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CONTROLLABILITY FOR THE VIBRATING STRING EQUATION WITH MIXED BOUNDARY CONDITIONS

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In this paper we study the controllability in $T = \pi$ of the vibrating string equation with mixed boundary conditions by solving a moment problem.

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1. Introduction

We consider the vibrating string equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, \pi), \quad t > 0, \tag{1}$$

with the following initial and mixed boundary conditions:

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad \frac{\partial u}{\partial x}(0, t) = f(t), \quad u(\pi, t) = 0, \tag{2}$$

where $f \in L^2(0, \pi)$.

In our problem, $u(x, t)$ denotes the vertical displacement and $\frac{\partial u}{\partial t}(x, t)$ the vertical velocity of the point at position x at time t. This problem is relevant for the modeling of the longitudinal vibration in a spring with the end fastened at $x = \pi$ and with a traction force exerted at the end $x = 0$, which allows to control vibrations.

There are some significant papers which studied the controllability of the wave equation with mixed boundary conditions, using the Hilbert Uniqueness Method: in [2] Cui and Gao proved the exact controllability for the one-dimensional case in a non-cylindrical domain, in [1] Cavalcanti showed the exact controllability for the time-dependent coefficients case, in [6] Heibig and Moussaoui proved the exact controllability in a plane domain with cracks, by using a boundary control.

In this paper, we obtain some new properties of the reachable sets, using the Fourier series method presented in [7], and we study the exact controllability of the mixed problem for the wave equation at $T = \pi$, using the moment problem approach.

Denote by u^f the solution of equation (1) with conditions (2).

Let us define $H_+ = \{\varphi \in H^1(0, \pi); \varphi(\pi) = 0\}$ and

$$u_k(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi u^f(x, t) \cos\left(k + \frac{1}{2}\right) x \, \mathrm{d}x.$$

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After some computations, detailed in [8], we obtain that the function $u_k(t)$ solves the following non-homogeneous second order differential equation:

$$u_k''(t) + \left(k + \frac{1}{2}\right)^2 u_k(t) = -\sqrt{\frac{2}{\pi}} f(t),$$

with the initial conditions: $u_k(0) = u_k'(0) = 0$. Therefore,

$$u_k(t) = -\frac{\sqrt{\frac{2}{\pi}}}{k + \frac{1}{2}} \int_0^t f(s) \sin\left(k + \frac{1}{2}\right)(t - s)\, \mathrm{d}s,$$

which implies

$$u^f(x,t) = -\sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{1}{k + \frac{1}{2}} \left[\int_0^t f(s) \sin\left(k + \frac{1}{2}\right)(t - s)\, \mathrm{d}s\right] \cos\left(k + \frac{1}{2}\right) x. \tag{3}$$

In [8] we have also shown that the map $t \mapsto u^f(\cdot, t)$ belongs to

$$C\left([0;\pi], H_+\right) \cap C^1([0;\pi], L^2(0;\pi)).$$

For the case:

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0, \tag{4}$$

with the following initial and mixed boundary conditions:

$$\varphi(x,0) = \xi(x), \quad \frac{\partial \varphi}{\partial t}(x,0) = \eta(x), \quad \frac{\partial \varphi}{\partial x}(0,t) = 0, \quad \varphi(\pi,t) = 0,$$

where $\xi, \eta \in L^2(0,\pi)$ with:

$$\xi(x) = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \xi_k \cos\left(k + \frac{1}{2}\right) x \quad \text{and} \quad \eta(x) = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \eta_k \cos\left(k + \frac{1}{2}\right) x,$$

we find the solution (see [8])

$$\varphi(x,t) = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \left[\xi_k \cos\left(k + \frac{1}{2}\right) t + \frac{\eta_k}{k + \frac{1}{2}} \sin\left(k + \frac{1}{2}\right) t\right] \cos\left(k + \frac{1}{2}\right) x. \tag{5}$$

The map $t \mapsto \varphi(\cdot, t)$ belongs to $C\left([0;T], \operatorname{dom}(-A)^{\frac{1}{2}}\right) \cap C^1([0;T], L^2(0;\pi))$, for all $T > 0$, where A is the operator $Ah = \dfrac{\mathrm{d}^2 h}{\mathrm{d}x^2}$ with

$$\operatorname{dom} A = \left\{h \in H^2(0,\pi), \frac{\mathrm{d}h}{\mathrm{d}x}(0) = 0, h(\pi) = 0\right\}.$$

It follows that

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(x,t) &= \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \left[-\left(k + \frac{1}{2}\right) \xi_k \sin\left(k + \frac{1}{2}\right) t + \eta_k \cos\left(k + \frac{1}{2}\right) t\right] \cdot \\ &\quad \cdot \cos\left(k + \frac{1}{2}\right) x. \end{aligned} \tag{6}$$

We define:

Definition 1.1. *The reachable set of the displacement at time T is*

$$\mathcal{R}_D(T) = \left\{u^f(\cdot, T), f \in L^2(0,T)\right\} \subseteq H_+$$

while

$$\mathcal{R}(T) = \left\{\left(u^f(\cdot, T), \frac{\partial u^f}{\partial t}(\cdot, T)\right), f \in L^2(0,T)\right\} \subseteq H_+ \times L^2(0,\pi)$$

is the reachable set of the pair $\left(u, \dfrac{\partial u}{\partial t}\right)$.

We remark that $\mathcal{R}(T)$ is the image of the map

$$\Lambda_T : L^2(0,\pi) \to H_+ \times L^2(0,\pi),$$

defined by:

$$\Lambda_T f = \begin{pmatrix} \Lambda_T^1 f \\ \Lambda_T^2 f \end{pmatrix} = \begin{pmatrix} u^f(x,T) \\ \dfrac{\partial u^f}{\partial t}(x,T) \end{pmatrix}.$$

Lemma 1.1. *The adjoint operator $\Lambda_T^* : H_+ \times L^2(0,\pi) \to L^2(0,\pi)$ is given by:*

$$\Lambda_T^* \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \left(\Lambda_T^{1*}, \Lambda_T^{2*}\right) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Lambda_T^{1*}\xi + \Lambda_T^{2*}\eta = f,$$

with:

$$\Lambda_T^{1*}\xi = \frac{\partial \varphi_1}{\partial t}(0, T-t)$$

and

$$\Lambda_T^{2*}\eta = \frac{\partial \varphi_2}{\partial t}(0, T-t),$$

where $\varphi_1(x,t)$ solves (4) *with the conditions: $\varphi_1(x,0) = \xi(x)$, $\dfrac{\partial \varphi_1}{\partial t}(x,0) = 0$, and $\varphi_2(x,t)$ solves* (4) *with the conditions: $\varphi_2(x,0) = 0$, $\dfrac{\partial \varphi_2}{\partial t}(x,0) = \eta(x)$.*

Proof. The inner product of H_+ is that of $H^1(0,\pi)$ and the next inner product gives us a norm equivalent with the standard norm in $H^1(0,\pi)$ (see [4]):

$$\langle \varphi, \psi \rangle = \varphi(\pi)\psi(\pi) + \int_0^\pi \varphi'(x)\psi'(x)\,\mathrm{d}x, \quad \varphi, \psi \in H^1(0,\pi).$$

We compute the following inner product:

$$\langle \Lambda_T^1 f, \xi(x)\rangle_{H_+} = \langle u^f(x,T), \xi(x)\rangle_{H_+} =$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\pi \xi'(x)\sum_{k=0}^{\infty}\sin\left(k+\frac{1}{2}\right)x\left[\int_0^T \sin\left(k+\frac{1}{2}\right)(T-s)f(s)\,\mathrm{d}s\right]\mathrm{d}x$$

$$= \int_0^T f(s)\left\{\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\left[\int_0^\pi \xi'(x)\sin\left(k+\frac{1}{2}\right)x\,\mathrm{d}x\right]\sin\left(k+\frac{1}{2}\right)(T-s)\right\}\mathrm{d}s$$

$$= \int_0^T f(s)\left\{-\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\left[\int_0^\pi \left(k+\frac{1}{2}\right)\xi(x)\cos\left(k+\frac{1}{2}\right)x\,\mathrm{d}x\right]\cdot\right.$$

$$\left.\cdot\sin\left(k+\frac{1}{2}\right)(T-s)\right\}\mathrm{d}s$$

$$= \int_0^T f(s)\left\{-\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)\xi_k\sin\left(k+\frac{1}{2}\right)(T-s)\right\}\mathrm{d}s.$$

So,

$$\left[\left(\Lambda_T^1\right)^*\xi\right](t) = -\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)\xi_k\sin\left(k+\frac{1}{2}\right)(T-t).$$

If we compare with (6), we see that

$$\left[\left(\Lambda_T^1\right)^*\xi\right](t) = \frac{\partial\varphi_1}{\partial t}(0,T-t),$$

where $\varphi_1(x,t)$ solves (4) with the conditions $\varphi_1(x,0)=\xi(x)$, $\dfrac{\partial\varphi_1}{\partial t}(x,0)=0$.

An analogue computation shows that

$$\left[\left(\Lambda_T^2\right)^*\eta\right](t) = \sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\eta_k\cos\left(k+\frac{1}{2}\right)(T-t).$$

We compare with (6) and we can write

$$\left[\left(\Lambda_T^2\right)^*\eta\right](t) = \frac{\partial\varphi_2}{\partial t}(0,T-t),$$

where $\varphi_2(x,t)$ solves (4) with the conditions $\varphi_2(x,0)=0$, $\dfrac{\partial\varphi_2}{\partial t}(x,0)=\eta(x)$. □

2. **Properties of the solution map**

The characterization of the sets $\mathcal{R}_D(T)$ and $\mathcal{R}(T)$ is very important for the study of the properties of controllability. Now, we present some general properties of these two sets, for $T>0$.

Theorem 2.1.

1) $\mathcal{R}^\perp(T) = \left\{(\xi,\eta)\in H_+\times L^2(0,\pi) : \dfrac{\partial\varphi}{\partial t}(0,t)=0, \forall t\in(0,T),\right.$ *where* φ *solves* (4) *with respect to* $(\xi,\eta)\}$.

2) $\mathcal{R}_D^{\perp}(T) = \Big\{\xi \in H_+ : \dfrac{\partial\varphi}{\partial t}(0,t) = 0, t \in (0,T),$ *and* φ *solves* (4) *with* $\eta = 0\}$.

3) If the solution of (4) *satisfies the following implication:*

$$\frac{\partial\varphi}{\partial t}(0,t) = 0, \forall t \in (0,T) \Rightarrow \varphi = 0,$$

then $\mathcal{R}(T)$ *is dense in* $H_+ \times L^2(0,\pi)$.

4) If the solution of (4) *satisfies the following implication:*

$$\frac{\partial\varphi}{\partial t}(0,t) = 0, \forall t \in (0,T) \Rightarrow \varphi = 0,$$

then $\mathcal{R}_D(T)$ *is dense in* H_+.

For $T \leq \pi$*:*

5) $\mathcal{R}(T)$ *is closed if*

$$\int_0^T \left|\frac{\partial\varphi}{\partial t}(0,t)\right|^2 \mathrm{d}t \geq m_0\left(|\xi|^2_{H_+} + |\eta|^2_{L^2(0,\pi)}\right), \tag{7}$$

for all $(\xi,\eta) \in \operatorname{Im}\Lambda_T$.

6) $\mathcal{R}_D(T)$ *is closed if*

$$\int_0^T \left|\frac{\partial\varphi}{\partial t}(0,t)\right|^2 \mathrm{d}t \geq m_0|\xi|^2_{H_+}, \tag{8}$$

for all $\xi \in \operatorname{Im}\Lambda_T^1$.

Proof. 1) and 2): We characterize $\mathcal{R}(T)^{\perp}$, which is $\operatorname{Ker}\Lambda_T^*$, so, $(\xi,\eta) \in H_+ \times L^2(0,\pi)$ belongs to $\mathcal{R}(T)^{\perp}$ if and only if $0 = \langle\Lambda_T f, (\xi,\eta)\rangle_{H_+\times L^2(0,\pi)} = \langle f, \Lambda_T^*(\xi,\eta)\rangle_{L^2(0,\pi)} = \langle f, \Lambda_T^{1*}\xi + \Lambda_T^{2*}\eta\rangle_{L^2(0,\pi)}$, for every $f \in L^2(0,\pi)$, i.e., $\Lambda_T^{1*}\xi + \Lambda_T^{2*}\eta = 0$. The previous lemma implies that $\Lambda_T^{1*}\xi(t) + \Lambda_T^{2*}\eta(t) = \dfrac{\partial\varphi}{\partial t}(0,T-t)$, where $\varphi(x,t)$ solves (3) with the conditions $\varphi(x,0) = \xi(x)$ and $\dfrac{\partial\varphi}{\partial t}(x,0) = \eta(x)$. So, 1) and 2) are proved.

3) and 4): This property implies that $R^{\perp}(T) = \{0\}$ and so $\mathcal{R}(T)$ is dense in $H_+ \times L^2(0,\pi)$. Then $\mathcal{R}_D(T)$ is dense in H_+.

5) and 6): Consider $\Lambda_T^*(\xi_n,\eta_n) \to (\psi_1,\psi_2)$, for $n \to \infty$. Then $\{\Lambda_T^*(\xi_n,\eta_n)\}_n$ is a Cauchy sequence and, using the inequalities (6) and (7), it follows that: $\|(\xi_{n+p},\eta_{n+p}) - (\xi_n,\eta_n)\|_{H_+\times L^2(0,\pi)} = \|(\xi_{n+p}-\xi_n, \eta_{n+p}-\eta_n)\|_{H_+\times L^2(0,\pi)} \leq \dfrac{1}{m_0}\|\Lambda_T^*(\xi_{n+p},\eta_{n+p}) - \Lambda_T^*(\xi_n,\eta_n)\|_{L^2(0,\pi)} < \varepsilon$, for all $n \geq n_\varepsilon$ and $p \geq 1$. Therefore, $\{(\xi_n,\eta_n)\}_n$ is a Cauchy sequence and there exists $\lim\limits_{n\to\infty}(\xi_n,\eta_n) = (\xi,\eta)$ which implies that $\lim\limits_{n\to\infty}\Lambda_T^*(\xi_n,\eta_n) = \Lambda_T^*(\xi,\eta)$, then $(\psi_1,\psi_2) = \Lambda_T^*(\xi,\eta)$.

Now, we prove that the transformation $f \mapsto \dfrac{\partial u^f}{\partial t}$ is continuous. From (3) we obtain

$$\frac{\partial u^f}{\partial t}(x,t) = -\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\left[\int_0^t f(s)\cos\left(k+\frac{1}{2}\right)(t-s)\,\mathrm{d}s\right]\cos\left(k+\frac{1}{2}\right)x$$

and

$$\left\| \frac{\partial u^f}{\partial t}(\cdot, T) \right\|^2_{L^2(0,\pi)} = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \left(\int_0^T f(s) \cos\left(k + \frac{1}{2}\right)(T - s)\, \mathrm{d}s \right)^2,$$

because the system of the normalized eigenfunctions $\left\{ \sqrt{\frac{2}{\pi}} \cos\left(k + \frac{1}{2}\right) x \right\}_{k \geq 0}$ is complete (see [5]).

If note $g_T(s) := f(T - s)$, we obtain:

$$\int_0^T \cos\left(k + \frac{1}{2}\right)(T - s) \cdot f(s)\, ds = \int_0^T \cos\left(k + \frac{1}{2}\right) s \cdot g_T(s)\, ds =$$

$$= \int_0^T \left[g_T(s) \cos ks \cos\frac{s}{2} - g_T(s) \sin ks \sin\frac{s}{2} \right] ds.$$

Now, consider the following functions: $g_{T1}(s) = \begin{cases} 0 & , T < s < \pi \\ g_T(s) \cos\frac{s}{2} & , 0 < s < T \end{cases}$ and $g_{T2}(s) = \begin{cases} 0 & , T < s < \pi \\ g_T(s) \sin\frac{s}{2} & , 0 < s < T \end{cases}$.

Therefore,

$$\sum_{k=0}^{\infty} \left| \int_0^T \cos\left(k + \frac{1}{2}\right)(T - s) \cdot f(s)\, ds \right|^2 \leq$$

$$\leq 2 \sum_{k=0}^{\infty} \left(\left| \int_0^T g_T(s) \cos ks \cos\frac{s}{2}\, ds \right|^2 + \left| \int_0^T g_T(s) \sin ks \sin\frac{s}{2}\, ds \right|^2 \right) =$$

$$= \pi \sum_{k=0}^{\infty} \left(\left| \int_0^\pi \sqrt{\frac{2}{\pi}} g_{T1}(s) \cos ks\, ds \right|^2 + \left| \int_0^\pi \sqrt{\frac{2}{\pi}} g_{T2}(s) \sin ks\, ds \right|^2 \right).$$

Using the Parseval identity we obtain:

$$\sum_{k=0}^{\infty} \left| \int_0^T \cos\left(k + \frac{1}{2}\right)(T - s) f(s)\, ds \right|^2 \leq \pi \left(\|g_{T1}\|^2_{L^2(0,\pi)} + \|g_{T2}\|^2_{L^2(0,\pi)} \right)$$

$$= \pi \int_0^T g_T^2(s) ds = \pi \int_0^T f^2(T - s) ds \leq \pi \|f\|^2_{L^2(0,\pi)} < \infty.$$

We obtain that the transformation $f \mapsto \frac{\partial u^f}{\partial t}$ is continuous. So the operator Λ_T is bounded and, because we proved above that the range of Λ_T^* is closed, it follows that the range of Λ_T, which is $\mathcal{R}(T)$, is also closed (see Banach Closed Range Theorem from [3], pp. 488-489). □

3. Controllability for $T = \pi$

Definition 3.1. ([10], p. 3) *The wave equation (1) is* controllable *at time T if, for every $\psi \in H_+$, there exists a control $f \in L^2(0, \pi)$ such that $u^f(\cdot, T) = \psi$.*

We conclude with the following theorem, which gives us the controllability at time $T = \pi$:

Theorem 3.1. *The reachable set* $\mathcal{R}_D(\pi) = H_+$.

Proof. We use the explicit formula (3) to deduce the controllability of the displacement at time $T = \pi$. Therefore, we have that

$$\begin{aligned} u^f(x,\pi) &= -\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{1}{k+\frac{1}{2}}\left[\int_0^{\pi} f(s)\sin\left(k+\frac{1}{2}\right)(\pi - s)\,\mathrm{d}s\right]\cos\left(k+\frac{1}{2}\right)x \\ &= -\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\left[\frac{1}{k+\frac{1}{2}}\int_0^{\pi} g(s)\sin\left(k+\frac{1}{2}\right)s\,\mathrm{d}s\right]\cos\left(k+\frac{1}{2}\right)x, \qquad (9)\end{aligned}$$

where $g(t) := f(\pi - t)$.

The controllability can be obtained by solving a moment problem. So, the problem of controllability is reduced to the possibility to find the identification of the Fourier coefficients

$$\left\{\frac{1}{k+\frac{1}{2}}\int_0^{\pi} g(s)\sin\left(k+\frac{1}{2}\right)s\,\mathrm{d}s\right\}_{k\geq 0},$$

with $g \in L^2(0,\pi)$.

The sequence $\left\{\sqrt{\frac{2}{\pi}}\sin\left(k+\frac{1}{2}\right)x\right\}_{k\geq 0}$ is the sequence of the normalized eigenfunctions of the selfadjoint operator $\widetilde{A}$ in $L^2(0,\pi)$, where:

$$\operatorname{dom}\widetilde{A} = \left\{h \in H^2(0,\pi), h(0) = 0, h'(\pi) = 0\right\} \quad \text{and} \quad \widetilde{A}h(x) = h''(x).$$

The system of eigenfunctions is a complete system in $L^2(0,\pi)$ (see [5]), so it follows that

$$\left\{\left\{\int_0^{\pi} g(s)\sin\left(k+\frac{1}{2}\right)s\,\mathrm{d}s\right\}, g \in L^2(0,\pi)\right\}_{k\geq 0} = l^2.$$

We know that the functions from $H^1(0,\pi)$ have the following property for the Fourier coefficients:

$$c_k = \frac{1}{k+\frac{1}{2}}\,\xi_k,$$

with $\{\xi_k\}_{k\geq 0} \in l^2$. Then, for every $\psi \in H_+(0,\pi)$, there exists $f \in L^2(0,\pi)$, such that $u^f(x,\pi) = \psi(x)$. □

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REFERENCES

[1] *M.M. Cavalcanti*, Exact controllability of the wave equation with mixed boundary condition and time-dependent coefficients, Arch. Math., Brno, vol. **35**, no. 1 (1999), 29-57.

[2] *L. Cui, H. Gao*, Exact controllability for a wave equation with mixed boundary conditions in a non-cylindrical domain, in Electron. J. Differ. Equ., vol. **2014**, paper no. 101 (2014), 12 p.

[3] *N. Dunford and J. Schwartz*, Linear operators. Part I: General theory, Wiley Classics Library, John Wiley&Sons, Inc., New York (1988).

[4] *O. van Gaans*, Topics in Analysis I (lecture notes), Mathematical Institute of Leiden University (2008).

[5] *D. S. Grebenkov and B.-T. Nguyen*, Geometrical structure of Laplacian eigenfunctions, SIAM Rev., vol. **55**, no. 4(2013), 601-667.

[6] *A. Heibig, M. Moussaoui*, Exact controllability of the wave equation for domains with slits and for mixed boundary conditions, Discrete Contin. Dyn. Syst., vol. **2**, no. 3(1996), 367-386.

[7] *V. Komornik and P. Loreti*, Fourier Series in Control Theory, Springer, New York (2005).

[8] *A. Negrescu*, Controllability for the mixed Neumann problem for the vibrating string, Regularity results, Proceedings of the 2015 20th International Conference on Control Systems and Computer Science (CSCS), IEEE (2015), 618-622.

[9] *L. Pandolfi*, Traction, deformation and velocity of deformation in a viscoelastic string, Evol. Equ. Control Theory, vol. **2**, no. 3(2013), 471-493.

[10] *L. Pandolfi*, Cosine operator and controllability of the wave equation with memory revisited, arXiv:1407.3706.

[11] *V. Vladimirov*, Equations of mathematical physics, Marcel Dekker, New York (1971).