PARAMETER MARCINKIEWICZ INTEGRAL ON NON-HOMOGENEOUS MORREY SPACE WITH VARIABLE EXPONENT

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In this paper, the author proves that the parameter Marcinkiewicz integral $\mathcal{M}^\rho$ is bounded on Morrey space with variable exponent $M^{q(\cdot)}_{p(\cdot)}(X)_N$ over the non-homogeneous space (X, d, μ). Meanwhile, the boundeness of commutator $\mathcal{M}^\rho_b$ generated by the regular bounded mean oscillation space $(= \mathrm{RBMO})$ and $\mathcal{M}^\rho$ on the $M^{q(\cdot)}_{p(\cdot)}(X)_N$ is also obtained. As corollaries, the boundeness of the $\mathcal{M}^\rho$ and commutator the $\mathcal{M}^\rho_b$ on the Lebesgue space with variable exponent is also obtained.

Keywords: Non-homogeneous space, parameter Marcinkiewicz integral, commutator, RBMO(μ), Morrey space with variable exponent.

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1. Introduction

During the past fifteen and twenty years, many authors have paid much attention to study the classical theory of harmonic analysis with non-doubling measure which only satisfies the polynomial growth condition. For example, in 2005, Sawano and Tanaka [13] gave out the definition of Morrey space, and respectively established the boundedness of the Hardy-Littlewood maximal operator, fractional integral and Marcinkiewicz integral. In 2007, Hu et. al showed that the Marcinkiewicz integral $\mathcal{M}$ is bounded on Lebesgue space $L^p(\mu)$ with $p \in (1, \infty)$, and bounded from the Hardy space $H^1(\mu)$ into the Lebesgue space $L^1(\mu)$ (see [3]). The further research about the integral operators or function spaces under non-doubling measure, we can see [6, 7, 13, 17, 18] and the references therein.

Regarding as a generation of the classical Lebesgue space, Orlicz in 1931 first introduced the definition of Lebesgue space with variable exponent (see [11]). Fortunately, Orlicz et.al have researched a wide class of space, namely, Orlicz space and Musielak-Orlicz space (see [8, 9]). However, up to 1991, the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ and the Sobolev space with variable exponent $W^{\kappa,p(\cdot)}(\mathbb{R}^n)$ are systematically studied by Kováčik and Rákosnik [5]. Since then, the spaces with variable exponent on $\mathbb{R}^n$ have been widely studied, for example, Wang [16] proved that Marcinkiewicz integral operator and its commutator are bounded on Herz space with variable exponent. The more redevelopment about the space, we can see [1, 2, 10].

Let $X := (X, d, \mu)$ be a *quasimetric measure space*, if μ is a complete measure, and there exists a non-negative real-valued function (quasimetric) d on $X \times X$ satisfying the following conditions:

(i) $d(x, x) = 0$ for all x in X; (ii) $d(x, y) > 0$ for all $x \neq y$, $x, y \in X$;

(iii) for all $x, y, z \in X$, there exists a constant $a_1 > 0$, such that $d(x, y) \leq a_1(d(x, z)+ d(y, z))$;

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(iv) there exists a constant $a_0 > 0$, such that $d(x,y) \le a_0 d(y,x)$ for all $x, y \in X$.

Moreover, we always assume that the balls $B(x,r) := \{y \in X : d(x,y) < r\}$ are measurable, $0 \le \mu(B(x,r)) < \infty$, $\mu(X) < \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$ and $r > 0$ in this paper.

A measure μ on X is said to satisfy the growth condition, if there exists a positive constant C such that, for all $x \in X$ and $r > 0$,

$$\mu(B(x,r)) \le Cr. \tag{1.1}$$

Then the (X, d, μ) is called a *non-homogeneous space*. In this setting, we will study the boundedness of the parameter Marcinkiewicz integral $\mathcal{M}_\rho$ and its commutator $\mathcal{M}_b^\rho$ on Morrey space with variable exponent over the (X, d, μ). In 2010, Vakhtang and Alexander [15] proved that the modified maximal function M and the fractional integral with variable exponent $I_{\alpha(\cdot)}$ is respectively bounded from the Morrey space with variable exponent $M_{p(\cdot)}^{q(\cdot)}(X)_N$ into the $M_{p(\cdot)}^{q(\cdot)}(X)_{N\bar{a}}$, where $N\bar{a}$ is a positive constant satisfying $N \ge 1$ and $\bar{a} := a_1(a_1(a_0+1)+1)$ with $a_1,\ a_0 > 0$. Motivated by this, we will mainly prove that the parameter Marcinkiewicz integral $\mathcal{M}^\rho$ is bounded from the $M_{p(\cdot)}^{q(\cdot)}(X)_N$ into the $M_{p(\cdot)}^{q(\cdot)}(X)_{N\bar{a}}$, and the commutator $\mathcal{M}_b^\rho$ generated by the regular bounded mean oscillation space (= RBMO) and $\mathcal{M}^\rho$ on the $M_{p(\cdot)}^{q(\cdot)}(X)_N$ is also obtained.

Suppose that p is a μ-measurable function on X. We denote

$$p_-(E) := \inf_E p(x) \qquad \text{and} \qquad p_+(E) := \sup_E p(x).$$

Moreover, we simply abbreviate $p_- := p_-(X)$ and $p_+ := p_+(X)$.

Definition 1.1. [15] *Let $N \ge 1$ be a constant. Suppose that p is a function on X such that $0 < p_- < p_+ < \infty$. We say that $p \in \mathcal{P}(N)$ if there is a constant $C > 0$ such that*

$$[\mu(B(x,Nr))]^{p_-(B(x,r)) - p_+(B(x,r))} \le C \tag{1.2}$$

for all $x \in X$ and $r > 0$.

Definition 1.2. [15] *Let $0 < p_- \le p_+ < \infty$. We say that the function p on X satisfies the log-Hölder continuity condition $p \in \mathrm{LH}(X)$ if*

$$|p(x) - p(y)| \le \frac{A}{-\log(d(x,y))}, \quad d(x,y) \le \frac{1}{2},$$

where the positive constant A does not depend on $x,\ y \in X$.

For any ball B, we respectively denote its center and radius by c_B and r_B (or $r(B)$). Let $\alpha > 1$ and $\beta > \alpha$, we say that ball B is an (α, β)-doubling ball if $\mu(\alpha B) \le \beta\mu(B)$, where αB denotes the ball with the same center as B and $r(\alpha B) = \alpha r(B)$. Especially, for any given ball B, we denote by $\widetilde{B}$ the smallest doubling ball which contains B and has the same center as B. Given two balls $B \subset S$ in X, define

$$K_{B,S} := 1 + \sum_{k=1}^{N_{B,S}} \frac{\mu(2^k B)}{r(2^k B)},$$

where $N_{B,S}$ is denoted by the smallest integer k such that $r(2^k B) \ge r(S)$.

The following definition of the regular bounded mean oscillation space is from [14].

Definition 1.3. [14] *Let $\tau \in (1, \infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space* $\mathrm{RBMO}(\mu)$ *if there exists a constant $C > 0$ such that for any ball B centered at some point of* $\mathrm{supp}(\mu)$,

$$\frac{1}{\mu(\tau B)} \int_B |f(y) - m_{\widetilde{B}}(f)| \mathrm{d}\mu(y) \le C \tag{1.3}$$

and

$$|m_B(f) - m_S(f)| \le CK_{B,S} \tag{1.4}$$

for any two doubling balls $B \subset S$, where $m_B(f)$ represents the mean value of f over ball B. Moreover, the minimal constant C satisfying (1.3) *and* (1.4) *is defined to be the norm of f in the space* $\mathrm{RBMO}(\mu)$ *and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.*

We give out the definition of parameter Marcinkiewicz integral on X.

Definition 1.4. *Let K be a locally integrable function on $X \times X \setminus \{(x,y) : x = y\}$. Assume that there exists a constant $C > 0$ such that for all $x,\ y \in X$ with $x \neq y$,*

$$|K(x,y)| \le \frac{C}{d(x,y)}, \tag{1.5}$$

and for any $y, y' \in X$, $d(x,y) \ge 2d(x,x')$ and $\epsilon \in (0,1]$,

$$\Big[|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')|\Big] \le C\frac{[d(x,x')]^{\epsilon}}{[d(x,y)]^{1+\epsilon}}. \tag{1.6}$$

The parameter Marcinkiewicz integral $\mathcal{M}^{\rho}(f)$ associated to the above kernel K is defined by, for $x \in X$ and $\rho \ge 0$,

$$\mathcal{M}^{\rho}(f)(x) = \left(\int_0^{\infty}\left|\frac{1}{t^{\rho}}\int_{d(x,y)\le t}K(x,y)[d(x,y)]^{\rho}f(y)\mathrm{d}\mu(y)\right|^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}. \tag{1.7}$$

Given a function $b \in \mathrm{RBMO}(\mu)$, the commutator $\mathcal{M}_b^{\rho}(f)$ associated with the $\mathcal{M}^{\rho}(f)$ is defined by

$$\mathcal{M}_b^{\rho}(f)(x) = \left(\int_0^{\infty}\left|\frac{1}{t^{\rho}}\int_{d(x,y)\le t}K(x,y)[d(x,y)]^{\rho}(b(x)-b(y))f(y)\mathrm{d}\mu(y)\right|^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}}. \tag{1.8}$$

We now recall the notation of Morrey space with variable exponent $M_{q(\cdot)}^{p(\cdot)}(X)_N$ in [15].

Definition 1.5. [15] *Let $N \ge 1$ be a constant. Suppose that $1 < q_- \le q(x) \le p(x) \le p_+ < \infty$. Then the Morrey space with variable exponent $M_{q(\cdot)}^{p(\cdot)}(X)_N$ is defined by*

$$M_{q(\cdot)}^{p(\cdot)}(X)_N = \{f \text{ is measurable} : \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)_N} < \infty\},$$

where

$$\|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)_N} := \sup_{x\in X,\ r>0}[\mu(B(x,Nr))]^{\frac{1}{p(x)}-\frac{1}{q(x)}}\|f\|_{L^{q(\cdot)}(B(x,r))}. \tag{1.9}$$

Remark 1.1. (1) *If we take $p(x) \equiv q(x)$ in (1.9), then the $M_{q(\cdot)}^{p(\cdot)}(X)_N$ is just the variable exponent Lebesgue space $L^{p(\cdot)}(X)$. Respectively, its definition as follows*

$$\|f\|_{L^{p(\cdot)}(X)} = \inf\{\lambda > 0 : S_p(f/\lambda) \le 1\},$$

where $1 \le p_-(\cdot) \le p(\cdot) < \infty$ and $S_p(f) = \int_X |f(x)|^{p(x)}\mathrm{d}\mu(x) < \infty$. For the other properties of $L^{p(\cdot)}$, we can see [5, 12].

(2) *If $(X,d,\mu) := (\mathbb{R}^n, |\cdot|, \mathrm{d}x)$, $p(x)$ and $q(x)$ are the constants, then the $M_{q(\cdot)}^{p(\cdot)}(X)_N$ as in* (1.9) *is just the Morrey space $M_q^p(\mathbb{R}^n)$ under non-doubling measure (see* [13]*).*

Finally, we make some conventions on notation. Throughout the whole paper, C represents a positive constant being independent of the main parameters. For any subset E of X, we use χ_E to denote its characteristic function. Given any $q \in (1,\infty)$, let $q' := \frac{q}{q-1}$ denote its conjugate index.

2. Parameter Marcinkiewicz integral $\mathcal{M}^\rho$

In this section, we mainly establish the boundedness of the parameter Marcinkiewicz integral $\mathcal{M}^\rho$ on the Lebesgue space with variable exponent $L^{p(\cdot)}(X)$ and on the Morrey space with variable exponent $M^{q(\cdot)}_{p(\cdot)}(X)_N$.

The main theorems of this section are stated as follows.

Theorem 2.1. *Let K satisfy the conditions (1.6) and (1.7), $1 < p_- \le p_+ < \infty$, N be a constant satisfying the conditions $N \ge 1$ and $p \in \mathcal{P}(N)$ defined as in (1.3). If there exists a constant $C > 0$ such that, for all $x \in X$ and $r > 0$, the following inequality*

$$[\mu(B(x,Nr))]^{p_-(B(x,r))-p(x)} \le C \tag{2.1}$$

holds, then $\mathcal{M}^\rho$ is bounded in $L^{p(\cdot)}(X)$.

Theorem 2.2. *Let K satisfy the conditions (1.6) and (1.7), $1 < q_- \le q(x) \le p(x) \le p_+ < \infty$. Suppose that $N := a_1(1+2a_0)$ and $p \in \mathcal{P}(N)$ defined as in (1.2), $q \in \mathcal{P}(1)$. Then $\mathcal{M}^\rho$ is bounded from the $M^{p(\cdot)}_{q(\cdot)}(X)_N$ into the $M^{p(\cdot)}_{q(\cdot)}(X)_{N\bar{a}}$.*

For any $f \in L^1_{\rm loc}(\mu)$, the modified maximal function $M_{s,N}$ (see [15]) is defined by

$$M_{s,N}f(x) := \sup_{r>0}\left(\frac{1}{\mu(B(x,Nr))}\int_{B(x,r)}|f(y)|^s \mathrm{d}\mu(y)\right)^{\frac{1}{s}},\ x \in X, \tag{2.2}$$

where N is a constant greater than of equal to 1. Moreover, when $s = 1$, we abbreviate $M_N := M_{1,N}$. Moreover, the sharp maximal function of $M^\sharp f$ (see [3]) is defined by

$$M^\sharp f(x) = \sup_{B\ni x}\frac{1}{\mu(\frac{3}{2}B)}\int_B |f(y) - m_{\widetilde{B}}(f)|\mathrm{d}\mu(y) + \sup_{(B,S)\in\Delta_x}\frac{|m_B(f) - m_S(f)|}{K_{B,S}}. \tag{2.3}$$

where $\Delta_x = \{(B,S) : x \in B \subset S \text{ and } B,\ S \text{ doubling balls}\}$ and $f \in L^1_{\rm loc}(\mu)$.

For $0 < r < \infty$, let $M^\sharp_r f(x) = [M^\sharp(|f|^r)(x)]^{\frac{1}{r}}$ for $x \in X$. A simple computation shows that if $0 < r < 1$, it has

$$M^\sharp_r f(x) \le C_r M^\sharp f(x), \quad x \in X. \tag{2.4}$$

Now we establish the following lemmas on the (X, d, μ).

Lemma 2.1. *Let $\tau \in (0,1)$, $g \in L^1_{\rm loc}(X)$ and μ-measurable function f satisfy the following condition*

$$\mu(\{x \in X : |f(x)| > t\}) < \infty, \quad \text{for all } t > 0,$$

then

$$\int_X |f(x)g(x)|\mathrm{d}\mu(x) \le \int_X M^\sharp_\tau(f)(x)M_N(g)(x)\mathrm{d}\mu(x). \tag{2.5}$$

Remark 2.1. *With a way similar to that used in the duality inequality (see [12]), it is not difficult to show that (2.5) holds. Hence, here we omit the proof.*

Lemma 2.2. *Let K satisfy the conditions (1.5) and (1.6), $N \ge 1$, $s \in (1,\infty)$ and $p_0 \in (1,\infty)$. If $\mathcal{M}^\rho$ is bounded on $L^2(\mu)$, then there exists a constant $C > 0$ such that, for all $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$,*

$$M^\sharp[\mathcal{M}^\rho(f)](x) \le CM_{s,N}(f)(x).$$

Proof of Lemma 2.2. By applying the idea of the Theorem 9.1 in [14], it only suffices to prove that

$$\frac{1}{\mu(\frac{3}{2}B)}\int_B |\mathcal{M}^\rho(f)(y) - h_B|\mathrm{d}\mu(y) \le CM_{s,N}(f)(x) \tag{2.6}$$

for all x and B with $x \in B$, and

$$|h_B - h_S| \le CK_{B,S}M_{s,N}(f)(x) \tag{2.7}$$

for all ball $B \subset S$ with $x \in B$, where B is arbitrary ball and S is doubling ball,

$$h_B := m_B[\mathcal{M}^\rho(f\chi_{X\setminus \frac{4}{3}B})] \qquad \text{and} \qquad h_S := m_S[\mathcal{M}^\rho(f\chi_{X\setminus \frac{4}{3}S})].$$

Now we give out the estimate for the (2.6). For a fixed ball B, $x \in B$ and $f \in L^\infty(\mu)$, decompose f as $f := f_1 + f_2 := f\chi_{\frac{4}{3}B} + f\chi_{X\setminus \frac{4}{3}B}$. Then, write

$$\begin{aligned}\frac{1}{\mu(\frac{3}{2}B)}\int_B |\mathcal{M}^\rho(f)(y) - h_B|\mathrm{d}\mu(y) &\le \frac{1}{\mu(\frac{3}{2}B)}\int_B |\mathcal{M}^\rho(f_1)(y)|\mathrm{d}\mu(y)\\ &+\frac{1}{\mu(\frac{3}{2}B)}\int_B |\mathcal{M}^\rho(f_2)(y) - h_B|\mathrm{d}\mu(y) =: \mathrm{D}_1 + \mathrm{D}_2.\end{aligned}$$

By applying Hölder inequality, (2.2) and the $(L^s(\mu), L^s(\mu))$-boundedness of $\mathcal{M}^\rho$ (see [7]), we can get

$$\mathrm{D}_1 \le \frac{1}{\mu(\frac{3}{2}B)}\bigg(\int_B |\mathcal{M}^\rho(f_1)(y)|^s \mathrm{d}\mu(y)\bigg)^{\frac{1}{s}}[\mu(B)]^{1-\frac{1}{s}} \le CM_{s,N}(f)(x).$$

Since

$$\mathrm{D}_2 \le \frac{1}{\mu(\frac{3}{2}B)}\frac{1}{\mu(B)}\int_B\int_B |\mathcal{M}^\rho(f_2)(y) - \mathcal{M}^\rho(f_2)(z)|\mathrm{d}\mu(y)\mathrm{d}\mu(z),$$

thus, for any $y,\ z \in B$, we only need to estimate $|\mathcal{M}^\rho(f_2)(y) - \mathcal{M}^\rho(f_2)(z)|$. By applying Minkowski inequality, we have

$$\begin{aligned}&|\mathcal{M}^\rho(f_2)(y) - \mathcal{M}^\rho(f_2)(z)|\\ &\le \bigg(\int_0^\infty \bigg|\int_{d(y,w)\le t} K(y,w)[d(y,w)]^\rho f_2(w)\mathrm{d}\mu(w)\\ &\qquad - \int_{d(z,w)\le t} K(z,w)[d(z,w)]^\rho f_2(w)\mathrm{d}\mu(w)\bigg|^2 \frac{\mathrm{d}t}{t^{1+2\rho}}\bigg)^{\frac{1}{2}}\\ &\le \bigg(\int_0^\infty \bigg|\int_{d(y,w)\le t<d(z,w)} K(y,w)[d(y,w)]^\rho f_2(w)\mathrm{d}\mu(w)\bigg|^2 \frac{\mathrm{d}t}{t^{1+2\rho}}\bigg)^{\frac{1}{2}}\\ &+\bigg(\int_0^\infty \bigg|\int_{d(z,w)\le t<d(y,w)} K(z,w)[d(z,w)]^\rho f_2(w)\mathrm{d}\mu(w)\bigg|^2 \frac{\mathrm{d}t}{t^{1+2\rho}}\bigg)^{\frac{1}{2}}\\ &+\bigg(\int_0^\infty \bigg|\int_{d(y,w)\le t} K(y,w)[d(y,w)]^\rho f_2(w)\mathrm{d}\mu(w)\\ &- \int_{d(z,w)\le t} K(z,w)[d(z,w)]^\rho f_2(w)\mathrm{d}\mu(w)\bigg|^2 \frac{\mathrm{d}t}{t^{1+2\rho}}\bigg)^{\frac{1}{2}} =: \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3.\end{aligned}$$

By (1.5), the Minkowski inequality, the Hölder inequality and (2.2), we have

$$\begin{aligned}
&\left(\int_0^\infty \left|\int_{d(y,w)\le t<d(z,w)} K(y,w)[d(y,w)]^\rho f_2(w)\mathrm{d}\mu(w)\right|^2 \frac{\mathrm{d}t}{t^{1+2\rho}}\right)^{\frac{1}{2}}\\
&\le C\int_X \frac{|f_2(w)|}{[d(y,w)]^{2-\rho}}\left(\frac{1}{[d(y,w)]^{2\rho}}-\frac{1}{[d(z,w)]^{2\rho}}\right)^{\frac{1}{2}}\mathrm{d}\mu(w)\\
&\le C\sum_{k=1}^\infty \int_{2^k\times\frac{4}{3}B\setminus(2^{k-1}\times\frac{4}{3}B)} \frac{[d(z,y)]^{\frac{1}{2}}}{[d(y,w)]^{\frac{3}{2}}}|f(w)|\mathrm{d}\mu(w)\\
&\le C\sum_{k=1}^\infty \frac{r^{\frac{1}{2}}}{[2^{k-1}\times\frac{4}{3}r]^{\frac{3}{2}}}\left(\int_{2^k\times\frac{4}{3}B}|f(w)|^s\mathrm{d}\mu(w)\right)^{\frac{1}{s}}[\mu(2^k\times\frac{4}{3}B)]^{1-\frac{1}{s}}\\
&\le CM_{s,N}(f)(x)\sum_{k=1}^\infty \frac{1}{2^{\frac{1}{2}(k-1)}}\le CM_{s,N}(f)(x).
\end{aligned}$$

Hence, we have $\mathrm{E}_1 \le CM_{s,N}(f)(x)$.

With an argument similar to that used in the estimate of E_1, it is not difficult to obtain that $\mathrm{E}_2 \le CM_{s,N}(f)(x)$.

For any $y,\ z\in B$, by applying (1.5), (1.6), the Minkowski inequality, the Hölder inequality and (2.2), we can deduce that

$$\begin{aligned}
\mathrm{E}_3 \le& \left(\int_0^\infty \left|\int_{\substack{d(y,w)\le t\\ d(z,w)\le t}} K(y,w)\Big([d(y,w)]^\rho-[d(z,w)]^\rho\Big)\mathrm{d}\mu(w)\right|^2\frac{\mathrm{d}t}{t^{1+2\rho}}\right)^{\frac{1}{2}}\\
&+\left(\int_0^\infty \left|\int_{\substack{d(y,w)\le t\\ d(z,w)\le t}} \frac{K(y,w)-K(z,w)}{[d(y,w)]^{-\rho}}f_2(w)\mathrm{d}\mu(w)\right|^2\frac{\mathrm{d}t}{t^{1+2\rho}}\right)^{\frac{1}{2}}\\
\le& C\int_{X\setminus\frac{4}{3}B}\frac{|f(w)|}{d(y,w)}d(y,z)[d(y,w)]^{\rho-1}\left(\int_{\substack{d(y,w)\le t\\ d(z,w)\le t}}\frac{\mathrm{d}t}{t^{1+2\rho}}\right)^{\frac{1}{2}}\mathrm{d}\mu(w)\\
&+\int_{X\setminus\frac{4}{3}B}\frac{|K(y,w)-K(z,w)|}{[d(y,w)]^{-\rho}}|f(w)|\left(\int_{\substack{d(y,w)\le t\\ d(z,w)\le t}}\frac{\mathrm{d}t}{t^{1+2\rho}}\right)^{\frac{1}{2}}\mathrm{d}\mu(w)\\
\le& C\sum_{k=1}^\infty\frac{r}{[2^{k-1}\times\frac{4}{3}r]^2}\left(\int_{2^k\times\frac{4}{3}B}|f(w)|^s\mathrm{d}\mu(w)\right)^{\frac{1}{s}}[\mu(2^k\times\frac{4}{3}B)]^{1-\frac{1}{s}}\\
&+C\sum_{k=1}^\infty\frac{r^\epsilon}{[2^{k-1}\times\frac{4}{3}r]^{1+\epsilon}}\left(\int_{2^k\times\frac{4}{3}B}|f(w)|^s\mathrm{d}\mu(w)\right)^{\frac{1}{s}}[\mu(2^k\times\frac{4}{3}B)]^{1-\frac{1}{s}}\\
\le& CM_{s,N}(f)(x)\sum_{k=1}^\infty\frac{1}{2^{k-1}}\le CM_{s,N}(f)(x).
\end{aligned}$$

Which, together with E_1 and E_2, we get (2.6).

We now estimate (2.7). For any balls $B\subset S$ with $x\in B$, where B is an arbitrary ball and S is a doubling ball, denote $H_{B,R}+1$ by H. Write

$$\begin{aligned}
|h_B-h_S|\le&\left|m_B[\mathcal{M}^\rho(f\chi_{X\setminus 2^HS})]-m_S[\mathcal{M}^\rho(f\chi_{X\setminus 2^HS})]\right|\\
&+\left|m_B[\mathcal{M}^\rho(f\chi_{2^HS\setminus\frac{4}{3}B})]\right|+\left|m_S[\mathcal{M}^\rho(f\chi_{2^HS\setminus\frac{4}{3}S})]\right|=:\mathrm{F}_1+\mathrm{F}_2+\mathrm{F}_3.
\end{aligned}$$

With an argument similar to that used in estimate of E_3, it is not difficult to get $F_1 \leq CM_{s,N}(f)(x)$.

We now turn to F_3. For any $y \in B$, by applying (1.5), the Minkowski inequality, the Hölder inequality and (2.2), we have

$$\begin{aligned}
&\mathcal{M}^\rho(f\chi_{2^H S\setminus\frac{4}{3}S})(y)\\
&\leq \int_{2^H S\setminus\frac{4}{3}S} \frac{|K(y,z)|}{[d(y,z)]^{-\rho}}|f(z)|\left(\int_{d(y,z)}^\infty \frac{\mathrm{d}t}{t^{1+2\rho}}\right)^{\frac{1}{2}}\mathrm{d}\mu(z)\\
&\leq C\sum_{k=1}^{H-1}\int_{2^k\times\frac{4}{3}S\setminus(2^{k-1}\times\frac{4}{3}S)} \frac{|f(z)|}{d(y,z)}\mathrm{d}\mu(z)\\
&\leq C\sum_{k=1}^{H-1}\frac{1}{(2^{k-1}\times\frac{4}{3}r_S)}\left(\int_{2^k\times\frac{4}{3}S}|f(z)|^s\mathrm{d}\mu(z)\right)^{\frac{1}{s}}[\mu(2^k\times\frac{4}{3}S)]^{1-\frac{1}{s}}\\
&\leq CK_{B,S}M_{s,N}(f)(x).
\end{aligned}$$

Similarly, we also have $F_2 \leq CK_{B,S}M_{s,N}(f)(x)$.

Combing the estimates for F_1, F_2 and F_3, imply (2.7). Hence, the proof of Lemma 2.2 is finished. □

Also, we need establish the following lemmas on the modified maximal operator $M_{s,N}$.

Lemma 2.3. *Let $\mu(X) < \infty$, $1 \leq s < p_- \leq p_+ < \infty$, $N \geq 1$ be a constant and the inequality* (2.1) *hold. Then $M_{s,N}$ as in* (2.2) *is bounded on $L^{p(\cdot)}(X)$.*

Lemma 2.4. *Let $\mu(X) < \infty$, $1 < s < q_- \leq q(x) \leq p(x) \leq p_+ < \infty$. Suppose that $N := a_1(1+2a_0)$ and $p \in \mathcal{P}(N)$ defined as in* (1.3), *$q \in \mathcal{P}(1)$. Then $M_{s,N}$ as in* (2.2) *is bounded from the $M^{p(\cdot)}_{q(\cdot)}(X)_N$ into the $M^{p(\cdot)}_{q(\cdot)}(X)_{N\bar{a}}$.*

The following lemma is from [12].

Lemma 2.5. [12] *Let E be a subset of X such that $\mu(E) < \infty$ and $1 \leq p_-(E) \leq p(\cdot) \leq q(\cdot) \leq q_+(E) < \infty$. Then the inequality*

$$\|f\|_{L^{p(\cdot)}(E)} \leq (1+\mu(E))\|f\|_{L^{q(\cdot)}(E)}$$

holds for all $f \in L^{q(\cdot)}(E)$.

Now we give out the proof of Lemmas 2.3 and 2.4, respectively.

Proof of Lemma 2.3. With the same spirit to the proof of the Theorem 3.1 in [15], we first prove that for all $f \in L^{p(\cdot)}(X)$ such that

$$(1+\mu(X))\|f\|_{L^{p(\cdot)}(X)} \leq 1 \tag{2.8}$$

and all $x \in X$,

$$[M_{s,N}f(x)]^{p(x)} \leq C\Big(M_N(|f|^{p(\cdot)})(x)+1\Big). \tag{2.9}$$

Let $NB := B(x, Nr)$. Then by applying Lemma 2.5 and (2.8), it is easy to see that

$$\|f\|_{L^{p_-(B)}(B)} \leq 1. \tag{2.10}$$

By using Hölder inequality, (2.1) and (2.10), we can deduce that

$$
\begin{aligned}
&[M_{s,N}f(x)]^{p(x)}\\
&\le \frac{1}{[\mu(NB)]^{p(x)/s}}\left[\left(\int_B |f(y)|^{s\times\frac{p_-(B)}{s}}\mathrm{d}\mu(y)\right)^{\frac{s}{p_-(B)}}[\mu(B)]^{1-\frac{s}{p_-(B)}}\right]^{\frac{p(x)}{s}}\\
&\le C\frac{[\mu(NB)]^{\frac{p(x)}{s}-\frac{p(x)}{p_-(B)}+1}}{[\mu(NB)]^{p(x)/s}}\left(\frac{1}{\mu(NB)}\int_B |f(y)|^{p_-(B)}\mathrm{d}\mu(y)\right)\\
&\le C\Big(M_N(|f|^{p(\cdot)})(x)+1\Big).
\end{aligned}
$$

Hence, we prove that (2.8) holds.

Further, let us set that $\|f\|_{L^{p(\cdot)}(X)}\le 1$ and prove that $\|M_{s,N}f\|_{L^{p(\cdot)}(X)}\le C\|f\|_{L^{p(\cdot)}(X)}$. In addition, we also assume that $(1+\mu(X))\|f\|_{L^{q(\cdot)}(X)}\le 1$. By applying the $L^{p_-}(X)$-boundedness of M_N in [4], we can get

$$
\begin{aligned}
\int_X [M_{s,N}(f)]^{p(x)}(x)\mathrm{d}\mu(x) &\le C\left[\left(\int_X [M_N(|f|^{\frac{p(\cdot)}{p_-}})]^{p_-}(x)\mathrm{d}\mu(x)\right)^{\frac{1}{p_-}}+[\mu(X)]^{\frac{1}{p_-}}\right]^{p_-}\\
&\le C\left[\left(\int_X |f(x)|^{p(x)}(x)\mathrm{d}\mu(x)\right)^{\frac{1}{p_-}}+[\mu(X)]^{\frac{1}{p_-}}\right]^{p_-}\le C\int_X |f(x)|^{p(x)}(x)\mathrm{d}\mu(x).
\end{aligned}
$$

What's more, for any f satisfying only the condition $\|f\|_{L^{p(\cdot)}(X)}\le 1$, the result holds. In special, if we take $g:=\frac{f}{(1+\mu(X))^2}$, then by Lemma 2.8, we get $(1+\mu(X))\|g\|_{L^{p(\cdot)}(X)}\le 1$. □

Proof of Lemma 2.4. With a slightly modified argument similar to that used in the proof of Theorem 3.4 and Lemma 2.3 in this section, it is not difficult to obtain Lemma 2.4. Here we do not give out the proof. □

Now let us turn to the Theorems 2.1 and 2.2.

Proof of Theorem 2.1. By applying the Lemma 2.3, Hölder inequality, (2.4) and Lemmas 2.5 and 2.6, we have

$$
\begin{aligned}
\int_X |\mathcal{M}^\rho(f)(x)g(x)|\mathrm{d}\mu(x) &\le C\int_X M_r^\sharp(\mathcal{M}^\rho(f))(x)M(g)(x)\mathrm{d}\mu(x)\\
&\le C\int_X M_{s,N}(f)(x)M_{s,N}(g)(x)\mathrm{d}\mu(x)\\
&\le C\|M_{s,N}(f)\|_{L^{p(\cdot)}(X)}\|M_{s,N}(g)\|_{L^{q(\cdot)}(X)}\le C\|f\|_{L^{p(\cdot)}(X)}\|g\|_{L^{q(\cdot)}(X)}.
\end{aligned}
$$

Further, we can get

$$
\|\mathcal{M}^\rho(f)\|_{L^{p(\cdot)}(X)}=\sup_{\|g\|_{L^{q(\cdot)}(X)}\le 1}\left|\int_X \mathcal{M}^\rho(f)(x)g(x)\mathrm{d}\mu(x)\right|\le C\|f\|_{L^{p(\cdot)}(X)}.
$$

Thus, the proof of the Theorem 2.1 is finished. □

Proof of Theorem 2.2. By applying (1.9), (2.2), (2.3) and Lemmas 2.5 and 2.7, we have

$$
\begin{aligned}
\|\mathcal{M}^\rho(f)\|_{M^{p(\cdot)}_{q(\cdot)}(X)_{N\bar{a}}} &\le \|M^\sharp[\mathcal{M}^\rho(f)]\|_{M^{p(\cdot)}_{q(\cdot)}(X)_{N\bar{a}}}\le C\|M^\sharp[\mathcal{M}^\rho(f)]\|_{M^{p(\cdot)}_{q(\cdot)}(X)_{N\bar{a}}}\\
&\le C\|M_{s,N}(f)\|_{M^{p(\cdot)}_{q(\cdot)}(X)_{N\bar{a}}}\le C\|f\|_{M^{p(\cdot)}_{q(\cdot)}(X)_N}.
\end{aligned}
$$

Thus, we complete the proof of the Theorem 2.2. □

3. Commutator $\mathcal{M}_b^\rho$ on Morrey space with variable exponent

In this section, we will prove that the commutator $\mathcal{M}_b^\rho$ generated by the $b \in \mathrm{RBMO}(\mu)$ and $\mathcal{M}^\rho$ is bounded on $L^{p(\cdot)}(X)$ via the sharp maximal function. Besides, the boundedness of the $\mathcal{M}_b^\rho$ on $M_{q(\cdot)}^{p(\cdot)}(X)_N$ is also obtained.

Theorem 3.1. *Let $b \in \mathrm{RBMO}(\mu)$, K satisfy the conditions* (1.5) *and* (1.6)*, $1 < p_- \le p_+ < \infty$ and $N \ge 1$ be a constant. If there exists a constant $C > 0$ such that, for all $x \in X$ and $r > 0$, the following inequality*

$$[\mu(B(x, Nr))]^{p_-(B(x,r))-p(x)} \le C$$

holds, then $\mathcal{M}_b^\rho$ is bounded in $L^{p(\cdot)}(X)$.

Theorem 3.2. *Let $b \in \mathrm{RBMO}(\mu)$, K satisfy the conditions* (1.5) *and* (1.6)*, $1 < q_- \le q(x) \le p(x) \le p_+ < \infty$. Suppose that $N := a_1(1 + 2a_0)$ and $p \in \mathcal{P}(N)$ defined as in* (1.2)*, $q \in \mathcal{P}(1)$. Then the $\mathcal{M}_b^\rho$ is bounded from the $M_{q(\cdot)}^{p(\cdot)}(X)_N$ into the $M_{q(\cdot)}^{p(\cdot)}(X)_{N\bar{a}}$ with $\bar{a} = a_1(a_1(a_0 + 1) + 1)$.*

By an argument similar to that used in the proof of the Lemma 3.4 in [3], we can get the following lemma .

Lemma 3.1. *Let $b \in L^\infty(\mu)$, K satisfy the conditions* (1.5) *and* (1.6)*, N be a constant satisfying the condition $N \ge 1$, $s \in (1, \infty)$ and $p_0 \in (1, \infty)$. If $\mathcal{M}^\rho$ is bounded on $L^2(\mu)$, then there exists a positive constant C such that for all $f \in L^\infty(\mu) \cap L^{p_0}(\mu)$ and $x \in X$,*

$$M^\sharp[\mathcal{M}_b^\rho(f)](x) \le C\|b\|_{\mathrm{RBMO}(\mu)}\left\{M_{s,N}[\mathcal{M}^\rho(f)] + \|f\|_{L^\infty(\mu)}\right\}.$$

Now we give out the proof of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. By using Theorem 2.1, Lemmas 2.2 and 2.3, we can deduce that

$$\begin{aligned}\|\mathcal{M}_b^\rho(f)\|_{L^{p(\cdot)}(X)} &\le \|M^\sharp[\mathcal{M}_b^\rho(f)]\|_{L^{p(\cdot)}(X)} \le C\|b\|_{\mathrm{RBMO}(\mu)}\|M_{s,N}[\mathcal{M}^\rho(f)]\|_{L^{p(\cdot)}(X)}\\ &\le C\|b\|_{\mathrm{RBMO}(\mu)}\|\mathcal{M}^\rho(f)\|_{L^{p(\cdot)}(X)} \le C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{L^{p(\cdot)}(X)}.\end{aligned}$$

So, the proof of the Theorem 3.1 is finished. □

Proof of Theorem 3.2. By using Theorem 2.2, Lemmas 2.4 and 3.1, we have

$$\begin{aligned}&\|\mathcal{M}_b^\rho(f)\|_{M_{q(\cdot)}^{p(\cdot)}(X)_{N\bar{a}}} \le \|M^\sharp[\mathcal{M}_b^\rho(f)]\|_{M_{q(\cdot)}^{p(\cdot)}(X)_{N\bar{a}}}\\ &\quad\le C\|b\|_{\mathrm{RBMO}(\mu)}\|M_{s,N}[\mathcal{M}^\rho(f)]\|_{M_{q(\cdot)}^{p(\cdot)}(X)_{N\bar{a}}} \le C\|b\|_{\mathrm{RBMO}(\mu)}\|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)_N}.\end{aligned}$$

Hence, the proof of Theorem 3.2 is completed. □

4. Conclusions

With the result obtained in [15], the author proves that the parameter Marcinkiewicz integral $\mathcal{M}^\rho$ and the commutator $\mathcal{M}_b^\rho$ which is generated by the $\mathcal{M}^\rho$ and the regular bounded mean oscillation space (=RBMO) is bouned on the Morrey space with variable exponent and the Lebesgue space with variable exponent, respectively.

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