_1) \leq H_\sigma(Tx_0, Tx_1) < M_\sigma(x_0, x_1),$$

which implies that $M_\sigma(x_0, x_1) = \sigma(x_0, x_1)$. It follows that

$$\sigma(x_1, Tx_1) \leq \sigma(x_0, x_1)\lambda(H_\sigma(Tx_0, Tx_1), \sigma(x_0, x_1)). \tag{8}$$

Combining (7) and (8), we get

$$\sigma(x_1, x_2) \leq \sqrt{\lambda(H_\sigma(Tx_0, Tx_1), \sigma(x_0, x_1))}\sigma(x_0, x_1).$$

Note that $x_2 \neq x_1$ because $x_1 \notin Tx_1$. If $x_2 \in Tx_2$, we conclude that x_2 is a fixed point of T and so the proof is finished. Now, we assume that $x_2 \notin Tx_2$. Since T is α-admissible and $x_2 \in Tx_1$, we have

$$\alpha(x_1, x_2) \geq 1.$$

Hence by (2)

$$\eta(H_\sigma(Tx_1, Tx_2), M_\sigma(x_1, x_2)) \geq 0,$$

where

$$\begin{aligned} M_\sigma(x_1, x_2) &= \max\{\sigma(x_1, x_2), \sigma(x_1, Tx_1), \sigma(x_2, Tx_2), \frac{1}{4}[\sigma(x_1, Tx_2) + \sigma(x_2, Tx_1)]\} \\ &= \max\{\sigma(x_1, x_2), \sigma(x_2, Tx_2)\}). \end{aligned}$$

Since $\sigma(x_2, Tx_2) > 0$, by using (6), we have

$$\sigma(x_2, Tx_2) < \frac{1}{\sqrt{\lambda(H_\sigma(Tx_1, Tx_2), M_\sigma(x_1, x_2))}} \sigma(x_1, Tx_2).$$

Lemma 1.2 implies the existence of a point $x_3 \in Tx_2$ such that

$$\sigma(x_2, x_3) < \frac{1}{\sqrt{\lambda(H_\sigma(Tx_1, Tx_2), M_\sigma(x_1, x_2))}} \sigma(x_2, Tx_2).$$

Similarly, we get $\alpha(x_2, x_3) \geq 1$ and

$$\sigma(x_2, x_3) \leq \sqrt{\lambda(H_\sigma(Tx_1, Tx_2), \sigma(x_1, x_2))} \sigma(x_1, x_2).$$

Continuing in this fashion, we construct a sequence $\{x_n\}$ in X such that for all $n \geq 1$

(i) $\alpha(x_n, x_{n+1}) \geq 1, \quad x_n \notin Tx_n, \quad x_n \neq x_{n+1}, \quad x_{n+1} \in Tx_n$;

(ii)
$$\sigma(x_n, x_{n+1}) \leq \sqrt{\lambda(H_\sigma(Tx_{n-1}, Tx_n), \sigma(x_{n-1}, x_n))} \sigma(x_{n-1}, x_n). \tag{9}$$

From (9) and (4), we get $0 < \sigma(x_n, x_{n+1}) < \sigma(x_{n-1}, x_n)$ for all n, which implies that $\{\sigma(x_{n-1}, x_n)\}$ is a non-increasing sequence of positive reals, then it is convergent. Also, we have

$$0 < H_\sigma(Tx_{n-1}, Tx_n) < \sigma(x_{n-1}, x_n),$$

for all n, which implies that $\{H_\sigma(Tx_{n-1}, Tx_n)\}$ is a bounded sequence. From (η_2), we have

$$\limsup_{n\to\infty} \lambda(H_\sigma(Tx_{n-1}, Tx_n), \sigma(x_{n-1}, x_n)) < 1. \tag{10}$$

Let

$$\lambda_n = \sqrt{\lambda(H_\sigma(Tx_{n-1}, Tx_n), \sigma(x_{n-1}, x_n))}, \quad \forall n \geq 1.$$

From (9), we get

$$\sigma(x_n, x_{n+1}) \leq \lambda_n \sigma(x_{n-1}, x_n), \quad \forall n \geq 1. \tag{11}$$

By (10), there exist $\gamma \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that

$$\lambda_n \leq \gamma, \quad \forall n \geq n_0.$$

Hence, by (11), we get

$$\sigma(x_n, x_{n+1}) \leq \gamma \sigma(x_{n-1}, x_n), \quad \forall n \geq n_0.$$

Thus

$$\sigma(x_n, x_{n+1}) \leq \gamma^{n-n_0+1} \sigma(x_{n_0-1}, x_{n_0}), \quad \forall n \geq n_0.$$

Now, for $m > n \geq n_0$, we have

$$\sigma(x_n, x_m) \quad \leq \quad \sum_{i=n}^{m-1} \sigma(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \gamma^{i-n_0+1} \sigma(x_{n_0-1}, x_{n_0}) \leq \sigma(x_{n_0-1}, x_{n_0}) \sum_{i=n}^{\infty} k^i \to 0n \text{ as } n \to \infty.$$

Thus,

$$\lim_{n,m\to\infty}\sigma(x_n,x_m)=0.$$

So $\{x_n\}$ is $\sigma-$Cauchy in the complete metric-like space (X,σ). Then there exists $u\in X$ such that

$$\lim_{n\to\infty}\sigma(x_n,u)=\sigma(u,u)=\lim_{n,m\to\infty}\sigma(x_n,x_m)=0.$$

We will show that u is a fixed point of T. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}=u$ or $Tx_{n_k}=Tu$ for all k, then $Tx_{n_k}=Tu$ for all k. Since $x_{n_k+1}\in Tx_{n_k}$ for all k, then $x_{n_k+1}\in Tu$ for all k. Hence $\sigma(u,Tu)\le\sigma(u,x_{n_k+1})$ for all k. Letting $k\to\infty$, we get $\sigma(u,Tu)\le 0$ and so by Lemma 1.1, we have $u\in\overline{Tu}=Tu$.
So, without loss of generality, we may suppose that $x_n\neq u$ and $x_n\neq Tu$ for all nonnegative integer n. By assumption (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},u)\ge 1$ for all k. Hence by (2), we have

$$\eta(H_\sigma(Tx_{n(k)},Tu),M_\sigma(x_{n(k)},u))\ge 0,\quad\forall k,$$

where

$$M_\sigma(x_{n(k)},u)=\max\{\sigma(x_{n(k)},u),\sigma(u,Tu),\sigma(x_{n(k)},Tx_{n(k)}),\frac{1}{4}[\sigma(x_{n(k)},Tu)+\sigma(u,Tx_{n(k)})]\}.$$

From (5), we have

$$H_\sigma(Tx_{n(k)},Tu)\le\lambda(H_\sigma(Tx_{n(k)},Tu),M_\sigma(x_{n(k)},u))M_\sigma(x_{n(k)},u)<M_\sigma(x_{n(k)},u),\quad\forall k.$$

Since

$$\sigma(u,Tu)\le\sigma(u,x_{n(k)+1})+\sigma(x_{n(k)+1},Tu)\le\sigma(u,x_{n(k)+1})+H_\sigma(Tx_{n(k)},Tu),$$

then

$$\begin{aligned}\sigma(u,Tu)&\le\sigma(u,x_{n(k)+1})+H_\sigma(Tx_{n(k)},Tu)\\&\le\sigma(u,x_{n(k)+1})+\lambda(H_\sigma(Tx_{n(k)},Tu),M_\sigma(x_{n(k)},u))M_\sigma(x_{n(k)},u),\quad\forall k.\end{aligned}$$

Suppose that $\sigma(u,Tu)>0$. Then, there exists $N\in\mathbb{N}$ such that

$$M_\sigma(x_{n(k)},u)=\sigma(u,Tu),\quad\forall k\ge N.$$

It follows that

$$\sigma(u,Tu)\le\sigma(u,x_{n(k)+1})+\lambda(H_\sigma(Tx_{n(k)},Tu),\sigma(u,Tu))\sigma(u,Tu),\quad\forall k\ge N.$$

Passing to $\limsup$ as $k\to\infty$, we get

$$\begin{aligned}\sigma(u,Tu)&\le\limsup_{k\to\infty}\sigma(u,x_{n(k)+1})+\sigma(u,Tu)\limsup_{k\to\infty}\lambda(H_\sigma(Tx_{n(k)},Tu),\sigma(u,Tu))\\&<\sigma(u,Tu),\end{aligned}$$

which is a contradiction. Hence $\sigma(u,Tu)=0$ and so $u\in Tu$, that is, u is a fixed point of T. □

By using the same techniques, we may state the following results in the setting of partial metric and metric-like spaces. Mention that the partial Hausdorff metric H_p written in Theorem 2.2 has been already introduced by Aydi et al. [3].

Theorem 2.2. *Let (X,p) be a complete partial metric space and $T:X\to CB^p(X)$ be a given multi-valued mapping. Suppose that there exist a manageable function $\eta\in\widehat{Man(\mathbb{R})}$ and $\alpha:X\times X\to[0,\infty)$ such that*

$$\eta(H_p(Tx,Ty),N_p(x,y))\ge 0\tag{12}$$

for all $x, y \in X$ *satisfying* $\alpha(x, y) \geq 1$, *where*

$$N_p(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\}.$$

Assume that

(*i*) T *is* α*-admissible mapping;*
(*ii*) *there exist elements* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq 1$*;*
(*iii*) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x$ *in* (X, σ) *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k.

Then T *has a fixed point.*

Theorem 2.3. *Let* (X, σ) *be a complete metric-like space and* $T : X \to CB^\sigma(X)$ *be a given multi-valued mapping. Suppose that there exist a manageable function* $\eta \in \widehat{Man(\mathbb{R})}$ *and* $\alpha : X \times X \to [0, \infty)$ *such that*

$$\eta(H_\sigma(Tx, Ty), \sigma(x, y)) \geq 0 \tag{13}$$

for all $x, y \in X$ *satisfying* $\alpha(x, y) \geq 1$. *Assume that*

(*i*) T *is* α*-admissible mapping;*
(*ii*) *there exist elements* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq 1$*;*
(*iii*) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x$ *in* (X, σ) *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k.

Then T *has a fixed point.*

Remark 2.1. *Theorem 2.1 is a generalization of Theorem 2 in [5]. Theorem 2.2 is a generalization of Theorem 2.2 in [4] (when considering one mapping).*

We give an example to illustrate the utility of Theorem 2.1.

Example 2.1. *Let* $X = [0, \infty)$ *and* $\sigma : X \times X \to [0, \infty)$ *defined by*

$$\sigma(x, y) = x + y, \quad \forall x, y \in X$$

Then (X, σ) *is a complete metric-like space. Define the map* $T : X \to CB^\sigma(X)$ *by*

$$Tx = \begin{cases} [2, \infty) & if x > 1 \\ \{0, \frac{x^2}{1+x}\} & if x \in [0, 1] \end{cases}$$

Note that Tx *is bounded and closed for all* $x \in X$ *in metric-like space* (X, σ). *Take the applications* $\alpha : X \times X \to [0, \infty)$ *and* $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined as follow*

$$\alpha(x, y) = \begin{cases} 2 & if x, y \in [0, 1] \\ 0 & if\ not \end{cases}$$

$\eta(t, s) = rs - t$ *for all* $s, t \in \mathbb{R}$ *with* $r \in [\frac{1}{2}, 1)$.

It is easy tho show that η *is a manageable function and* T *is an* α*-admissible mapping.*

Let $x, y \in X$ *such that* $\alpha(x, y) \geq 1$. *This implies that* $x, y \in [0, 1]$. *We shall show that*

$$H_\sigma(Tx, Ty) \leq \frac{1}{2} M_\sigma(x, y), \quad \forall x, y \in [0, 1].$$

For this, we consider the following cases:
Case 1: $x = y$. *We have*

$$H_\sigma(Tx,Ty) = \max\{\sigma(0,Tx),\, \sigma(\frac{x^2}{1+x},Tx)\}$$
$$= \max\{\min\{\sigma(0,0),\sigma(0,\frac{x^2}{1+x})\},\ \min\{\sigma(0,\frac{x^2}{1+x}),\sigma(\frac{x^2}{1+x},\frac{x^2}{1+x})\}\}$$
$$= \max\{0,\frac{x^2}{1+x}\} = \frac{x^2}{1+x} \leq x = \frac{1}{2}\sigma(x,x) \leq \frac{1}{2}M_\sigma(x,y).$$

Case 2: $x \neq y$. *Since* σ *is symmetric, it suffices to consider the case where* $x > y$. *We have*

$$H_\sigma(Tx,Ty) = H_\sigma(\{0,\frac{x^2}{1+x}\},\{0,\frac{y^2}{1+y}\})$$
$$= \max\{\max\{\sigma(0,\{0,\frac{y^2}{1+y}\}),\sigma(\frac{x^2}{1+x},\{0,\frac{y^2}{1+y}\})\},$$
$$\max\{\sigma(0,\{0,\frac{x^2}{1+x}\}),\sigma(\frac{y^2}{1+y},\{0,\frac{x^2}{1+x}\})\}\}$$
$$= \max\{\sigma(\frac{x^2}{1+x},\{0,\frac{y^2}{1+y}\})\},\ \sigma(\frac{y^2}{1+y},\{0,\frac{x^2}{1+x}\})\}\}$$
$$= \max\{\frac{x^2}{1+x},\frac{y^2}{1+y}\} = \frac{x^2}{1+x} \leq \frac{1}{2}x \leq \frac{1}{2}(x+y) = \frac{1}{2}\sigma(x,y) \leq \frac{1}{2}M_\sigma(x,y).$$

Thus

$$\eta(H_\sigma(Tx,Ty),M_\sigma(x,y)) = rM_\sigma(x,y) - H_\sigma(Tx,Ty) \geq (r-\frac{1}{2})M_\sigma(x,y) \geq 0.$$

Moreover, the conditions (ii) and (iii) of Theorem 2.1 are verified. Indeed, for $x_0 = 0$ *and* $x_1 = 0$, *we have* $\alpha(x_0,x_1) = 2 > 1$. *Also, if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n,x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x$ *in* (X,σ) *as* $n \to \infty$, *we get* $\{x_n\} \subseteq [0,1]$ *and* $|x_n - x| \to 0$ *as* $n \to \infty$. *So,* $x \in [0,1]$. *Hence* $\alpha(x_n,x) = 2 \geq 1$ *for all* n. *Then all required hypotheses of Theorem 2.1 are satisfied. Here* $u = 0$ *is a fixed point of* T.

3. Fixed point theory in ordered metric-like spaces

The study of fixed points in partially ordered sets was developed in [10,20–22,24]. In this section, we give some fixed point results for multi-valued mappings in the concept of metric-like spaces endowed with a partial order. Finally, we say that $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$ holds. Moreover, for $A, B \subseteq X$, we have $A \preceq B$ whenever for each $x \in A$ there exists $y \in B$ such that $x \preceq y$.

First, we introduce the following concept.

Definition 3.1. *Let* (X,σ) *be a metric-like space and* $T : X \to CB^\sigma(X)$ *be a multi-valued mapping. The pair* $(X,\preceq)$ *is said to be regular if the following condition holds: for any sequence* $\{x_n\}$ *in* X *with* $Tx_n \preceq Tx_{n+1}$, *for all* $n \in \mathbb{N}$ *and* $x_n \to x \in (X,\sigma)$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $Tx_{n(k)} \preceq Tx$, *for all* $k \in \mathbb{N}$.

We also have the following results.

Theorem 3.1. *Let* $(X,\sigma,\preceq)$ *be a complete partially ordered metric-like space. Suppose that* $T : X \to CB^\sigma(X)$ *is a multi-valued mapping. Suppose that there exists a manageable function* $\eta \in \widehat{Man}(\mathbb{R})$ *such that*

$$\eta(H_\sigma(Tx,Ty),M_\sigma(x,y)) \geq 0 \tag{14}$$

for all $x, y \in X$, *with* $Tx \preceq Ty$, *where*

$$M_\sigma(x,y) = \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{1}{4}[\sigma(x,Ty) + \sigma(Tx,y)]\}.$$

Assume that

(*i*) *for each* $x \in X$ *and* $y \in Tx$ *with* $Tx \preceq Ty$, *we have* $Ty \preceq Tz$ *for all* $z \in Ty$;
(*ii*) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $Tx_0 \preceq Tx_1$;
(*iii*) $(X, \preceq)$ *is regular.*

Then T *has a fixed point.*

Proof. Take $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x,y) = \begin{cases} 1 \text{ if } Tx \preceq Ty \\ 0 \quad otherwise. \end{cases}$$

The multi-valued mapping T is α-admissible. In fact, if $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$, then $Tx \preceq Ty$. By condition (i), we have $Ty \preceq Tz$ for all $z \in Ty$, then $\alpha(y,z) = 1$. Also, by (16), T verifies (2) of Theorem 2.1. Proceeding as in proof of Theorem 2.1, we may construct a sequence $\{x_n\}$ which converges to $x \in (X, \sigma)$ and $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. Finally, by condition (iii) and Lemma 1.1, we conclude that x is a fixed point of T. □

Theorem 3.2. *Let* $(X, p, \preceq)$ *be a complete partially ordered partial metric space. Suppose that* $T : X \to CB^p(X)$ *is a multi-valued mapping. Suppose that there exists a manageable function* $\eta \in \widehat{Man(\mathbb{R})}$ *such that*

$$\eta(H_p(Tx,Ty), N_p(x,y)) \geq 0 \tag{15}$$

for all $x, y \in X$, *with* $Tx \preceq Ty$, *where*

$$N_p(x,y) = \max\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(Tx,y)]\}.$$

Assume that

(*i*) *for each* $x \in X$ *and* $y \in Tx$ *with* $Tx \preceq Ty$, *we have* $Ty \preceq Tz$ *for all* $z \in Ty$;
(*ii*) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $Tx_0 \preceq Tx_1$;
(*iii*) $(X, \preceq)$ *is regular.*

Then T *has a fixed point.*

Theorem 3.3. *Let* $(X, \sigma, \preceq)$ *be a complete partially ordered metric-like space. Suppose that* $T : X \to CB^\sigma(X)$ *is a multi-valued mapping. Suppose that there exists a manageable function* $\eta \in \widehat{Man(\mathbb{R})}$ *such that*

$$\eta(H_\sigma(Tx,Ty), \sigma(x,y)) \geq 0 \tag{16}$$

for all $x, y \in X$, *with* $Tx \preceq Ty$. *Assume that*

(*i*) *for each* $x \in X$ *and* $y \in Tx$ *with* $Tx \preceq Ty$, *we have* $Ty \preceq Tz$ *for all* $z \in Ty$;
(*ii*) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $Tx_0 \preceq Tx_1$;
(*iii*) $(X, \preceq)$ *is regular.*

Then T *has a fixed point.*

4. **Application**

In this section, we consider the following two-point boundary value problem for second order differential equation:

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t, x(t)), t \in [0, 1] \\ x(0) = x(1) = 0, \end{cases} \tag{17}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The Green's function associated to (17) is

$$\begin{cases} G(t, s) = t(1 - s) & \text{if } 0 \leq t \leq s \leq 1 \\ G(s, t) = s(1 - t) & \text{if } 0 \leq s \leq t \leq 1. \end{cases} \tag{18}$$

Let us take $X = \mathcal{C}(I)(I = [0, 1])$ the space of all continuous functions defined on I. Consider the metric-like σ given by

$$\sigma(x, y) = \|x\|_\infty + \|y\|_\infty \quad \text{for all } x, y \in X,$$

where $\|u\|_\infty = \max\limits_{t\in[0,1]} |u(t)|$ for each $u \in X$. Clearly, (X, σ) is complete. Note that σ is not a partial metric.

It is well known that $x \in C^2(I)$ is a solution of (17) if and only if $x \in X = C(I)$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t, s)f(s, x(s))ds, t \in I. \tag{19}$$

Inspired from [6], we state the following result.

Theorem 4.1. *Suppose the following conditions hold:*

- *there exists a continuous function $\beta : I \to [0, \infty)$ such that*
$$|f(s, a)| \leq 8\,\beta(s)\,|a|,$$
for each $s \in I$ and $a \in \mathbb{R}$;
- $\sup\limits_{s\in I} \beta(s) = k \in (0, 1)$.

Then the problem (17) has a solution $u \in X$.

Proof. Consider the mapping $T : X \to X$ defined by

$$Tx(t) = \int_0^1 G(t, s)f(s, x(s))ds,$$

for all $x \in X$ and $t \in I$. Note that problem (17) is equivalent to finding $u \in X$ that is a fixed point of T. For $x, y \in X$, we have

$$\begin{aligned} |Tx(t)| &= |\int_0^1 G(t, s)\,f(s, x(s))ds| \\ &\leq \int_0^1 G(t, s)\,|f(s, x(s))|\,ds \\ &\leq 8\int_0^1 G(t, s)\,\beta(s)\,|x(s)|\,ds \\ &\leq 8k\,\|x\|_\infty \sup_{t\in I}\int_0^1 G(t, s)\,ds \\ &= k\,\|x\|_\infty. \end{aligned}$$

We have used the fact that for each $t \in I$, we have $\int_0^1 G(t,s)\,ds = -\frac{t^2}{2} + \frac{t}{2}$, and so $\sup_{t\in I} \int_0^1 G(t,s)\,ds = \frac{1}{8}$. Thus

$$\|Tx\|_\infty \leq k\,\|x\|_\infty. \tag{20}$$

Proceeding similarly, one can get

$$\|Ty\|_\infty \leq k\,\|y\|_\infty. \tag{21}$$

Summing (20) to (21), we find

$$\begin{aligned}\sigma(Tx,Ty) &= \|Tx\|_\infty + \|Ty\|_\infty \\ &\leq k(\,\|x\|_\infty + \|y\|_\infty) \\ &= k\,\sigma(x,y) \leq kM(x,y).\end{aligned}$$

Thus

$$\eta(H_\sigma(Tx,Ty), M_\sigma(x,y)) =: kM(x,y) - H_\sigma(Tx,Ty) \geq 0.$$

So all hypotheses of Theorem 2.1 are satisfied (with $\alpha(x,y)=1$), and so T has a fixed point $u \in X$, that is, the problem (17) has a solution $u \in C^2(I)$. □

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