U.P.B. Sci. Bull., Series A, Vol. 79, Iss. 3, 2017 ISSN 1223-7027

SOME LOWER AND UPPER BOUNDS FOR RELATIVE OPERATOR ENTROPY

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The relation between related operator entropy $S(A|B)$ and Tsallis relative operator entropy $T_p(A|B)$ was firstly considered by Fujii and Kamei, as follows:

$$T_{-p}(A|B) \le S(A|B) \le T_p(A|B), \qquad p \in (0,1]$$

where A, B are positive invertible operators. The aim of this paper is to establish some refinements for above inequality.

Keywords: Relative operator entropy, Tsallis relative operator entropy, operator inequality.

MSC2010: 47A63, 46L05, 47A60.

1. Introduction

We begin with introducing some (more general) standard terminology and giving a short account of some related results. A capital letter means an operator on a Hilbert space $\mathscr{H}$. For two positive invertible operators A, B and $p \in [0,1]$, p-power mean $A\#_pB$ is defined by

$$A\#_pB = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^pA^{\frac{1}{2}},$$

and we remark that $A\#_pB = A^{1-p}B^p$ if A commutes with B. The weighted operator arithmetic mean asserts that

$$A\nabla_pB = (1-p)A + pB,$$

for any $p \in [0,1]$. Let f be a continuous function defined on the interval I of real numbers, B a self-adjoint operator on the Hilbert space $\mathscr{H}$ and A a positive invertible operator on $\mathscr{H}$. Assume that the spectrum $Sp\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \subset \overset{o}{I}$ (the interior of I). Then by using the continuous functional calculus, we can

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define the perspective $\mathscr{P}_f(B|A)$, by setting

$$\mathscr{P}_f(B|A) = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

For related works the reader can refer to [14].

Tsallis entropy $S_q(X) = -\sum_x p(x)^q \ln_q p(x)$, was defined in [12] for the probability distribution $p(x)$, where q-logarithm function is defined by $\ln_q(x) = \frac{x^{1-q}-1}{1-q}$ for any non-negative real numbers x and $q \neq 1$. It is easily seen that Tsallis entropy is one parameter extension of Shannon entropy $S_1(X) = -\sum_x p(x) \log p(x)$ and converges to it as $q \to 1$. Tsallis entropy plays an essential role in non-extensive statistics, which is often called Tsallis statistics, so that many important results have been published from the various points of view [7, 10].

Very recently, Tsallis relative operator entropy $T_p(A|B)$ in Yanagi-Kuriyama-Furuichi [8] is defined by

$$T_p(A|B) := \frac{A^{\frac{1}{2}}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^p A^{\frac{1}{2}} - A}{p},$$

where A, B are two positive invertible operators on a Hilbert space $\mathscr{H}$ and $p \in (0,1]$. Notice that $T_p(A|B)$ can be written by using the notation of $A\#_p B$ as follows:

$$T_p(A|B) = \frac{A\#_p B - A}{p}, \quad p \in (0,1]. \tag{1}$$

The related operator entropy $S(A|B)$ in [4] is defined by

$$S(A|B) = A^{\frac{1}{2}}\left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}},$$

as an extension of [15]. The relation between $S(A|B)$ and $T_p(A|B)$ and $T_{-p}(A|B)$ was considered in [4] and the following inequalities was proved

$$T_{-p}(A|B) \leq S(A|B) \leq T_p(A|B), \tag{2}$$

$$A - AB^{-1}A \leq T_p(A|B) \leq B - A. \tag{3}$$

Furuichi et al. [8] proved that

$$A\natural_r B - \frac{1}{a} A\natural_{r-1} B + \left(\ln_r \frac{1}{a}\right) A \leq T_r(A|B) \leq \frac{1}{a} B - \left(\ln_r \frac{1}{a}\right) A\natural_r B - A, \tag{4}$$

where $A\natural_r B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^r A^{\frac{1}{2}}$ is defined for $r \in \mathbb{R}$.

As an extension of (4), Zou in [16] obtained that

$$\begin{aligned} & c_3(a,r,t) A\natural_r B - c_1(a,r,t) A\natural_{r-1} B + c_2(a,r,t) A \\ & \leq T_r(A|B) \\ & \leq c_1(a,r,t) B - c_2(a,r,t) A\natural_r B - c_3(a,r,t) A, \end{aligned}$$

where

$$c_1(a,r,t)=\frac{ra^{r-1}}{d(a,r,t)},\ c_2(a,r,t)=\frac{t(a^r-1)}{d(a,r,t)},\ c_2(a,r,t)=\frac{ra^r+(t-1)(a^r-1)}{d(a,r,t)}$$

with $d(a,r,t)=r(ta^r+(1-t))$.

For recent results related to Tsallis relative operator entropy inequality, see [5, 6, 9] and the references therein.

Motivated by the above facts, we obtain some inequalities for operator non-commutative perspectives. As a consequence, on basis of the Tsallis operator entropy proposed recently by Furuichi, Yanagi and Kuriyama, in this paper we present some upper and lower bound for Tsallis relative operator entropy.

2. Main Results

Dragomir has proved in [3, Lemma 1] the following multiplicative inequality between the weighted arithmetic and harmonic means: If $a,b>0$ and $p\in[0,1]$ then

$$\begin{aligned}&\left[p(1-p)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2+1\right]H_p(a,b)\\ &\le A_p(a,b)\\ &\le\left[p(1-p)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2+1\right]H_p(a,b),\end{aligned}\tag{5}$$

for any $a,b>0$ and $p\in[0,1]$, where $A_p(a,b)$ and $H_p(a,b)$ are the scalar weighted arithmetic mean and harmonic mean, respectively, namely

$$A_p(a,b)=(1-p)a+pb,\quad H_p(a,b)=\frac{ab}{(1-p)b+pa}.$$

Utilizing (5), the following theorem gives us a new bound for $T_{-p}(A|B)$.

Theorem 2.1. *Assume that A,B are positive invertible operators and the constants $M>m>0$ are such that $m^{-\frac{1}{p}}A\le B\le M^{-\frac{1}{p}}A$, where $p\in(0,1]$. Then*

$$\begin{aligned}&\frac{1}{-p^2}\left[\left(p(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\right.\\ &\left.\left(A^{\frac12}(A\nabla_p(A\#_pB))^{-1}A^{\frac12}\right)-1_{\mathscr{H}}\right]\\ &\ge T_{-p}(A|B)\\ &\ge\frac{1}{-p^2}\left[\left(p(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\right.\\ &\left.\left(A^{\frac12}(A\nabla_p(A\#_pB))^{-1}A^{\frac12}\right)-1_{\mathscr{H}}\right],\end{aligned}\tag{6}$$

for any $p \in (0,1]$.

Proof. We consider the case $a = 1$ and $b = x \in (0,\infty)$ in the inequality (5), we have

$$\begin{aligned}&\left[p(1-p)\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2+1\right]\times\left((1-p)+px^{-1}\right)^{-1}\\&\le(1-p)+px\\&\le\left[p(1-p)\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2+1\right]\times\left((1-p)+px^{-1}\right)^{-1},\end{aligned}$$

for any $p \in (0,1]$.

If $x \in [m,M] \subset (0,\infty)$, then $\max\{m,1\} \le \max\{x,1\} \le \max\{M,1\}$ and $\min\{m,1\} \le \min\{x,1\} \le \min\{M,1\}$. We have

$$\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2\le\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2,$$

and

$$\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2\le\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2,$$

for any $x \in [m,M] \subset (0,\infty)$. From what has been discussed above, the inequality

$$\begin{aligned}&\left[p(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right]\left((1-p)+px^{-1}\right)^{-1}\\&\le(1-p)+px\\&\le\left[p(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right]\left((1-p)+px^{-1}\right)^{-1},\end{aligned}$$

always true. Note that this is equivalent to saying

$$\begin{aligned}&\frac{1}{-p^2}\left[\left(p(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\left((1-p)+px^{-1}\right)^{-1}-1\right]\\&\ge\frac{(x-1)}{-p}\\&\ge\frac{1}{-p^2}\left[\left(p(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\left((1-p)+px^{-1}\right)^{-1}-1\right],\end{aligned}$$

for any $x \in [m, M]$ and any $p \in (0,1]$. The following inequality follows immediately from the assumptions and by setting $x = \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-p}$,

$$\begin{aligned}
&\frac{1}{-p^2}\left[\left(p(1-p)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right)\right.\\
&\left.\left((1-p)+p\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p\right)^{-1}-1_{\mathscr{H}}\right]\\
&\geq \frac{\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-p}-1_{\mathscr{H}}}{-p}\\
&\geq \frac{1}{-p^2}\left[\left(p(1-p)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right)\right.\\
&\left.\left((1-p)+p\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p\right)^{-1}-1_{\mathscr{H}}\right],
\end{aligned}$$

for any $p \in (0,1]$. Multiplying both sides of the above inequality by $A^{\frac{1}{2}}$, we can deduce the required inequality (6). Since

$$\begin{aligned}
&A^{-\frac{1}{2}}\left((1-p)A+pA^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p A^{\frac{1}{2}}\right)A^{-\frac{1}{2}}\\
&= A^{-\frac{1}{2}}\left((1-p)A+p\left(A\#_p B\right)\right)A^{-\frac{1}{2}}\\
&= A^{-\frac{1}{2}}\left(A\nabla_p\left(A\#_p B\right)\right)A^{-\frac{1}{2}}.
\end{aligned}$$

Hence Theorem 2.1 is proved. □

Remark 2.1. *The case of equality in Theorem 2.1 holds when p tends to zero.*

In [2], third author established some inequalities for the quantity $\frac{b-a}{a} - \ln b + \ln a$, where $a, b > 0$. Among other results, he proved that

$$\frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 \leq \frac{b-a}{a}-\ln b+\ln a \leq \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2 \tag{7}$$

for any $a, b > 0$.

The following theorem is an operator version of (7) in the case of operators satisfying the condition $mA \leq B \leq MA$ with $1 < m < M$.

Theorem 2.2. *Let A and B be positive invertible operators such that $mA \leq B \leq MA$ and $1 < m$. Then*

$$\frac{1}{2M^2}\left(BA^{-1}B-2B+A\right) \leq (B-A)-S(A|B) \leq \frac{1}{2m^2}\left(BA^{-1}B-2B+A\right). \tag{8}$$

Proof. Assume that $a, b \in [m, M] \subset (0, \infty)$, then we have the local bounds

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(b-a)^2}{m^2}. \tag{9}$$

When $a = 1$ and $b = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, the inequality (9) says

$$\begin{aligned} &\frac{\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^2 - 2A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + 1_{\mathscr{H}}}{2M^2} \\ &\le \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) - 1_{\mathscr{H}} - \left(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \\ &\le \frac{\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^2 - 2A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + 1_{\mathscr{H}}}{2m^2}. \end{aligned}$$

Multiplying $A^{\frac{1}{2}}$ from the both sides, we reached the desired inequality (8). □

Another result can be seen as follows.

Theorem 2.3. *Let $f : [0, \infty) \to \mathbb{R}$ be an increasing function and A, B are positive invertible operators and the constants $M > m > 0$ are such that $mA \le B \le MA$. Then*

$$f(m)A \le \mathscr{P}_f(B|A) \le f(M)A. \tag{10}$$

Proof. Since $f : [0, \infty) \to \mathbb{R}$ is an increasing function we have

$$f(m) = \min_{x \in [m,M]} f(x) \le f(x) \le \max_{x \in [m,M]} f(x) = f(M). \tag{11}$$

Hence

$$f(m)1_{\mathscr{H}} \le f(X) \le f(M)1_{\mathscr{H}},$$

for the positive operator X such that $m1_{\mathscr{H}} \le X \le M1_{\mathscr{H}}$. Substituting $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ for X in the above inequality

$$f(m)1_{\mathscr{H}} \le f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \le f(M)1_{\mathscr{H}}. \tag{12}$$

Finally, if we multiply both sides of (12) by $A^{\frac{1}{2}}$, we get the desired result (10). □

For particular case of Theorem 2.3, we have the following corollary, which gives an upper and lower bound on the Tsallis relative entropy.

Corollary 2.1. *All as in Theorem 2.3, then*

$$\left(\frac{m^p - 1}{p}\right)A \le T_p(A|B) \le \left(\frac{M^p - 1}{p}\right)A, \tag{13}$$

for any $p \in (0, 1]$.

Proof. Consider the function $f_p : [0, \infty) \to \mathbb{R}$ defined by $f_p(x) := x^p$, for any $p \in (0, 1]$. Obviously, f_p is increasing function on $(0, \infty)$. Therefore from (11) we have

$$f_p(m) \le x^p \le f_p(M).$$

Replacing x with the positive operator $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$, and then multiplying $A^{\frac{1}{2}}$ from left hand side and right hand side, to get the desired inequality (13). □

Based on Corollary 2.1 we have the following result.

Remark 2.2. *Taking the limit as* $p \to 0$ *in* (13)*, then*

$$(\log m) A \le S(A|B) \le (\log M) A.$$

We note that this result is based on the fact that

$$\lim_{p \to 0} T_p(A|B) = S(A|B).$$

Remark 2.3. *It is natural to ask which upper and lower bound, provided by Furuichi, Yanagi and Kuriyama* (3) *and our inequality* (13) *is better? In order to answer this question, for each* m, M *which satisfies in the condition*

$$\frac{p}{M} + m^p - (p+1) \ge 0,$$

we have that

$$A - AB^{-1}A \le \left(\frac{m^p - 1}{p}\right) A \le T_p(A|B).$$

This is a refinement of the left-hand side of inequality (3)*. Furthermore, for each* m, M *which satisfies in the condition*

$$-M^p + pm + 1 - p \ge 0,$$

we have that

$$T_p(A|B) \le \left(\frac{M^p - 1}{p}\right) A \le B - A.$$

This result presents a refinement of the right-hand side of inequality (3)*.*

It is worthwhile to mention that we can easily find examples

$$A - AB^{-1}A \le \left(\frac{m^p - 1}{p}\right) A \le T_p(A|B) \le \left(\frac{M^p - 1}{p}\right) A \le B - A,$$

holds. For instance, if $p = \frac{1}{2}$ *and* $[m, M] \subseteq [3, 4]$*, the above series of inequalities is true.*

We conclude this section with the following theorem.

Theorem 2.4. *Let* I *be an open interval and* $f : I \to \mathbb{R}$ *be a convex and differentiable function on* I *whose derivative* f' *is continuous on* I*. Assume that* A, B *are positive invertible operators and the constants* $M > m > 0$ *are such that* $mA \le B \le MA$*. Then*

$$\mathscr{P}_f(B|A) \ge f(t) A + f'(t)(B - A). \tag{14}$$

Proof. Since f is convex and differentiable we have

$$f(x) \geq f(t) + (x-t) f'(t),$$

for any $x \in [m, M]$ and $t \in I$.

The operator $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum. According to monotonicity principle for operator functions, we can insert $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the above inequality, i.e., we have

$$f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \geq f(t) 1_{\mathscr{H}} + \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - t1_{\mathscr{H}}\right) f'(t)$$

Finally, by multiplying both sides by $A^{\frac{1}{2}}$ we deduce the desired result (14). □

Remark 2.4. *Of course if f is concave then inequality* (14) *is reversed.*

Using arguments similar to those in the proof of Corollary 2.1, we obtain a different lower bound on the Tsallis relative entropy.

Corollary 2.2. *All as in Theorem 2.4, then*

$$T_p(A|B) \geq \left(\frac{t^p - 1}{p}\right) A + t^{p-1}(B - tA), \tag{15}$$

for any $p \in (0, 1]$.

Remark 2.5. *Taking the limit as $p \to 0$ in* (15)*, then*

$$S(A|B) \geq (\log t) A + \frac{1}{t} B.$$

The famous Young inequality for scalars says that if $a, b > 0$ and $p \in [0, 1]$, then

$$a^{1-p}b^p \leq (1-p) a + pb.$$

Dragomir in [1, Theorem 1] obtained the following reverse of Young's inequality:

$$0 \leq (1-p) a + pb - a^{1-p}b^p \leq p(1-p)(a-b)(\ln a - \ln b), \tag{16}$$

where $a, b > 0$ and $p \in [0, 1]$. Further refinements and generalizations of Young's inequality have been obtained in [2, 11].

The following application of inequality (16) leads to a new reverse of AM-GM inequality.

Theorem 2.5. *Let A, B be positive invertible operators. Then*

$$\frac{A+B}{2} - (A\#B) \leq \frac{1}{4}\left((A\#B) A^{-1} S(A|B) A^{-1} (A\#B) - S(A|B)\right). \tag{17}$$

Proof. It can be inferred from the case $p = \frac{1}{2}$ and $b = 1$ of the inequality (16) that

$$\frac{a+1}{2} - a^{\frac{1}{2}} \leq \frac{a \ln a - \ln a}{4}.$$

Note that the above inequality holds for all $a > 0$. On the other hand, the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \geq 0$ is positive, which, by virtue of monotonicity principle for operator functions, yields the inequality

$$\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + 1_{\mathscr{H}}}{2} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left(\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}\left(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} - \ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right).$$

Multiplying $A^{\frac{1}{2}}$ to the above inequality from left hand side and right hand side, we deduce the desired result (17). □

3. Conclusion

Various generalizations of the Shannon inequalities have played an important role in classical information theory. Surprisingly, it has been discovered that many of these inequalities have operator generalizations, in which one replaced random variables by Hilbert space operators. In this paper, we find upper and lower bounds of relative operator entropy and Tsallis relative operator entropy according to operator p-power mean.

Acknowledgments

We would like to express our cordial thanks to the referee for his valuable comments.

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