FUZZY TRIGONOMETRIC KOROVKIN TYPE APPROXIMATION VIA POWER SERIES METHODS OF SUMMABILITY

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We prove a fuzzy trigonometric Korovkin type approximation theorem via power series methods of summability and give a related approximation result for periodic fuzzy continuous functions by means of fuzzy modulus of continuity. An illustrative example concerning fuzzy Abel-Poisson convolution operator is also constructed.

Keywords: Fuzzy set theory, Korovkin type approximation, power series methods of summability

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1. Introduction

Fuzzy set theory was introduced by Zadeh [1] as an extension of classical set theory which lacks the means to define imprecise knowledge and to allow approximate reasoning. Unlike the classical set theory, fuzzy set theory provides researchers with means to cope with uncertainty and imprecision which are intrinsic to many real-world problems. Since its invention, the theory has been used in many areas of science as a smart tool to handle problems involving fuzziness. Mathematical foundations of the theory have also developed in different ways and many concepts in classical setting have been extended to fuzzy setting [2–9]. In particular different sequence spaces are defined and corresponding convergence properties are investigated [10–16]. In the light of these developments, in this study we aim to investigate the approximation of fuzzy continuous functions by sequences of fuzzy positive linear operators via the concept of power series methods of summability which is defined recently [17]. Before stating the motivation, goals and results of the paper in more detail we need to give some preliminaries concerning fuzzy numbers.

A *fuzzy number* is a fuzzy set on the real axis, i.e. u is normal, fuzzy convex, upper semi-continuous and $\operatorname{supp} u = \overline{\{t \in \mathbb{R} : u(t) > 0\}}$ is compact [1]. $\mathbb{R}_{\mathcal{F}}$ denotes the space of fuzzy numbers. *α-level set* $[u]_\alpha$ is defined by

$$[u]_\alpha := \begin{cases} \{t \in \mathbb{R} : u(t) \geq \alpha\} \quad , & if \quad 0 < \alpha \leq 1, \\ \overline{\{t \in \mathbb{R} : u(t) > \alpha\}} \quad , & if \quad \alpha = 0. \end{cases}$$

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$r \in \mathbb{R}$ may be seen as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(t) := \begin{cases} 1 & , \quad if \quad t = r, \\ 0 & , \quad if \quad t \neq r. \end{cases}$$

Let $u, v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$. The addition and scalar multiplication are defined by

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+] \quad , \quad [ku]_\alpha = k[u]_\alpha$$

where $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, for all $\alpha \in [0,1]$. Partial ordering on $\mathbb{R}_{\mathcal{F}}$ is defined as follows:

$$\mu \preceq \nu \Longleftrightarrow [\mu]_\alpha \preceq [\nu]_\alpha \Longleftrightarrow \mu_\alpha^- \leq \nu_\alpha^- \text{ and } \mu_\alpha^+ \leq \nu_\alpha^+ \text{ for all } \alpha \in [0,1].$$

The metric D on $\mathbb{R}_{\mathcal{F}}$ is defined as

$$D(u,v) := \sup_{\alpha \in [0,1]} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}$$

and metric space $(\mathbb{R}_{\mathcal{F}}, D)$ is complete [18]. Let $f, g : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be fuzzy-number-valued functions. The distance between f and g is defined by

$$D^*(f,g) := \sup_{x \in I} D(f(x), g(x)).$$

Let $L : C_{\mathcal{F}}(\mathbb{R}) \to C_{\mathcal{F}}(\mathbb{R})$ be an operator where $C_{\mathcal{F}}(\mathbb{R})$ denotes the space of all fuzzy continuous functions on $\mathbb{R}$. Then we call L a *fuzzy linear operator* iff

$$L(c_1 f_1 + c_2 f_2) = c_1 L(f_1) + c_2 L(f_2)$$

for any $c_1, c_2 \in \mathbb{R}, f_1, f_2 \in C_{\mathcal{F}}(\mathbb{R})$. Also operator L is called *fuzzy positive linear operator* if it is fuzzy linear and the condition $L(f;x) \preceq L(g;x)$ is satisfied for any $f, g \in C_{\mathcal{F}}(\mathbb{R})$ with $f(x) \preceq g(x)$ and for all $x \in \mathbb{R}$.
A function $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ is 2π-periodic if $f(x) = f(x + 2\pi)$ for all $x \in \mathbb{R}$. The space of all 2π-periodic and fuzzy continuous functions on $\mathbb{R}$ is denoted by $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. Besides the space of all 2π-periodic and real valued continuous functions on $\mathbb{R}$ is denoted by $C_{2\pi}(\mathbb{R})$ and equipped with the supremum norm $\|\cdot\|$.

Anastassiou [19] proved the first fuzzy Korovkin approximation theorem in the space $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ as an extension of the classical trigonometric Korovkin theorem [20]. His theorem states as follows:

Theorem 1.1. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ into itself. Assume that there exists a corresponding sequence $\left\{\widetilde{L}_n\right\}_{n\in\mathbb{N}}$ of positive linear operators defined on $C_{2\pi}(\mathbb{R})$ with the property*

$$\{L_n(f;x)\}_\alpha^\pm = \widetilde{L}_n\left(f_\alpha^\pm; x\right) \tag{1}$$

for all $x \in \mathbb{R}, \alpha \in [0,1], n \in \mathbb{N}$ and $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. Assume further that

$$\lim_{n\to\infty} \|\widetilde{L}_n(f_i) - f_i\| = 0 \tag{2}$$

for $i = 0, 1, 2$ *with* $f_0(x) = 1, f_1(x) = \cos x, f_2(x) = \sin x$. *Then for all* $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ *we have*

$$\lim_{n\to\infty} D^*(L_n(f), f) = 0.$$

Due to the nature of the concept of convergence, a fuzzy positive linear operator with property (1) and converging in the sense (2) is an occasional case. Considering this issue, authors recently utilized summability theory to recover the convergence of fuzzy positive linear operators which fail to converge in the sense (2) and achieved the approximation in some cases by means of regular summability matrices. Duman and Anastassiou [21] studied the trigonometric approximation in the space $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ by means of $A-$statistical convergence and illustrated obtained results on fuzzy Fejer operators. Anastassiou *et al.* [22] presented a fuzzy trigonometric Korovkin type approximation theorem via $\mathcal{A}-$summation process and studied the rate of convergence of approximating fuzzy positive linear operators by the help of fuzzy modulus of continuity. Following these studies we now prove a fuzzy Korovkin type approximation theorem in the space $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ by using power series methods of summability and obtain a related approximation by the help of fuzzy modulus of continuity. We also introduce the concept fuzzy Abel-Poisson convolution operator and construct an example concerning fuzzy Abel-Poisson convolution operators such that our approximation results via power series methods of summability work but Theorem 1.1 does not work.

Power series summability method (J, p) is recently extended to fuzzy number space by Sezer and Çanak [17] followingly: Suppose that $p = (p_n)$ is a sequence of nonnegative real numbers with $p_0 > 0$ such that $\sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$ and associated power series $p(r) = \sum_{n=0}^{\infty} p_n r^n$ is convergent for $r \in (0, 1)$. A sequence (u_n) of fuzzy numbers is said to be summable to some fuzzy number μ by power series method determined by p if $\sum_{n=0}^{\infty} u_n p_n r^n$ converges for $r \in (0, 1)$ and

$$\lim_{r\to 1^-} \frac{1}{p(r)} \sum_{n=0}^{\infty} u_n p_n r^n = \mu$$

Then we write $u_n \to \mu(J_p)$. Let $\{L_n\}$ be a sequence of fuzzy positive linear operators from $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ into itself with property (1) and let for each $r \in (0, 1)$

$$\sum_{n=0}^{\infty} \left\| \widetilde{L}_n(f_0) \right\| p_n r^n < \infty. \tag{3}$$

Then for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$, series $\sum_{n=0}^{\infty} \widetilde{L}_n(f_\alpha^{\pm}) p_n r^n$ and series $\sum_{n=0}^{\infty} L_n(f) p_n r^n$ converge for $r \in (0, 1)$.

2. Main Results

Theorem 2.1. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ into itself. Suppose that there exists a corresponding sequence $\left\{\widetilde{L}_n\right\}_{n\in\mathbb{N}}$ of positive linear operators defined on $C_{2\pi}(\mathbb{R})$ with the property* (1) *and satisfying* (3)*. Suppose further that*

$$\lim_{r\to 1^-}\left\|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_i)p_nr^n - f_i\right\| = 0 \tag{4}$$

for $i = 0,1,2$ with $f_0(x) = 1, f_1(x) = \cos x, f_2(x) = \sin x$. Then for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ we have

$$\lim_{r\to 1^-} D^*\left(\frac{1}{p(r)}\sum_{n=0}^{\infty}L_n(f)p_nr^n, f\right) = 0.$$

Proof. Suppose that I is a closed subinterval of $\mathbb{R}$ with length 2π and $x \in I$ be fixed. Let $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|f_\alpha^{\pm}(y) - f_\alpha^{\pm}(x)\right| \le \varepsilon + 2M_\alpha^{\pm}\frac{\sin^2\left(\frac{y-x}{2}\right)}{\sin^2\frac{\delta}{2}}$$

holds for all $\alpha \in [0,1]$ and $y \in \mathbb{R}$ where $M_\alpha^{\pm} = \|f_\alpha^{\pm}\|$(see [20]). Using linearity and positivity of operators $\widetilde{L}_n$ we obtain

$$\begin{aligned}
&\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_\alpha^{\pm};x)p_nr^n - f_\alpha^{\pm}(x)\right| \\
&\quad\le \frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n\left(\left|f_\alpha^{\pm}(y) - f_\alpha^{\pm}(x)\right|;x\right)p_nr^n + M_\alpha^{\pm}\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(1;x)p_nr^n - 1\right| \\
&\quad\le \frac{\varepsilon}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(1;x)p_nr^n + \frac{2M_\alpha^{\pm}}{\sin^2\left(\frac{\delta}{2}\right)p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n\left(\sin^2\left(\frac{y-x}{2}\right);x\right)p_nr^n \\
&\qquad + M_\alpha^{\pm}\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(1;x)p_nr^n - 1\right| \\
&\quad\le \varepsilon + \left(\varepsilon + M_\alpha^{\pm}\right)\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(1;x)p_nr^n - 1\right| \\
&\qquad + \frac{M_\alpha^{\pm}}{\sin^2\left(\frac{\delta}{2}\right)}\left\{\left[\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(1;x)p_nr^n - 1\right]\right. \\
&\qquad\qquad - \cos x\left[\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(\cos y;x)p_nr^n - \cos x\right] \\
&\qquad\qquad \left. - \sin x\left[\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(\sin y;x)p_nr^n - \sin x\right]\right\}
\end{aligned}$$

$$\leq \varepsilon + K_{\alpha}^{\pm}(\varepsilon)\left\{\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(1;x)p_nr^n - 1\right| + \left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(\cos y;x)p_nr^n - \cos x\right| + \left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(\sin y;x)p_nr^n - \sin x\right|\right\}$$

where $K_{\alpha}^{\pm}(\varepsilon) = \varepsilon + M_{\alpha}^{\pm} + \frac{M_{\alpha}^{\pm}}{\sin^2\left(\frac{\delta}{2}\right)}$. Hence in view of the property (1) and definition of the metric D we get

$$D\left(\frac{1}{p(r)}\sum_{n=0}^{\infty}L_n(f;x)p_nr^n, f(x)\right)$$
$$\leq \varepsilon + K\left\{\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0;x)p_nr^n - f_0\right| + \left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_1;x)p_nr^n - f_1\right| + \left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_2;x)p_nr^n - f_2\right|\right\}$$

where $K := K(\varepsilon) = \sup_{\alpha\in[0,1]}\max\left\{K_{\alpha}^{-}(\varepsilon), K_{\alpha}^{+}(\varepsilon)\right\}$. By taking supremum over x we conclude

$$D^*\left(\frac{1}{p(r)}\sum_{n=0}^{\infty}L_n(f)p_nr^n, f\right)$$
$$\leq \varepsilon + K\left\{\left\|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0)p_nr^n - f_0\right\| + \left\|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_1)p_nr^n - f_1\right\| + \left\|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_2)p_nr^n - f_2\right\|\right\}.$$

Finally taking limit as $r \to 1^-$ and considering (4) we complete the proof. □

Now we give another approximation theorem for 2π-periodic fuzzy continuous functions on $\mathbb{R}$ by means of fuzzy modulus of continuity. For $f \in C_{2\pi}^{\mathcal{F}}(\mathbb{R})$ the (first) fuzzy modulus of continuity of f is defined by

$$\omega_1^{\mathcal{F}}(f;\delta) := \sup_{x,y\in\mathbb{R};|x-y|\leq\delta} D(f(x), f(y))$$

for any $\delta > 0$ [23, 24]

Lemma 2.1. *[25] Let $f \in C_{2\pi}^{\mathcal{F}}(\mathbb{R})$. Then for any $\delta > 0$*

$$\omega_1^{\mathcal{F}}(f;\delta) = \sup_{\alpha\in[0,1]}\max\left\{\omega_1(f_{\alpha}^{-};\delta), \omega_1(f_{\alpha}^{+};\delta)\right\}.$$

Theorem 2.2. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ into itself. Assume that there exists a corresponding sequence $\left\{\widetilde{L}_n\right\}_{n\in\mathbb{N}}$*

of positive linear operators defined on $C_{2\pi}(\mathbb{R})$ *with the properties* (1) *and* (3). *Assume also that*

(i) $\lim_{r\to 1^-}\left\|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0)p_nr^n - f_0\right\| = 0,$

(ii) $\lim_{r\to 1^-}\omega_1^{\mathcal{F}}(f;\gamma(r)) = 0$

where $\gamma(r) = \sqrt{\left\|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(\varphi)p_nr^n\right\|}$ *with* $\varphi(y) = \sin^2\left(\frac{y-x}{2}\right)$. *Then for all* $f\in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ *we have*

$$\lim_{r\to 1^-} D^*\left(\frac{1}{p(r)}\sum_{n=0}^{\infty}L_n(f)p_nr^n, f\right) = 0.$$

Proof. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ into itself and corresponding sequence $\left\{\widetilde{L}_n\right\}_{n\in\mathbb{N}}$ of positive linear operators defined as in the statement of the theorem. Let $f\in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ and let $x\in[-\pi,\pi]$ be fixed. By property (4) of [26] it follows that

$$\begin{aligned}
&\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_\alpha^{\pm};x)p_nr^n - f_\alpha^{\pm}(x)\right| \\
&\quad\le \frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n\left(\left|f_\alpha^{\pm}(y)-f_\alpha^{\pm}(x)\right|;x\right)p_nr^n + M_\alpha^{\pm}\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(1;x)p_nr^n - 1\right| \\
&\quad\le \frac{\omega_1(f_\alpha^{\pm};\delta)}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n\left(1+(\pi/\delta)^2\sin^2\left(\frac{y-x}{2}\right);x\right)p_nr^n \\
&\qquad + M_\alpha^{\pm}\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0;x)p_nr^n - f_0\right| \\
&\quad\le \left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0;x)p_nr^n - f_0\right|\omega_1(f_\alpha^{\pm};\delta) + \omega_1(f_\alpha^{\pm};\delta) \\
&\qquad + \frac{\pi^2}{\delta^2}\frac{\omega_1(f_\alpha^{\pm};\delta)}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n\left(\varphi(y);x\right)p_nr^n + M_\alpha^{\pm}\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0;x)p_nr^n - f_0\right|
\end{aligned}$$

where $M_\alpha^{\pm} = \|f_\alpha^{\pm}\|$. Then considering property (1), Lemma (2.1) and definition of the metric D we obtain

$$\begin{aligned}
&D\left(\frac{1}{p(r)}\sum_{n=0}^{\infty}L_n(f;x)p_nr^n, f(x)\right) \\
&\quad\le \left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0;x)p_nr^n - f_0\right|\omega_1^{\mathcal{F}}(f;\delta) + \omega_1^{\mathcal{F}}(f;\delta)
\end{aligned}$$

$$+\frac{\pi^2}{\delta^2}\frac{\omega_1^{\mathcal{F}}(f;\delta)}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n\left(\varphi(y);x\right)p_nr^n+M\left|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0;x)p_nr^n-f_0\right|$$

where $M:=\sup_{\alpha\in[0,1]}\max\left\{M_\alpha^-,M_\alpha^+\right\}$. Taking supremum over x and choosing $\delta=\gamma(r)$ reveal

$$\begin{aligned}
D^*&\left(\frac{1}{p(r)}\sum_{n=0}^{\infty}L_n(f)p_nr^n,f\right)\\
&\leq \left\|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0)p_nr^n-f_0\right\|\omega_1^{\mathcal{F}}(f;\gamma(r))+(1+\pi^2)\omega_1^{\mathcal{F}}(f;\gamma(r))\\
&\quad +M\left\|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0)p_nr^n-f_0\right\|\\
&\leq K\left\{\left\|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0)p_nr^n-f_0\right\|\omega_1^{\mathcal{F}}(f;\gamma(r))+\omega_1^{\mathcal{F}}(f;\gamma(r))\right.\\
&\quad\left.+\left\|\frac{1}{p(r)}\sum_{n=0}^{\infty}\widetilde{L}_n(f_0)p_nr^n-f_0\right\|\right\}
\end{aligned}$$

where $K=\max\{1+\pi^2,M\}$. Finally taking limit as $r\to1^-$ and considering (i) and (ii) of theorem we complete the proof. □

3. **Illustrative Example**

Now we construct an example concerning fuzzy Abel-Poisson convolution operator such that fuzzy trigonometric Korovkin Theorem 1.1 does not work but our new results work. Before continuing with the example we shortly mention about the concept of fuzzy Riemann integral. Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$. We say that f is fuzzy-Riemann integrable [23] to $I\in\mathbb{R}_{\mathcal{F}}$ if, for any $\varepsilon>0,\exists\delta>0$: for any division $P=\{[u,v];\xi\}$ of $[a,b]$ with norms $\Delta(P)<\delta$, we have

$$D\left(\sum_P(v-u)f(\xi),I\right)<\varepsilon.$$

We write

$$I:=(FR)\int_a^b f(x)dx.$$

By corollary 13.2 of [23, p.644], if $f\in C_{\mathcal{F}}[a,b]$(fuzzy continuous on $[a,b]$) then f is fuzzy-Riemann integrable on $[a,b]$.

Theorem 3.1. *[27] Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ be fuzzy continuous function. Then*

$$\left[(FR)\int_a^b f(x)dx\right]_\alpha=\left[\int_a^b f_\alpha^-(x)dx,\int_a^b f_\alpha^+(x)dx\right],\quad\forall\alpha\in[0,1].$$

Following [25] and [21], we now introduce the fuzzy Abel-Poisson convolution operators P_θ as follows:

$$P_\theta(f;x) = \frac{1-\theta^2}{2\pi}\left\{(FR)\int_{-\pi}^{\pi}\frac{f(x-y)}{1-2\theta\cos y+\theta^2}dy\right\}$$

where and $0 \leq \theta < 1, x \in \mathbb{R}$ and $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. Now consider the fuzzy positive linear operators

$$L_n(f;x) = (1+(-1)^n)P_{\frac{n}{n+1}}(f;x) \tag{5}$$

where $P_{\frac{n}{n+1}}$ is the $\frac{n}{n+1}-$th fuzzy Abel-Poisson convolution operator. Then it follows from Theorem 3.1 that

$$\begin{aligned}\{L_n(f;x)\}_\alpha^\pm &= \widetilde{L}_n\left(f_\alpha^\pm;x\right)\\ &= (1+(-1)^n)\left\{\frac{2n+1}{2\pi(n+1)^2}\int_{-\pi}^{\pi}\frac{f_\alpha^\pm(x-y)}{1-\left(\frac{2n}{n+1}\right)\cos y+\left(\frac{n}{n+1}\right)^2}dy\right\}\end{aligned}$$

where $f_\alpha^\pm \in C_{2\pi}(\mathbb{R})$. Now we apply Theorem 2.1 with special case $p_n = 1$. Since

$$\widetilde{L}_n(f_0;x) = (1+(-1)^n), \quad \widetilde{L}_n(f_1;x) = (1+(-1)^n)\left(\frac{n}{n+1}\right)\cos x,$$

$$\widetilde{L}_n(f_2;x) = (1+(-1)^n)\left(\frac{n}{n+1}\right)\sin x,$$

we get

$$\begin{aligned}\left\|(1-r)\sum_{n=0}^{\infty}\widetilde{L}_n(f_0)r^n - f_0\right\| &= \frac{1-r}{1+r} \to 0,\\ \left\|(1-r)\sum_{n=0}^{\infty}\widetilde{L}_n(f_1)r^n - f_1\right\| &= \left|\frac{1-r}{1+r}+\frac{(1-r)\ln(1-r)}{r}-\frac{(1-r)\ln(1+r)}{r}\right| \to 0\\ \left\|(1-r)\sum_{n=0}^{\infty}\widetilde{L}_n(f_2)r^n - f_2\right\| &= \left|\frac{1-r}{1+r}+\frac{(1-r)\ln(1-r)}{r}-\frac{(1-r)\ln(1+r)}{r}\right| \to 0\end{aligned}$$

as $r \to 1^-$. Then by Theorem 2.1 we conclude

$$\lim_{r\to 1^-} D^*\left((1-r)\sum_{n=0}^{\infty}L_n(f)r^n, f\right) = 0.$$

Now we obtain the approximation of fuzzy positive linear operators $\{L_n\}_{n\in\mathbb{N}}$ defined in (5) by using Theorem 2.2. Since

$$\left\|(1-r)\sum_{n=0}^{\infty}\widetilde{L}_n(f_0)r^n - f_0\right\| = \frac{1-r}{1+r} \to 0 \quad as \quad r \to 1^-,$$

(i) of Theorem 2.2 is satisfied. Besides by regularity of power series method (J_1) [17, Theorem 2] we have

$$\gamma(r) = \sqrt{\left\| (1-r)\sum_{n=0}^{\infty} \widetilde{L}_n(\varphi)\, r^n \right\|} = \sqrt{\left| (1-r)\sum_{n=0}^{\infty} \left\{ \frac{1+(-1)^n}{2(n+1)} \right\} r^n \right|} \to 0$$

as $r \to 1^-$. Since f is fuzzy continuous and 2π-periodic, it is fuzzy uniformly continuous on $\mathbb{R}$ [19, Lemma3]. As result we have $\lim_{r\to 1^-} \omega_1^{\mathcal{F}}(f;\gamma(r)) = 0$ in view of the fact $\lim_{r\to 1^-} \gamma(r) = 0$. So (ii) of Theorem 2.2 is also satisfied. Then by Theorem 2.2 we conclude that

$$\lim_{r\to 1^-} D^* \left((1-r)\sum_{n=0}^{\infty} L_n(f) r^n, f \right) = 0.$$

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