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RELIABILITY LOGISTIC FUNCTION COMPARED TO OTHER RELIABILITY FUNCTIONS

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Our main results include: (i) the introduction and study of the reliability logistic function, (ii) the relation with the dilogarithm function, (iii) the finding of parameters b and c according to the mean and the standard deviation, and (iv) comparison between logistic distribution and other well-known distributions.

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1. Reliability logistic function

The *reliability* or *survival* function is the complement of the cumulative distribution function [1]-[7].

From the authors' point of view, the origins of reliability functions are some engineering problems and their theory is made by mathematicians according to the truism: "the mathematician makes what he can do - how he must do, meanwhile the engineer does what must be done - how he can done". In this context, we introduce and study a new reliability function, conceived by two mathematicians and two engineers, namely the *reliability logistic function.*

1.1. General statements

At the beginning of the activity of any "enterprise", the initiating action is made beginning with a certain level of achievements. The first period of the enterprise activity may be characterized by slow development, but after a certain period of time, the activity will intensify. However to the intensification of the enterprise activity, subsequent in time, a more and more consistent resistance will be opposed. If the analyzed term is denoted by y, its growth rate in time is dy/dt which is proportional to y and also to the "distance"

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towards the level of saturation a. Taking into consideration a proportionality factor c/a, the described mathematical model is given by the ODE

$$\frac{dy}{dt} = \frac{c}{a}\, y(a-y).$$

The nonzero solutions of this ODE

$$y(t) = \frac{a}{1+be^{-ct}},$$

are called *logistic functions*.

Modifying this function properly, it can relatively easily become a probability repartition function (with the codomain $[0;1)$). For $b > -1$ and $c > 0$ one obtains:

- Logistic distribution function

$$U(t) = \begin{cases} \dfrac{1-e^{-ct}}{1+be^{-ct}} & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0. \end{cases} \tag{1}$$

- Reliability logistic function

$$R(t) = 1 - U(t) = \begin{cases} \dfrac{1+b}{b+e^{ct}} & \text{if} \quad t \geq 0 \\ 1 & \text{if} \quad t < 0. \end{cases}$$

- Probability density function

$$u(t) = \begin{cases} \dfrac{1+b}{(b+e^{ct})^2} ce^{ct} & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0. \end{cases}$$

- Mean time between failures

$$\theta = \begin{cases} \dfrac{1+b}{bc} \ln(1+b) & \text{if} \quad b \neq 0 \\ \dfrac{1}{c} & \text{if} \quad b = 0. \end{cases} \tag{2}$$

One has $\lim\limits_{b\to 0} \dfrac{1+b}{bc} \ln(1+b) = \dfrac{1}{c}$.

- The standard deviation: $\sigma^2 = \displaystyle\int_0^\infty t^2 u(t)\, \mathrm{d}t - \theta^2$.

1.2. **Evaluation of standard deviation**

To evaluate the standard deviation, we follow the next steps.

Step 1. For $b \neq 0$, we have

$$\sigma^2 = 2(1+b) \int_0^\infty \frac{t}{b+e^{ct}}\, \mathrm{d}t - \theta^2 = \frac{2(1+b)}{bc} \int_0^\infty \frac{t\, bce^{-ct}}{1+be^{-ct}}\, \mathrm{d}t - \theta^2$$

$$= -\frac{2(1+b)}{bc}\, t \ln(1+be^{-ct}) \Big|_0^\infty + \frac{2(1+b)}{bc} \int_0^\infty \ln(1+be^{-ct})\, \mathrm{d}t - \theta^2.$$

By changing the variable $x = be^{-ct}$, one obtains

$$\sigma^2 = \frac{2(1+b)}{bc^2} \int_0^b \frac{\ln(1+x)}{x}\, \mathrm{d}x - \theta^2. \tag{3}$$

Step 2. For $b = 0$, we find $\sigma^2 = 2\int\limits_0^\infty te^{-ct}\,\mathrm{d}t - \theta^2 = \frac{2}{c^2}\Gamma(2) - \frac{1}{c^2} = \frac{1}{c^2}$.
$\lim\limits_{b\to 0}\Big(\frac{2(1+b)}{bc^2}\int\limits_0^b \frac{\ln(1+x)}{x}\,\mathrm{d}x - \theta^2\Big) = \frac{2}{c^2}\lim\limits_{b\to 0}\frac{\ln(1+b)}{b} - \frac{1}{c^2} = \frac{1}{c^2}$.
It follows that formula (3) is also true for $b = 0$, extending by continuity the function from the right side (with respect to the variable b).

To discuss further, we consider the function

$$\psi\colon (-1,\infty) \to \mathbb{R}, \quad \psi(x) = \int\limits_0^x \frac{\ln(1+t)}{t}\,\mathrm{d}t, \quad \forall x > -1.$$

For any $x \in (-1, 1]$, the function $\psi(x)$ admits the development

$$\psi(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n^2}. \tag{4}$$

It follows that $\psi(1) = \sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$ and $\lim\limits_{\substack{x\to -1\\ x>-1}}\psi(x) = -\frac{\pi^2}{6}$.
To write the function ψ in another way, we use the *dilogarithm functions*

$$Li_2\colon (-\infty, 1) \to \mathbb{R}, \quad Li_2(y) = -\int\limits_0^y \frac{\ln(1-t)}{t}\,\mathrm{d}t, \quad \forall y < 1;$$

$$\mathrm{dilog}\colon (0,\infty) \to \mathbb{R}, \quad \mathrm{dilog}\, y = \int\limits_1^y \frac{\ln t}{1-t}\,\mathrm{d}t, \quad \forall y > 0.$$

The function ψ can be expressed by using the functions Li_2 or dilog as follows

$$\psi(x) = \int\limits_0^x \frac{\ln(1+t)}{t}\,\mathrm{d}t = -Li_2(-x) = -\mathrm{dilog}\,(1+x), \quad \forall x > -1.$$

Now we come back to the properties of the functions U and R.

Proposition 1.1. *i) For $b \leq 1$, the function U is strictly concave on the interval $[0,\infty)$ and R is strictly convex on the interval $[0,\infty)$.*

ii) For $b > 1$, the point $t_0 = \frac{\ln b}{c}$ (> 0) represents the inflection point of the functions U and R. In this case, U is strictly convex on the interval $[0, t_0]$ and strictly concave on the interval $[t_0,\infty)$; R is strictly concave on the interval $[0, t_0]$ and strictly convex on the interval $[t_0,\infty)$.

Proof. The statements follow immediately, observing that

$$U''(t) = -R''(t) = \frac{(1+b)c^2e^{ct}}{(b+e^{ct})^3}\,(b - e^{ct}), \quad \forall t > 0. \qquad \square$$

2. **Finding parameters b and c according to the mean and standard deviation**

For any $x > -1$, $x \neq 0$, we have $\psi'(x) = \dfrac{\ln(1+x)}{x}$ and $\psi'(0) = 1$. It can easily be noted that $\psi'(x) > 0$, $\forall x > -1$.

From the relation (2) we deduce that $\theta = \dfrac{1+b}{c}\,\psi'(b)$; since $b > -1$ and $c > 0$, it follows that $\theta > 0$. Being given $\theta > 0$ and $\sigma > 0$, we want to determine the parameters $b > -1$ and $c > 0$ for which the relations (2) and (3) are fulfilled, namely

$$\begin{cases} \dfrac{1+b}{c}\,\psi'(b) = \theta \\ \dfrac{2(1+b)}{bc^2}\,\psi(b) - \theta^2 = \sigma^2. \end{cases} \tag{5}$$

The left sides of the expressions (5) are consider being extended by continuity in 0 (with respect to the variable b); hence, for $b = 0$ the two equalities from (5) become $\dfrac{1}{c} = \theta$, $\dfrac{2}{c^2} - \theta^2 = \sigma^2$.

The system (5) is equivalent to

$$\begin{cases} c = \dfrac{1+b}{\theta}\,\psi'(b) \\ \dfrac{2}{b(1+b)}\,\dfrac{\psi(b)}{(\psi'(b))^2} - 1 = \dfrac{\sigma^2}{\theta^2} \end{cases} \tag{6}$$

(similar remark for the case $b = 0$).

The next result can be proven without difficulty.

Lemma 2.1. *a) For any $y > 0$, $y \neq 1$, $\ln y < y - 1$.*
b) The next inequalities are satisfied

i) $\forall y > 1,\ \ln y < \dfrac{y^2-1}{2y}$; *ii)* $\forall y \in (0,1),\ \ln y > \dfrac{y^2-1}{2y}$.

c) For any $x > -1$, $x \neq 0$, $x\psi'(x) > \dfrac{x}{1+x}$.

d) For any $x > -1$, $x \neq 0$, $\psi'(x) < \dfrac{1}{\sqrt{1+x}} \leq \dfrac{x+2}{2(x+1)}$.

e) For any $x > -1$, $x \neq 0$,

$$\psi''(x) = \frac{1}{x}\Big(\frac{1}{1+x} - \psi'(x)\Big). \tag{7}$$

f) For any $x > -1$, we deduce $\psi''(x) < 0$ and $\psi''(0) = -\dfrac{1}{2}$. Therefore, the function ψ' is strictly decreasing, and the function ψ is strictly concave.

$$\lim_{x\to 0}\Big(\frac{2}{x(1+x)}\,\frac{\psi(x)}{(\psi'(x))^2} - 1\Big) = \frac{2}{(\psi'(0))^2}\lim_{x\to 0}\frac{\psi(x)}{x} - 1 = 2\psi'(0) - 1 = 1. \tag{8}$$

We define the function

$$f\colon(-1,\infty)\to\mathbb{R},\quad f(x)=\frac{2}{x(1+x)}\,\frac{\psi(x)}{(\psi'(x))^2}-1,\quad \forall x>-1, x\neq 0;$$

and $f(0)=1$. From (8) it follows that f is continuous (it is even of class $\mathcal{C}^\infty$).

Lemma 2.2. *a) For any $x>-1$, we have $f'(x)<0$.*

b) $\lim\limits_{\substack{x\to-1\\x>-1}} f(x)=\infty$ and $\lim\limits_{x\to\infty} f(x)=0$.

c) The image of the function f is the interval $(0,\infty)$.

Proof. Let $v_1\colon(-1,\infty)\to\mathbb{R}$, $v_1(x)=2\psi(x)-x(1+x)(\psi'(x))^2$, $\forall x>-1$.

$v_1'(x)=2\psi'(x)-(1+2x)(\psi'(x))^2-2x(1+x)\psi'(x)\psi''(x)$, $(\forall x>-1)$.

We use the equality (7) and it follows that for any $x>-1$, $x\neq 0$, we have: $v_1'(x)=\psi'(x)\big(2-(1+2x)\psi'(x)-2+2(1+x)\psi'(x)\big)=(\psi'(x))^2$; and for $x=0$, $v_1'(0)=2\psi'(0)-(\psi'(0))^2=1=(\psi'(0))^2$. Consequently

$$v_1'(x)=(\psi'(x))^2>0,\quad \forall x>-1.$$

Therefore v_1 is strictly increasing; for $x>0\Longrightarrow v_1(x)>g(0)=0$; for $-1<x<0\Longrightarrow v_1(x)<g(0)=0$.

We have $f(x)=\dfrac{v_1(x)}{2\psi(x)-v_1(x)}$, $\forall x>-1, x\neq 0$; and

$$f'(x)=\frac{2(\psi'(x))^2}{\big(2\psi(x)-v_1(x)\big)^2}\left(\psi(x)-\frac{v_1(x)}{\psi'(x)}\right),\quad \forall x>-1, x\neq 0. \tag{9}$$

Also $\lim\limits_{x\to 0} f'(x)=-\dfrac{1}{2}$. Therefore $f'(0)=-\dfrac{1}{2}$.

Let $v_2\colon(-1,\infty)\to\mathbb{R}$, $v_2(x)=\psi(x)-\dfrac{v_1(x)}{\psi'(x)}$, $\forall x>-1$.

$$v_2'(x)=\psi'(x)+\frac{v_1(x)}{(\psi'(x))^2}\,\psi''(x)-\frac{v_1'(x)}{\psi'(x)}=\frac{v_1(x)\psi''(x)}{(\psi'(x))^2}.$$

Since $\forall x\in(-1,0)$, $v_1(x)<0$; $\forall x>0$, $v_1(x)>0$; $\forall x>-1$, $\psi''(x)<0$, one gets: $\forall x\in(-1,0)$, $v_2'(x)>0$; $\forall x>0$, $v_2'(x)<0$. One obtains that v_2 is strictly increasing on the interval $(-1,0]$ and it is strictly decreasing on the interval $[0,\infty)$.

For $-1<x<0\Longrightarrow v_2(x)<v_2(0)=0$; for $x>0\Longrightarrow v_2(x)<v_2(0)=0$.

Therefore for any $x\in(-1,\infty)$, we have $v_2(x)<0$. From relation (9) it follows that for any $x>-1$, $x\neq 0$, we have $f'(x)<0$. We deduce that f is a strictly decreasing function.

We have $\lim\limits_{\substack{x\to-1\\x>-1}} f(x)=\infty$ and $\lim\limits_{x\to\infty} f(x)=0$. Since f is a continuous and strictly decreasing function, it follows that the image of the function f is the interval $(0,\infty)$. □

Proposition 2.1. *Let $g\colon(-1,\infty)\to(0,\infty)$ be the function*

$$g(x)=(1+x)\psi'(x)=\begin{cases}\dfrac{(1+x)\ln(1+x)}{x} & \text{if } x>-1, x\neq 0;\\ 1 & \text{if } x=0.\end{cases} \tag{10}$$

a) For any $x \in (-1, 1]$, $g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}\, x^n$.

b) For any $x > -1$, $g(x) = \psi'\Big(\frac{-x}{1+x}\Big)$.

c) For any $x \geq -\frac{1}{2}$, $g(x) = \sum_{n=0}^{\infty} \frac{x^n}{(1+n)(1+x)^n}$.

d) For any $x > -1$, $x \neq 0$, $g(x) < \sqrt{1+x} \leq 1 + \frac{x}{2}$.

e) For each $x \in (-1, 0)$, $g(x) > 1 + x$.

f) For any $x > -1$, $g'(x) > 0$, $g''(x) < 0$. *Therefore the function* g *is strictly increasing and strictly concave. The function* g *is bijective.*

g) For any $x > -1$, $g''(x) + (g'(x))^2 < 0$.

Proof. a) The statement follows easily using the equality

$$\ln(1+y) = \sum_{n=1}^{\infty} (-1)^{n-1}\, \frac{y^n}{n}, \quad \forall y \in (-1, 1]. \tag{11}$$

b) One has $\forall x > -1 \Longrightarrow \frac{-x}{1+x} > -1$. The respective equality is obtained by direct calculation.

c) If $x \geq -\frac{1}{2}$ then $\frac{-x}{1+x} \in (-1, 1]$. Further one uses the step *b)* and the conclusion is obtained by replacing $y = \frac{-x}{1+x}$ in the equality (11).

d) The statement follows immediately from Lemma 2.1, *d)*.

e) If $x \in (-1, 0)$, then $\psi'(x) > \psi'(0) = 1$, equivalent to $g(x) > 1 + x$.

f) and *g)*: Using the equality (11) it follows that: $g'(0) = \frac{1}{2}$ and $g''(0) = -\frac{1}{3}$. By direct calculation one obtains

$$g'(x) = \frac{x - \ln(1+x)}{x^2} = \frac{1 + x - g(x)}{x(1+x)}, \quad \forall x > -1,\, x \neq 0. \tag{12}$$

From Lemma 2.1, *a)*, it follows that $g'(x) > 0$ $(x > -1, x \neq 0)$.

The relation (12) is equivalent to $x(1+x)g'(x) = 1 + x - g(x)$. We take the derivative and obtain: $x(1+x)g''(x) + (2x+1)g'(x) = 1 - g'(x)$ or $x(1+x)g''(x) = 1 - 2(x+1)g'(x)$; we use (12) and obtain

$$g''(x) = \frac{2g(x) - 2 - x}{(1+x)x^2}, \quad \forall x > -1, x \neq 0. \tag{13}$$

Since $g(x) > 0$, $\lim_{x \to -1} g(x) = 0$, $\lim_{x \to \infty} g(x) = \infty$, and g is a continuous function it follows that the image of the function g is the interval $(0, \infty)$. Therefore the function g is surjective.

The function g is strictly increasing ($\forall x > -1$, $g'(x) > 0$), therefore the function g is injective.

Let $x > -1$, $x \neq 0$. Using (12) and (13) we deduce

$$g''(x) + (g'(x))^2 = \frac{(g(x))^2 - (1+x)}{x^2(1+x)^2} < 0 \quad \big(\text{step } d)\big).$$

For $x = 0$: $g''(0) + (g'(0))^2 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12} < 0$.

From $g''(x) + (g'(x))^2 < 0$ it follows that $g''(x) < 0$. □

Theorem 2.1. *Let $f\colon (-1,\infty) \to (0,\infty)$ be the function*

$$f(x) = \begin{cases} \dfrac{2x}{(1+x)\big(\ln(1+x)\big)^2} \displaystyle\int_0^x \frac{\ln(1+t)}{t}\,\mathrm{d}t - 1 & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases} \tag{14}$$

Let $g\colon (-1,\infty) \to (0,\infty)$ be the function which is defined by (10).

a) The function f is well defined, strictly decreasing, bijective and continuous (it is even of the class $\mathcal{C}^\infty$).

b) Let be $y_0 = \dfrac{\pi^2}{12(\ln 2)^2} - 1 = 0.7118573712...$. We have $f^{-1}(y_0) = 1$.

The following statements are fulfilled: if $y \in (1,\infty)$, then $f^{-1}(y) \in (-1,0)$; if $y \in (0,1)$, then $f^{-1}(y) \in (0,\infty)$; if $y \in (0,y_0)$, then $f^{-1}(y) \in (1,\infty)$; if $y \in (y_0,1)$, then $f^{-1}(y) \in (0,1)$.

c) For any $\theta > 0$ and $\sigma > 0$, the parameters $b > -1$ and $c > 0$ exist and are unique such that, the relations (2) *and* (3) *are fulfilled (i.e., the relations* (5) *hold). These parameters are*

$$b = f^{-1}\Big(\frac{\sigma^2}{\theta^2}\Big); \qquad c = \frac{g(b)}{\theta} = \frac{1}{\theta}\, g\Big(f^{-1}\Big(\frac{\sigma^2}{\theta^2}\Big)\Big). \tag{15}$$

The necessary and sufficient condition for b to be strictly positive is $\sigma < \theta$. The necessary and sufficient condition for b to be zero is $\sigma = \theta$. In this case one obtains the exponential distribution with the parameter $c = \dfrac{1}{\theta}$. Therefore the exponential distribution is a particular case of logistic distribution.

The necessary and sufficient condition for $b = 1$ is $\sigma = \theta\sqrt{y_0}$. In this case we have $c = \dfrac{2\ln 2}{\theta}$ $\big(\sqrt{y_0} = 0.8437164045...\big)$.

Proof. a) From Lemma 2.2, c), it follows that the function $f\colon (-1,\infty) \to (0,\infty)$, defined by the relation (14), is well defined and surjective.

From Lemma 2.2, a), it follows that f is strictly decreasing; therefore the function f is injective.

b) The image of the function f^{-1} is the interval $(-1,\infty)$. Therefore $\forall y > 0 \Rightarrow f^{-1}(y) > -1$. On the other hand, $f(1) = \dfrac{\psi(1)}{(\ln 2)^2} - 1$; since $\psi(1) = \dfrac{\pi^2}{12}$, we deduce that $f(1) = y_0$, $f^{-1}(y_0) = 1$. From $f(0) = 1$ it follows that $f^{-1}(1) = 0$.

Since f is strictly decreasing, it follows that also f^{-1} is strictly decreasing.

If $y > 1$, then $f^{-1}(y) < f^{-1}(1) = 0$; if $y < 1$, then $f^{-1}(y) > f^{-1}(1) = 0$; if $y < y_0$, then $f^{-1}(y) > f^{-1}(y_0) = 1$; if $y_0 < y < 1$, then $1 = f^{-1}(y_0) > f^{-1}(y) > f^{-1}(1) = 0$.

c) The equations of the system (6) are written as $f(b) = \frac{\sigma^2}{\theta^2}$, $c = \frac{g(b)}{\theta}$. They are equivalent to the relations (15). It can be noted that c is really strictly positive, because $g(b) > 0$.

The other statements are immediately checked. □

Remark 2.1. *We conclude that for the determination of* b*, one needs to know only the ratio* $\frac{\sigma}{\theta}$.

Remark 2.2. *Let be* $b > 1$. *According to Proposition* 1.1, $t_0 = \frac{\ln b}{c}$ *is the only inflection point of the functions* U *and* R *in the interval* $(0, \infty)$.

Taking into account the relations (15), (10) *it follows that*

$$t_0 = \theta \, \frac{b \ln b}{(b+1)\ln(b+1)}.$$

The inflection point t_0 *is strictly smaller than* θ *and* $\lim_{b \to \infty} t_0(b) = \theta$.

According to relation (15), to determine the parameter b we have to know the values of the function f^{-1}.

3. Comparison between the logistic distribution and the exponential distribution

Let X_1, X_2 be two random variables, with the distribution functions F_1, respectively F_2, and reliability functions R_1, respectively R_2. The inequality $F_1(t) \leq F_2(t)$, $t \in I$, is equivalent to $R_1(t) \geq R_2(t)$, $t \in I$.

Definition 3.1. *The random variable* X_1 *is called better than* X_2 *on the interval* I, *if* $R_1(t) \geq R_2(t)$, $\forall t \in I$.

We consider, the exponential and logistic distributions, both, with the same average $\theta > 0$. The logistic distribution function is (1), with $c = \frac{g(b)}{\theta}$; and for exponential distribution we have the following distribution function

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0; \\ 0 & \text{if } t < 0, \end{cases} \qquad \text{with } \lambda = \frac{1}{\theta}.$$

In the above conditions, the following result is obtained.

Proposition 3.1. *i*) *If* $b \in (-1, 0)$, *then,* $\forall t \in (0, \theta]$, *we have* $F(t) < U(t)$. *Hence, the exponential repartition is better than the logistic repartition, on* $(-\infty, \theta]$.

ii) *If* $b > 0$, *then, for any* $t \in (0, \theta]$, *one gets* $U(t) < F(t)$. *Hence, the logistic repartition is better than the exponential repartition, on* $(-\infty, \theta]$.

iii) *If* $b = 0$, *then,* $\forall t \in \mathbb{R}$, *we have* $U(t) = F(t)$.

Proof. If $b = 0$ then it is obvious that $U(t) = F(t)$.

If $b \in (-1, 0)$, then $g(b) < g(0) = 1$. From Proposition 2.1, we obtain $g(b) > 1 + b$. It follows that $b < g(b) - 1 < 0$ and $\dfrac{b}{g(b) - 1} > 1$.

If $b > 0$, then $g(b) > g(0) = 1$. From Proposition 2.1, we deduce $g(b) < \sqrt{1 + b} \leq 1 + b$. This implies $0 < g(b) - 1 < b$ and $\dfrac{b}{g(b) - 1} > 1$.

We have showed that $\dfrac{b}{g(b) - 1} > 1$, $\forall b \in (-1, 0) \cup (0, \infty)$. From the above, it follows that to get the results *i)* and *ii)* it is sufficient to show that: $\forall b \in (-1, 0) \cup (0, \infty)$ and $\forall t \in (0, \theta]$, $\dfrac{U(t) - F(t)}{g(b) - 1} < 0$.

Let $v \colon [0, \infty) \to \mathbb{R}$, $v(t) = \dfrac{e^{\lambda(g(b)-1)t} + b(e^{-\lambda t} - 1) - 1}{g(b) - 1}$, $\forall t \geq 0$;
$v'(t) = \dfrac{\lambda(g(b) - 1)e^{\lambda(g(b)-1)t} - \lambda b e^{-\lambda t}}{g(b) - 1} = \lambda e^{-\lambda t}\Big(e^{\lambda g(b)t} - \dfrac{b}{g(b) - 1}\Big)$.

Let $t_1 = \dfrac{\theta}{g(b)} \ln\Big(\dfrac{b}{g(b) - 1}\Big)$. Since $\dfrac{b}{g(b) - 1} > 1$, we deduce $t_1 > 0$. One observes that $e^{\lambda g(b) t_1} = \dfrac{b}{g(b) - 1}$; hence $v'(t_1) = 0$.

If $t \in [0, t_1)$, then $v'(t) < 0$, and hence the function v is strictly decreasing on $[0, t_1]$. It follows that, for any $t \in (0, t_1]$, we have $v(t) < v(0) = 0$.

We have shown that v is strictly negative on the interval $(0, t_1]$. Particularly, $v(t_1) < 0$.

If $t \geq t_1$, then $v'(t) > 0$, hence v is strictly increasing on $[t_1, \infty)$; if $b \in (-1, 0)$, then $g(b) - 1 < 0$ and $\lim\limits_{t\to\infty} v(t) = \dfrac{-(b + 1)}{g(b) - 1} > 0$; if $b > 0$, then $g(b) - 1 > 0$ and $\lim\limits_{t\to\infty} v(t) = \infty$. Hence, for any $b > -1$, $b \neq 0$, we have $\lim\limits_{t\to\infty} v(t) > 0$. Since $v(t_1) < 0$, we deduce that there exists $t_2 \in (t_1, \infty)$, such that $v(t_2) = 0$. This t_2 is the only number in (t_1, ∞) where v vanishes, since v is strictly increasing (hence injective) on this interval. In fact, t_2 is the only number in $(0, \infty)$ where v vanishes, since v is strictly negative on $(0, t_1]$.

Hence, on each of the intervals $(0, t_2)$, (t_2, ∞), the function v has constant sign. Since $t_1 \in (0, t_2)$, $v(t_1) < 0$, and $\lim\limits_{t\to\infty} v(t) > 0$, it follows that: $\forall t \in (0, t_2)$, $v(t) < 0$; and $\forall t \in (t_2, \infty)$, $v(t) > 0$.

We shall show that $\theta < t_2$, hence $(0, \theta] \subseteq (0, t_2)$; from the above discussion it follows that: $\forall t \in (0, \theta]$, $v(t) < 0$; equivalent to $\dfrac{U(t) - F(t)}{g(b) - 1} < 0$.

The inequality $\theta < t_2$ is equivalent to $v(\theta) < 0$, i.e.,

$$e^{g(b)} + b(1 - e) - e > 0 \quad \text{if } b \in (-1, 0) \tag{16}$$
$$e^{g(b)} + b(1 - e) - e < 0 \quad \text{if } b > 0. \tag{17}$$

Let $v_1 \colon (-1, \infty) \to \mathbb{R}$, $v_1(x) = e^{g(x)} + x(1 - e) - e$, $\forall x > -1$;
$v_1'(x) = g'(x)e^{g(x)} + 1 - e$;

$v_1''(x) = g''(x)e^{g(x)} + \big(g'(x)\big)^2 e^{g(x)} = \big(g''(x) + (g'(x))^2\big)e^{g(x)}$.

From Proposition 2.1, it follows that v_1'' is strictly negative, hence v_1' is strictly decreasing. Since $\lim\limits_{\substack{x\to -1\\ x>-1}} v_1'(x) = \infty$ and $v_1'(0) = \dfrac{2-e}{2} < 0$, there exists $b_0 \in (-1, 0)$, such that $v_1'(b_0) = 0$.

If $-1 < x < b_0$, then $v_1'(x) > v_1'(b_0) = 0$. If $x > b_0$, then $v_1'(x) < v_1'(b_0) = 0$. Hence the function v_1 is strictly increasing on $(-1, b_0]$ and strictly decreasing on $[b_0, \infty)$.

If $x \in (-1, b_0]$, then $v_1(x) > \lim\limits_{\substack{y\to -1\\ y>-1}} v_1(y) = 0$. Particularly, $v_1(b_0) > 0$.

We have $v_1(0) = 0$; 0 is the only point in $[b_0, \infty)$ where v_1 vanishes, since v_1 is strictly decreasing (injective) on $[b_0, \infty)$. Hence v_1 has constant sign on each of the intervals $[b_0, 0)$ and $(0, \infty)$.

Since $v_1(b_0) > 0$, we deduce that v_1 is strictly positive on $[b_0, 0)$.

We have proved that v_1 is strictly positive on $(-1, 0)$, i.e., (16).

Due to $\lim\limits_{x\to\infty} v_1(x) = -\infty$, it follows that v_1 is strictly negative on $(0, \infty)$, i.e., (17). □

4. **Comparison between the logistic distribution and the Weibull distribution**

Consider a random variable with Weibull distribution; the repartition function is

$$F_{\beta,\lambda}(t) = \begin{cases} 1 - e^{-\lambda t^\beta} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0, \end{cases} \tag{18}$$

with $\lambda > 0$, $\beta > 0$, $\beta \neq 1$ (for $\beta = 1$ one obtains the exponential repartition).

Proposition 4.1. *i) If $\beta \in (0, 1)$, then there exists $q > 0$, such that, for any $t \in (0, q)$, we have $U(t) < F_{\beta,\lambda}(t)$. Consequently the logistic repartition is better than Weibull repartition on $(-\infty, q)$.*

ii) If $\beta > 1$, then there exists $q > 0$, such that, for any $t \in (0, q)$, we have $F_{\beta,\lambda}(t) < U(t)$. Hence the Weibull repartition is better than logistic repartition on $(-\infty, q)$.

Proof. Let $w\colon [0, \infty) \to \mathbb{R}$, $w(t) = e^{ct-\lambda t^\beta} + be^{-\lambda t^\beta} - b - 1$,

$w'(t) = \big(c - \lambda\beta t^{\beta-1}\big)e^{ct-\lambda t^\beta} - b\lambda\beta t^{\beta-1}e^{-\lambda t^\beta}$, $\forall t > 0$.

i) $\beta \in (0, 1)$. We have $t^{1-\beta}w'(t) = \big(ct^{1-\beta} - \lambda\beta\big)e^{ct-\lambda t^\beta} - b\lambda\beta e^{-\lambda t^\beta}$ and $\lim\limits_{\substack{t\to 0\\ t>0}} t^{1-\beta}w'(t) = -\lambda\beta(1 + b) < 0$; it follows that there exists $q > 0$ such that, $\forall t \in (0, q)$ we have $w'(t) < 0$, hence w is strictly decreasing on $[0, q)$.

For any $t \in (0, q)$, $w(t) < w(0) = 0$. It is easily seen that inequality $w(t) < 0$ is equivalent to $U(t) < F_{\beta,\lambda}(t)$.

ii) We have $\lim\limits_{\substack{t\to 0\\ t>0}} w'(t) = c > 0$; it follows that there exists $q > 0$ such that, $\forall t \in (0, q)$, we have $w'(t) > 0$, hence w is strictly increasing on $[0, q)$.

For any $t \in (0, q)$, $w(t) > w(0) = 0$. It follows that $F_{\beta,\lambda}(t) < U(t)$. □

Further, we study what is equivalent to condition $\beta \in (0,1)$.

Let $\tilde{\theta}$, $\tilde{\sigma}^2$ be the mean and dispersion of the random variable which has the Weibull repartition, i.e., $\tilde{\theta} = \eta\Gamma\Big(\dfrac{1}{\beta}+1\Big)$, $\tilde{\sigma}^2 = \eta^2\Gamma\Big(\dfrac{2}{\beta}+1\Big) - \tilde{\theta}^2$, where $\eta = \dfrac{1}{\lambda^{\frac{1}{\beta}}}$. It follows that: $\dfrac{\tilde{\sigma}^2}{\tilde{\theta}^2}+1 = \Gamma\Big(\dfrac{2}{\beta}+1\Big)\Big(\Gamma\Big(\dfrac{1}{\beta}+1\Big)\Big)^{-2}$.

We introduce three functions $w_1, w_2, w_3\colon (0,\infty) \to \mathbb{R}$,

$w_1(x) = \ln\Gamma(x)$, $w_2(x) = \dfrac{\Gamma(2x+1)}{\big(\Gamma(x+1)\big)^2}$, $w_3(x) = \ln w_2(x)$.

For any $x > 0$, we have $w_3'(x) = 2w_1'(2x+1) - 2w_1'(x+1)$. According to Lagrange Theorem it follows that there exists $\xi \in (x+1, 2x+1)$, such that $w_3'(x) = 2xw_1''(\xi)$. One knows that $w_1''(y) > 0$ $(\forall y > 0)$. We obtain $w_3'(x) > 0$, hence w_3 is strictly increasing, equivalent to w_2 is strictly increasing. The inequality $\beta < 1$ is equivalent to $w_2\Big(\dfrac{1}{\beta}\Big) > w_2(1)$ or $\dfrac{\tilde{\sigma}^2}{\tilde{\theta}^2}+1 > 2$, i.e., $\tilde{\sigma} > \tilde{\theta}$.

Furthermore we assume that repartitions considered above have the same mean and the same dispersion, i.e., $\tilde{\theta} = \theta$ and $\tilde{\sigma} = \sigma$. From Theorem 2.1, it follows that $\sigma > \theta$ is equivalent to $b \in (-1,0)$.

One obtains the following result.

Proposition 4.2. *Suppose that the logistic and Weibull repartitions have the same mean θ and the same dispersion σ^2, the repartition functions being* (1) *and* (18) *(with $\lambda > 0$, $\beta > 0$, $\beta \neq 1$).*

The following statements are equivalent.

i) $\beta \in (0,1)$ (respectively $\beta > 1$).

ii) $b \in (-1,0)$ (respectively $b > 0$).

iii) $\sigma > \theta$ (respectively $\sigma < \theta$).

iv) There exists $q > 0$, such that, for any $t \in (0,q)$, we have $U(t) < F_{\beta,\lambda}(t)$ $\big($respectively $F_{\beta,\lambda}(t) < U(t)\big)$.

5. Comparison between the logistic distribution and the uniform distribution

We consider a logistic repartition, with $c = \dfrac{g(b)}{\theta}$, and a uniform repartition on the interval $[0,2\theta]$. Both have the same mean $\theta > 0$. The logistic repartition is (1); the uniform repartition, on the interval $[0,2\theta]$, has the formula: $V(t) = \begin{cases} 0 & \text{if } t < 0; \\ \dfrac{t}{2\theta} & \text{if } t \in [0,2\theta]; \\ 1 & \text{if } t > 2\theta. \end{cases}$

Lemma 5.1. *There exists a unique solution b_0 of the equation $\psi'(x) = \dfrac{1}{2}$. This solution b_0 belongs to the interval* $(2,3)$.

Proof. According to Lemma 2.1, $\psi'(3) < \dfrac{1}{\sqrt{1+3}} = \dfrac{1}{2}$; $\psi'(2) = \dfrac{\ln 3}{2} > \dfrac{1}{2}$.

Hence there exists $b_0 \in (2,3)$, such that $\psi'(b_0) = \dfrac{1}{2}$. The solution is unique since the function ψ' is strictly decreasing. □

Proposition 5.1. *Let* b_0 *be the solution of the equation* $\psi'(x) = \dfrac{1}{2}$.

i) If $b > b_0$*, then there exists* $q > 0$*, such that, for any* $t \in (0, q)$*, we have* $U(t) < V(t)$*. Hence, on the interval* $(-\infty, q)$*, the logistic repartition is better than the uniform repartition.*

ii) If $b \in (-1, b_0]$*, then there exists* $q > 0$*, such that, for any* $t \in (0, q)$*, we have* $V(t) < U(t)$*. Hence, on the interval* $(-\infty, q)$*, the uniform repartition is better than the logistic repartition.*

Proof. Let $h\colon [0, 2\theta] \to \mathbb{R}$, $h(t) = U(t) - \dfrac{t}{2\theta}$, $\forall t \in [0, 2\theta]$. For any $t > 0$, we get $h'(t) = u(t) - \dfrac{1}{2\theta} = \dfrac{1+b}{(b+e^{ct})^2} ce^{ct} - \dfrac{1}{2\theta}$. Hence $h'(0) = \dfrac{1}{\theta}\Big(\psi'(b) - \dfrac{1}{2}\Big)$.

i) If $b > b_0$, then $\psi'(b) < \psi'(b_0) = \dfrac{1}{2}$; hence $h'(0) < 0$. Hence, there exists $q \in (0, 2\theta)$ such that, for any $t \in (0, q)$, we have $h'(t) < 0$, i.e., h is strictly decreasing on $[0, q)$. It follows that, for any $t \in (0, q)$, the inequality $h(t) < h(0) = 0$ holds, equivalent to $U(t) < V(t)$.

ii) If $b \in (-1, b_0)$, then $\psi'(b) > \psi'(b_0) = \dfrac{1}{2}$; hence $h'(0) > 0$. Consequently, there exists $q \in (0, 2\theta)$ such that, for any $t \in (0, q)$, we have $h'(t) > 0$, i.e., h is strictly increasing on $[0, q)$. It follows that, for any $t \in (0, q)$, the inequality $h(t) > h(0) = 0$ holds, equivalent to $V(t) < U(t)$.

If $b = b_0$, then $h'(0) = 0$; $\lim\limits_{\substack{t\to 0\\ t>0}} \dfrac{h(t)}{t^2} = \text{(twice l'Hospital)} = \dfrac{c^2(b_0 - 1)}{2(b_0 + 1)^2} > 0$ $(b_0 > 2)$. Since $\lim\limits_{\substack{t\to 0\\ t>0}} \dfrac{h(t)}{t^2} > 0$, it follows that there exists $q \in (0, 2\theta)$ such that, for any $t \in (0, q)$, we have $h(t) > 0$, equivalent to $V(t) < U(t)$. □

REFERENCES

[1] *D. G. Kleinbaum and M. Klein*, Survival Analysis: a Self-learning Text, Third ed. Springer, 2012.

[2] *M. Tableman and J.S. Kim*, Survival Analysis Using S: Analysis of Time-to-event Data, First ed. Chapman and Hall/CRC, 2003.

[3] *C. Ebeling*, An Introduction to Reliability and Maintainability Engineering, Second ed. Waveland Press, 2010.

[4] *J. Klein and M. Moeschberger*, Survival Analysis: Techniques for Censored and Truncated Data, Second ed. Springer, 2005.

[5] *W. Mendenhall and S. Terry*, Statistics for Engineering and the Sciences, 5th ed. Pearson/Prentice Hall, 2007.

[6] *B. Efron and T. Hastie*, Computer Age Statistical Inference: Algorithms, Evidence, and Data Science, First ed. Cambridge University Press, 2016.

[7] *J. Lawless*, Statistical Models and Methods for Lifetime Data, Second ed. Wiley, 2002.