FRÉCHET α-LIPSCHITZ VECTOR-VALUED OPERATOR ALGEBRA

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Let (X, d) be a metric space with at least two elements and (A, p_l), be a Fréchet algebra over the scalar field F and take $\alpha \in \mathbb{R}$ with $\alpha > 0$. In this paper, at first we define the big and little Fréchet α-Lipschitz vector-valued operator algebras of order α and examine how this concept in Banach algebra can be generalized for Fréchet algebra. Furthermore, we study some properties and ideal amenability of Fréchet α-Lipschitz vector-valued operator algebras.

Keywords: Vector-valued Lipschitz algebra, Fréchet algebra, Ideal amenability of Fréchet algebra, Metric space.

1. Introduction

Some of the notions related to Banach algebras, have been introduced and studied for Fréchet algebra. For example, the notion of amenability of a Fréchet algebra was studied by Pirkovskii [22]. Lawson and Read introduced and studied the notions of approximate amenability and approximate contractibility of Fréchet algebras in [21]. Furthermore in [1], Abtahi and et al introduced and studied the notion of weak amenability of Fréchet algebra. Moreover, according to the basic definition of Segal algebras and abstract Segal algebras [2], recently they introduced the Segal- Fréchet algebra for the Fréchet algebra (A, p_l). Rejali and Ranjbari generalized the concept of ideal amenability for Freéchet algebras in [20].

In this paper we introduce and study the notion of Lipschitz algebra, for Fréchet algebra. Let (X, d) be a metric space and $B(X)$ indicates the Banach space consisting of all bounded complex valued functions on X, endowed with the norm

$$||f||_{\sup} = \sup_{x\in X} |f(x)| \quad \Big(f \in B(X)\Big)$$

Take $\alpha \in \mathbb{R}$ with $\alpha > 0$. Then $\operatorname{Lip}_\alpha X$ is the subspace of $B(X)$, consisting of all of bounded complex-valued functions f on X such that

$$p_\alpha(f) := \sup\Big\{\frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y\Big\} < \infty$$

It is known that $\operatorname{Lip}_\alpha X$ endowed with the norm $||.||_\alpha$ given by

$$||f||_\alpha := p_\alpha(f) + ||f||_{\sup}$$

and pointwise product is a unital commutative Banach algebra, which is called Lipschitz algebra of order α.

If E is a Banach space, for a constant $\alpha > 0$ and a function $f : X \to E$, the Lipschitz constant of f is defined by

$$p_{\alpha,E}(f) := \sup\Big\{\frac{||f(x) - f(y)||}{d(x, y)^\alpha} : x, y \in X, x \neq y\Big\}$$

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and the vector-valued big Lipschitz algebra (of order α), or simply, the vector-valued Lipschitz algebra is defined by

$$\mathrm{Lip}_\alpha(X,E)=\{f:X\to E: f\,\text{is bounded and}\, p_{\alpha,E}(f)<\infty\}.$$

Similarly, for $\alpha>0$ the vector-valued little Lipschitz algebra (of order α) is defined by

$$\mathrm{lip}_\alpha(X,E)=\{f\in \mathrm{Lip}_\alpha(X,E):\frac{||f(x)-f(y)||}{d(x,y)^\alpha}\to 0\ as\ d(x,y)\to 0\}.$$

These Lipschitz algebras were first studied by Sherbert in [24, 25].

Let E be a Banach algebra, and $||f||_{\alpha,E}=||f||_{\infty,E}+p_{\alpha,E}(f)$. In [9] it has been shown that $\Big(\mathrm{Lip}_\alpha(X,E), ||\cdot||_{\alpha,E}\Big)$ is complete and it is, in fact, a Banach subalgebra of $C_b(X,E)$, and moreover, $\mathrm{lip}_\alpha(X,E)$ is a closed subalgebra of $\mathrm{Lip}_\alpha(X,E)$.

Many results on amenability and weak amenability of Lipschitz algebras are given in [4, 14, 18, 26]. For any $\phi\in\Delta(\mathrm{lip}_\alpha X)$, ϕ-amenability of $\mathrm{Lip}_\alpha X$, considered by Kaniuth and et al. (see [19], Example 5.3), and some results about character amenability of these algebras are studied in [5, 6, 7, 10]. Many results on Lipschitz algebra are given in [27, 28, 17, 3, 12].

Gordji and et al. (see [14],Theorem 2.4) proved that if (X,d) is a metric space. Then, the following statements are equivalent:

(i) $\mathrm{Lip}_\alpha X$ is character amenable;
(ii) (X,d) is uniformly discrete;
(iii) $\mathrm{Lip}_\alpha X$ is amenable.

2. Preliminaries and introduction

In this section, we recall and review some of the basic terminologies about Fréchet algebra and Lipschitz algebras. For further details, see [13, 28].

A Fréchet algebra (A,p_l) is a topological algebra A whose topology can be defined by a sequence (p_l) of separating and submultiplicative seminorms, i.e., $p_l(fg)\le p_l(f)p_l(g)$ for all $f,g\in A$ and which is complete with respect to this topology.

Without loss of generality we can assume that $p_l\le p_{l+1}$ and that $p_l(1)=1$ if A has identity 1. Note that a sequence $(a_n)_{n\in N}$ in the Fréchet algebra (A,p_l) converges to $a\in A$ if and only if $p_l(a_n-a)\to 0$ for each $l\in N$,as $n\to\infty$. A locally convex space E is a linear space over the field $K(\mathbb{R}\,\text{or}\,\mathbb{C})$ together with a compatible topology (i.e. addition $E\times E\to E$ and scalar multiplication $K\times E\to E$ are continuous) and which has a 0-neighborhood basis consisting of (absolutely) convex sets.

A locally convex space is a Fréchet space, if and only if, it is metrizable and complete. Locally convex spaces provide the general framework for the Hahn-Banach theorem and its consequences.

Let (A,p_l) be a locally convex space. Then $B\subseteq A$ is bounded if and only if for each $l\in\mathbb{N}$

$$\sup_{x\in B} p_l(x)<\infty$$

Proposition 2.1. [23, Proposition 22.12] *Let E be a locally convex space, F be a subspace of E and p be a continuous seminorm on E. Then*

(a) *For each $y\in F^*$ there exists a $Y\in E^*$ with $Y|_F=y$;*
(b) *For each $z\in E$ there exists a $\sigma\in E^*$ with $\sigma(z)=p(z)$ and $|\sigma|\le p$;*
(c) *For each $x\in E$ with $x\neq 0$ there exists a $\sigma\in E^*$ with $\sigma(x)\neq 0$.*

Let E, F be linear spaces and ξ,η linear forms on E and F, respectively. Then the map $(x,y)\to\xi(x)\eta(y)$ is bilinear on $E\times F$. Hence for $u=\sum_{j=1}^n x_j\otimes y_j\in E\otimes F$, the map $u\to\sum_{j=1}^m\xi(x_j)\eta(y_j)$ is a linear form on $E\otimes F$.

Let p on E and q on F be seminorms. Then for $u \in E \otimes F$, we set

$$p \otimes_\varepsilon q(u) := \sup\{|\sum_{j=1}^{m} \xi(x_j)\eta(y_j)| : p^*(\xi) \leq 1 \text{ and } q^*(\eta) \leq 1\}$$

As usually $p^*(\xi) := \sup\{|\xi(x)| : p(x) \leq 1\}$ and $q^*(\eta)$ analogous.
$p \otimes_\varepsilon q$ is a seminorm on $E \otimes F$ and $p \otimes_\varepsilon q(x \otimes y) = p(x)q(y)$, for all $x \in E$ and $y \in F$.
Since $p \otimes_\varepsilon q$ increases when p and q are increased, the following definition makes sense. The ε-tensor product of two locally convex spaces E and F is their tensor product equipped with the uniquely defined locally convex topology on $E \otimes F$ given by the seminorms $p \otimes_\varepsilon q$ where p runs through the continuous seminorms on E, q those on F. It is denoted by $E \otimes_\varepsilon F$ and its completion is denoted by $E\hat{\otimes}_\varepsilon F$.
The injective tensor product $E \otimes_\varepsilon F$ is metrizable (resp. normable) if E and F are.

3. Fréchet α-lipschitz vector-valued operator algebras

Let (X, d) be a metric space with at least two elements and (A, p_l) be a Fréchet algebra over the scalar field F , for a constant $\alpha > 0$ and a function $f : X \to A$, set

$$q_{\alpha,l}(f) := \sup_{x \in X} p_l(f(x)) \quad \text{and} \quad p_{\alpha,l}(f) := \sup_{x \neq y} \frac{p_l(f(x) - f(y))}{d(x, y)^\alpha}$$

Definition 3.1. *Let (X, d) be a metric space with at least two elements, (A, p_l) be a Fréchet algebra and $\alpha > 0$. We define, the vector-valued Fréchet Lipschitz algebra, $\mathrm{Lip}_\alpha(X, A)$ of all functions such $f : X \to A$ satisfy in the following conditions:*
i) *$q_{\alpha,l}(f) < \infty$, for each $l \in \mathbb{N}$;*
ii) *$p_{\alpha,l}(f) < \infty$, for each $l \in \mathbb{N}$.*
The Lipschitz algebra $\mathrm{lip}_\alpha(X, A)$ is the subalgebra of $\mathrm{Lip}_\alpha(X, A)$ defined by

$$\mathrm{lip}_\alpha(X, A) := \Big\{f : X \to A \Big| \frac{p_l\Big(f(x) - f(y)\Big)}{d(x, y)^\alpha} \to 0 \text{ as } d(x, y) \to 0\Big\}$$

Let (X, d) be a metric space with at least two elements and (A, p_l) be a Fréchet algebra and $\alpha > 0$. We define

$$r_{\alpha,l}(f) := q_{\alpha,l}(f) + p_{\alpha,l}(f) \quad (f \in \mathrm{Lip}_\alpha(X, A))$$

We denote the set of all bounded continuous operators from X into A by $C_b(X, A)$, and define $q_l(f) := \sup_{x \in X} p_l(f(x))$ for each $f \in C_b(X, A)$.
Let $f, g \in C_b(X, A)$, and $\lambda \in \mathbb{C}$. Define

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x) \ (x \in X)$$

It is easy to see that $(C_b(X, A), q_l)$ becomes a Fréchet space over $\mathbb{C}$ and $\mathrm{Lip}_\alpha(X, A)$ is a linear subspace of $C_b(X, A)$.

Lemma 3.1. $(\mathrm{Lip}_\alpha(X, A), r_{\alpha,l})$ *is a Fréchet subalgebra of $C_b(X, A)$.*

Proof. Let $x \in X$ and $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $x_n \to x$. Then $d(x_n, x) \to 0$. Let $f \in \mathrm{Lip}_\alpha(X, A)$, since $p_{\alpha,l}(f) < \infty$ (for each $l \in \mathbb{N}$), thus there exist M_l such that $\frac{p_l(f(x_n) - f(x))}{d(x_n, x)^\alpha} \leq M_l$ for each $l \in \mathbb{N}$.
Hence $p_l(f(x_n) - f(x)) \leq M_l d(x_n, x)^\alpha \to 0$, so $p_l(f(x_n) - f(x)) \to 0$.
Thus $f(x_n) \to f(x)$ in the topology of A. It follows that f is continuous.
It is obvious that the sequence $(r_{\alpha,l})$ is a countable family of seminorms. We show that the sequence $(r_{\alpha,l})$ are submultiplicative seminorms i. e.

$$q_{\alpha,l}(fg) = \sup_{x \in X} p_l(fg(x)) \leq \sup_{x \in X} p_l(f(x)) \sup_{x \in X} p_l(g(x)) = q_{\alpha,l}(f)q_{\alpha,l}(g)$$

Also

$$\begin{aligned} p_{\alpha,l}(fg) &:= \sup_{x\neq y} \frac{p_l(fg(x)-fg(y))}{d(x,y)^\alpha} \\ &= \sup_{x\neq y} \frac{p_l(f(x)g(x)-f(y)g(x)+f(y)g(x)-f(y)g(y))}{d(x,y)^\alpha} \\ &\leq \sup_{x\neq y} \frac{p_l(f(x)-f(y))p_l(g(x))+p_l(f(y))p_l(g(x)-g(y))}{d(x,y)^\alpha} \\ &= q_{\alpha,l}(g)p_{\alpha,l}(f)+q_{\alpha,l}(f)p_{\alpha,l}(g) \end{aligned}$$

Hence we have:

$$\begin{aligned} r_{\alpha,l}(fg) &:= q_{\alpha,l}(fg)+p_{\alpha,l}(fg) \\ &\leq q_{\alpha,l}(f)q_{\alpha,l}(g)+q_{\alpha,l}(g)p_{\alpha,l}(f)+q_{\alpha,l}(f)p_{\alpha,l}(g) \\ &\leq (q_{\alpha,l}(f)+p_{\alpha,l}(f))(q_{\alpha,l}(g)+p_{\alpha,l}(g)) = r_{\alpha,l}(f)+r_{\alpha,l}(g) \end{aligned}$$

□

The Fréchet algebra $(\mathrm{Lip}_\alpha(X,A), r_{\alpha,l})$ will be called Fréchet Lipschitz algebra of order α on X. The elements of $\mathrm{Lip}_\alpha(X,A)$ and $\mathrm{lip}_\alpha(X,A)$ are called big and little Fréchet α-Lipschitz operators, respectively. Sometimes we call them Lipschitz operators, for short.

It is easy to show that if (X,d) be a metric space and (A,p_l) be a Fréchet algebra. Then $\mathrm{Lip}_\alpha(X,A)$ is a commutative (unital) Fréchet algebra if and only if A is a commutative (unital) Fréchet algebra.

Lemma 3.2. *Let (A,p_n) and (B,q_m) be Fréchet algebras and $\phi : A \to B$ be a continuous homomorphic, with dense range, and (e_α) be a bounded approximate identity for A. Then $(\phi(e_\alpha))$ is a bounded approximate identity for B.*

Proof. Suppose that $b \in B$, so there exist $(a_n) \subseteq A$, such that $\phi(a_n) \to b$, hence for all $m \in \mathbb{N}$, $q_m(\phi(a_n)-b) \to 0$. Let $a_{n_0} \in A$, since (e_α) is a bounded approximate identity for A, hence $a_{n_0}e_\alpha - a_{n_0} \to 0$, so $p_n(a_{n_0}e_\alpha - a_{n_0}) \to 0$ for all $n \in \mathbb{N}$. Also there exist $M > 0$, such that $p_n(e_\alpha) \leq M$.

Since ϕ is continuous, hence for all $m,n \in \mathbb{N}$ there exist $k > 0$ such that $q_m(\phi(e_\alpha)) \leq kp_n(e_\alpha) \leq kM$. Thus $(\phi(e_\alpha))$ is bounded. Moreover we have

$$\begin{aligned} q_m(b\phi(e_\alpha)-b) &= q_m(b\phi(e_\alpha)-\phi(a_{n_0})\phi(e_\alpha)+\phi(a_{n_0})\phi(e_\alpha)-\phi(a_{n_0})+\phi(a_{n_0})-b) \\ &\leq q_m(\phi(e_\alpha))q_m(b-\phi(a_{n_0})+q_m(\phi(a_{n_0}e_\alpha-a_{n_0}))+q_m(\phi(a_{n_0})-b) \\ &\leq kMq_m(b-\phi(a_{n_0})+kp_n(a_{n_0}e_\alpha-a_{n_0})+q_m(\phi(a_n)-b) \to 0 \end{aligned}$$

Hence for each $b \in B$, $b\phi(e_\alpha) \to b$. Similarly $\phi(e_\alpha)b \to b$. Therefore $(\phi(e_\alpha))$ is a bounded approximate identity for B. □

It is worth mentioning that, if (E,p_l) be a Fréchet algebra and $(A,\|\cdot\|)$ be a Banach algebra. Then $(A\hat{\otimes}E, s_l)$ is a Fréchet algebra where $s_l(a\otimes e) = \|a\|p_l(e)$, for all $a \in A$ and $e \in E$.

Lemma 3.3. *Let (X,d) be metric space, (A,p_l) be a Fréchet algebra and $\alpha > 0$. If (A,p_l) has a bounded approximate identity. Then $(\mathrm{Lip}_\alpha(X)\hat{\otimes}A, s_l)$ has a bounded approximate identity.*

Proof. Let (z_γ) be a bounded approximate identity for (A,p_l). Put $e_\gamma := 1\otimes z_\gamma$ we now show that (e_γ) is a bounded approximate identity for $\mathrm{Lip}_\alpha(X)\hat{\otimes}A$.

Let $f \in \mathrm{Lip}_\alpha(X)\hat{\otimes}A$. Then $f = \sum_{n=1}^\infty g_n \otimes a_n$, for some $g_n \in \mathrm{Lip}_\alpha(X)$ and $a_n \in A$ such that $\sum_{n=1}^\infty \|g_n\|_\alpha p_l(a_n) < \infty$.

Thus for each $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \|g_n\|_\alpha p_l(a_n) < \epsilon$.
So for each $\epsilon > 0$ we have:

$$\begin{aligned}
& s_l(\sum_{n=N+1}^{n} (g_n \otimes a_n) \cdot (1 \otimes z_\gamma) - \sum_{n=N+1}^{\infty} (g_n \otimes a_n)) \\
& \leq \sum_{n=N+1}^{\infty} s_l(g_n \otimes (a_n z_\gamma - a_n)) \\
& \leq \sum_{n=N+1}^{\infty} \|g_n\|_\alpha p_l(a_n z_\gamma - a_n) \\
& \leq \sum_{n=N+1}^{\infty} \|g_n\|_\alpha p_l(a_n z_\gamma) + \sum_{n=N+1}^{\infty} \|g_n\|_\alpha p_l(a_n) \\
& \leq (p_l(z_\gamma) + 1)\epsilon \\
& \leq (M + 1)\epsilon
\end{aligned}$$

where $M := \sup_\gamma\{p_l(z_\gamma) : l \in \mathbb{N}\} < \infty$.
Also we have

$$\begin{aligned}
& s_l(\sum_{n=1}^{N}(g_n \otimes a_n) \cdot (1 \otimes z_\gamma - \sum_{n=1}^{N}(g_n \otimes a_n)) \\
& \leq \sum_{n=1}^{N} s_l(g_n \otimes (a_n z_\gamma - a_n)) \\
& = \sum_{n=1}^{N} \|g_n\|_\alpha p_l(a_n z_\gamma - a_n) \to 0
\end{aligned}$$

Hence for each $f \in \mathrm{Lip}_\alpha(X)\hat{\otimes}A$, $s_l(f \cdot e_\gamma - f) \to 0$. Therefore (e_γ) is a bounded approximate for $\mathrm{Lip}_\alpha(X)\hat{\otimes}A$. □

Proposition 3.1. *Let (X, d) be metric space , (A, p_l) be a Fréchet algebra and $\alpha > 0$. Then the following statements are equivalent*
i) $\mathrm{Lip}_\alpha(X, A)$ *has a bounded approximate identity;*
ii) (A, p_l) *has a bounded approximate identity.*

Proof. (i) $\to$ (ii): Suppose that (e_β) be a bounded approximate identity for $\mathrm{Lip}_\alpha(X, A)$. For $x_0 \in X$, we define $z_\beta := e_\beta(x_0)$.

Let $f_z(x) = z$, for $x \in X$. Then $f_z \in \mathrm{Lip}_\alpha(X, A)$ and

$$\begin{aligned}
p_l(z_\beta.z - z) &= p_l(e_\beta \cdot f_z(x_0) - f_z(x_0)) \\
&\leq q_{\alpha,l}(e_\beta \cdot f_z - f_z) \\
&\leq r_{\alpha,l}(e_\beta \cdot f_z - f_z) \to 0
\end{aligned}$$

Hence, $z_\beta \cdot z \to z$, for all $z \in A$. Similarly $z \cdot z_\beta \to z$. Also, $p_l(z_\beta) = p_l(e_\beta(x_0)) \leq q_{\alpha,l}(e_B) \leq r_{\alpha,l}(e_\beta) \leq M$, for some $M > 0$ and each β. Hence (z_β) is a bounded approximate identity of A.
(ii) $\to$ (i): Let (e_β) be a bounded approximate identity for (A, p_l). For every $h \in \mathrm{Lip}_\alpha(X)\hat{\otimes}A$, there exist, $(f_i)_{i\in\mathbb{N}} \in \mathrm{Lip}_\alpha(X)$ and $(a_i)_{i\in\mathbb{N}} \in A$ such that, $h = \sum_{i=1}^{\infty}(f_i \otimes a_i)$. Define

$$T : \mathrm{Lip}_\alpha(X)\hat{\otimes}A \to \mathrm{Lip}_\alpha(X, A)$$

$$T(h) = \sum_{i=1}^{\infty} f_i a_i$$

and for all $x \in X$, we have $f\cdot a(x) = f(x)a$. Then T is continuos homorphic, with dense range. Thus by using lemma 3.3, $(1 \otimes e_\beta)$ is a bounded approximate identity for $\mathrm{Lip}_\alpha(X, A)$. □

The following result is a generalization of [Proposition 2.1] for the Fréchet case.

Lemma 3.4. *Let (A, p_l) be a Fréchet algebra and $z \in A$. Then*

$$p_l(z) = \sup\{|\sigma(z)| : \sigma \in A^* \text{ and } |\sigma| \le p_l\}$$

Proof. Since p_l is a seminorm on A, so there is a $\sigma_0 \in A^*$ such that

$$\sigma_0(z) = p_l(z) \quad \text{and} \quad |\sigma_0| \le p_l$$

Hence we have

$$p_l(z) = \sigma_0(z) \le \sup\{|\sigma(z)| : \sigma \in A^* \text{ and } |\sigma| \le p_l\} \tag{1}$$

Conversly, suppose that $|\sigma| \le p_l$. Then we have $|\sigma(z)| \le p_l(z)$, so

$$\sup\{|\sigma(z)| : \sigma \in A^* \text{ and } |\sigma| \le p_l\} \le p_l(z) \tag{2}$$

By using (1) and (2), we have $p_l(z) = \sup\{|\sigma(z)| : \sigma \in A^* \text{ and } |\sigma| \le p_l\}$. □

The following result is a generalization of [5, Lemma 1.1], for the Fréchet case.

Theorem 3.1. *Let (X, d) be a metric space with at least two elements, (A, p_l) be a Fréchet space, $\alpha > 0$ and $f \in B(X, A)$. Then the following statements are equivalent.*
(i) *$f \in \mathrm{Lip}_\alpha(X, A)$;*
(ii) *$\sigma o f \in \mathrm{Lip}_\alpha(X)$, for all $\sigma \in A^*$.*

Proof. $(i) \to (ii)$ Suppose that $f \in \mathrm{Lip}_\alpha(X, A)$ and $\sigma \in A^*$. Then there exists $c > 0$ and $l \in \mathbb{N}$ such that $|\sigma(z)| \le c p_l(z)$, for all $z \in A$. Especially for all $x \in X$, we have $|\sigma(f(x))| \le c p_l(f(x))$.
On the other hand $q_{\alpha,l}(f) := \sup_{x\in X} p_l(f(x)) < \infty$. Hence $|\sigma(f(x))| \le c q_{\alpha,l}(f) < \infty$ and $\sigma o f$ is bounded. Also

$$\begin{aligned} p_\alpha(\sigma o f) &= \sup_{x\neq y} \frac{|\sigma o f(x) - \sigma o f(y)|}{d(x,y)^\alpha} = \sup_{x\neq y} \frac{|\sigma(f(x) - f(y))|}{d(x,y)^\alpha} \\ &\le \sup_{x\neq y} \frac{c p_l(f(x) - f(y))}{d(x,y)^\alpha} = c p_{\alpha,l}(f) < \infty \end{aligned}$$

Therefore $\sigma o f \in \mathrm{Lip}_\alpha(X)$.

$(ii) \to (i)$ Suppose that $\sigma \in A^*$. We define $T_\sigma : \mathrm{Lip}_\alpha(X, A) \to \mathrm{Lip}_\alpha(X)$ by $T_\sigma(f) = \sigma o f$, for all $f \in \mathrm{Lip}_\alpha(X, A)$. Then $\{T_\sigma\}_{\sigma\in A^*}$ is a family of continuous linear maps. Since $\sigma \in A^*$, so there exists $c > 0$ and $l \in \mathbb{N}$ such that for all $x \in X$,

$$|\sigma(f(x))| \le c p_l(f(x))$$

Hence $\|\sigma o f\|_\infty \le c q_{\alpha,l}(f)$ and we have

$$p_\alpha(\sigma o f) = \sup_{x\neq y} \frac{|(\sigma o f)(x) - (\sigma o f)(y)|}{d(x,y)^\alpha} \le c \sup_{x\neq y} \frac{p_l\Big(f(x) - f(y)\Big)}{d(x,y)^\alpha} = c p_{\alpha,l}(f)$$

Thus for each $f \in \mathrm{Lip}_\alpha(X, A)$,

$$\begin{aligned} ||T_\sigma(f)||_\alpha &= ||\sigma o f||_\alpha \\ &= ||\sigma o f||_\infty + p_\alpha(\sigma o f) \\ &\le (c q_{\alpha,l}(f) + c p_{\alpha,l}(f)) \\ &= c r_{\alpha,l}(f) \end{aligned}$$

Therefore T_σ is continuous.
Now, we show that $f \in \mathrm{Lip}_\alpha(X,E)$. For each $l \in \mathbb{N}$ there exists $\sigma_l \in A^*$:

$$p_l(f(x)-f(y)) = \sigma_l(f(x)-f(y))$$

Thus:

$$\begin{aligned} p_{\alpha,l}(f) &= \sup_{x\neq y} \frac{p_l(f(x)-f(y))}{d(x,y)^\alpha} \\ &= \sup_{x\neq y} \frac{|\sigma_l(f(x))-\sigma_l(f(y))|}{d(x,y)^\alpha} = p_\alpha(\sigma_l o f) < \infty \end{aligned}$$

Also

$$\begin{aligned} q_{\alpha,l}(f) &= \sup_{x\in X} p_l(f(x)) \\ &= \sup_{\substack{\sigma\in E^* \\ |\sigma|\leq p_l}} |\sigma(f(x))| = q_\alpha(\sigma o f) < \infty \end{aligned}$$

Therefore $f \in \mathrm{Lip}_\alpha(X,E)$. □

Corollary 3.1. *Let (X,d) be metric space and (A,p_l) be a Fréchet algebra. Then $\mathrm{Lip}_\alpha(X,A)$ is the maximal subalgebra of $C_b(X,A)$ satisfying*

$$A^* \circ \mathrm{Lip}_\alpha(X,A) \subseteq \mathrm{Lip}_\alpha(X)$$

Proof. Suppose that E is a subalgebra of $C_b(X,A)$ satisfying $A^* \circ E \subseteq \mathrm{Lip}_\alpha(X)$, we show that E is contained in $\mathrm{Lip}_\alpha(X,A)$. To see this, let $f \in E$ then for every $\sigma \in E$ we have, $\sigma o f \in \mathrm{Lip}_\alpha(X)$ by Theorem 3.1, so $f \in \mathrm{Lip}_\alpha(X,A)$, which implies that $E \subseteq \mathrm{Lip}_\alpha(X,A)$. □

Let (A,p_l) and (B,q_l) be Fréchet space and $\phi : A \to B$ be a linear bijecion map such that $q_l(\phi(x)) = p_l(x)$ for each $x \in A$. Then the metric induced by (p_l) and (q_l) are isometric.

Proposition 3.2. *Let (X,d) be a compact metric space and (A,p_l) be a Fréchet algebra. Then $C_b(X) \otimes_\varepsilon A$ is isometrically isomorphic to $C_b(X,A)$.*

Proof. For all $f \in C_b(X)$ and $a \in A$, $f.a \in C_b(X,A)$ where by $f.a(x) = f(x)a$, $(x \in X)$.The mapping $(f,a) \to f.a$ from $C_b(X) \times A$ into $C_b(X,A)$ is bilinear. Hence there exists a unique linear map $\phi : C_b(X) \otimes A \to C_b(X,A)$ such that $\phi(f \otimes a)(x) = f(x)a$, for all $f \in C_b(X)$, $x \in X$ and $a \in A$. Clearly, ϕ is a homomorphism.
For $u = \sum_{j=1}^n f_i \otimes a_i \in C_b(X) \otimes A$ and $l \in \mathbb{N}$ we have

$$\begin{aligned} q_l(\phi(u)) &= \sup\{p_l(\phi(u)(x)) : x \in X\} \\ &= \sup\{p_l(\sum_{i=1}^n f_i(x)a_i) : x \in X\} \\ &= \sup_{x\in X}\{\sup\{|\sigma(\sum_{i=1}^n f_i(x)a_i| : \sigma \in A^* \text{ and } |\sigma| \leq p_l\} \\ &= \sup_{x\in X}\{\sup\{|\sum_{i=1}^n f_i(x)\sigma(a_i)| : \sigma \in A^* \text{ and } |\sigma \leq p_l\} \\ &= \sup_{\sigma\in A^*} \sup_{x\in X}\{|\sum_{i=1}^n f_i(x)\sigma(a_i)| : |\sigma| \leq p_l\} \\ &= \sup_{\sigma\in A^*}\{||\sum_{i=1}^n f_i\sigma(a_i)||_\infty : |\sigma| \leq p_l\} \end{aligned}$$

$$= \sup_{\sigma \in A^*} \sup_{\mu \in C(X)_1^*} \{|\sum_{i=1}^{n} \sigma(a_i)\mu(f_i)| : |\sigma| \leq p_l\}$$
$$= \sup\{|\sum_{i=1}^{n} \sigma(a_i)\mu(f_i) : p_l^*(\sigma) \leq 1 \text{ and } \mu \in C_b(X)_1^*\}$$
$$= (||.|| \otimes p_l)(u)$$

Thus ϕ is an isometry. It remains to show that $\phi(C_b(X) \otimes A)$ is dense in $C_b(X, A)$. Choose a seminorm p on A and $\varepsilon > 0$. For $f \in C_b(X, A)$, we can find $a_1, \cdots, a_m \in A$ such that $X = \cup_{j=1}^{m}\{x|p(a_j - f(x)) < \varepsilon\}$. Put $\phi_j(x) = \max\{0, \varepsilon - p(a_j - f(x))\}$ and $\phi = \sum_{j=1}^{m} \phi_j$. Note that $\phi(x) > 0$, for all $x \in X$. Put

$$u = \sum_{j=1}^{m} \frac{\phi_j}{\phi} a_j \in C_b(X) \otimes A$$

Then we have for $x \in X$

$$p(u(x) - f(x)) = p(\sum_{j=1}^{m} \frac{\phi_j(x)}{\phi(x)}(a_j - f(x))$$
$$\leq \sum_{j=1}^{m} \frac{\phi_j(x)}{\phi(x)} p(a_j - f(x)) < \varepsilon$$

Hence $\overline{C_b(X) \otimes_\epsilon A} = C_b(X, A)$ □

Now, as a consequence of Theorem 3.2, we generalize the following result on vector-valued Lipschitz algebras for Fréchet algebra. Although, this result has already been proved in [26] for Banach algebra.

Corollary 3.2. *Let (X, d) be a compact metric space with at least two elements, (A, p_l) be a Fréchet algebra and $\alpha > 0$. Then*

$$\overline{\mathrm{Lip}_\alpha(X, A)} = C_b(X, A)$$

Proof. Let $h \in C_b(X, A)$. Since $C_b(X) \otimes_\varepsilon A = C_b(X, A)$, thus there exists $f_1, f_2, \ldots \in C_b(X)$ and $a_1, a_2, \ldots \in A$ such that $h = \sum_{i=1}^{\infty} f_i \cdot a_i$.
In other hand we have $C_b(X) = \overline{\mathrm{Lip}_\alpha(X)}$. Hence for each $i \in \mathbb{N}$ there exist $g_i \in \mathrm{Lip}_\alpha(X)$ such that

$$||g_i - f_i||_\infty < \frac{\varepsilon}{2^i p_l(a_i)}$$

Therefore for each $\varepsilon > 0$ and for each $x \in X$,

$$|g_i(x) - f_i(x)| < \frac{\varepsilon}{2^i p_l(a_i)}$$

Now, we define $\phi_i : \mathbb{C} \to A$, by $\phi_i(\lambda) = \lambda \cdot a_i$.
Then $\phi_i o g_i \in \mathrm{Lip}_\alpha(X, A)$. Since

$$q_{\alpha,l}(\phi_i o g_i) = \sup_{x \in X} p_l(\phi_i(g_i(x)) = \sup_{x \in X} p_l(g_i(x) \cdot a_i)$$
$$= ||g_i||_\infty p_l(a_i) < \infty$$

We have

$$p_{\alpha,l}(\phi_i o g_i) = \sup_{x \neq y} \frac{p_l((\phi_i o g_i)(x) - (\phi_i o g_i)(y))}{d(x, y)^\alpha}$$
$$= \sup_{x \neq y} \frac{p_l(g_i(x) \cdot a_i - g_i(y) \cdot a_i)}{d(x, y)^\alpha}$$

$$= \sup_{x \neq y} \frac{|g_i(x) - g_i(y)|}{d(x,y)^\alpha} p_l(a_i)$$
$$\leq ||g_i||_\infty p_l(a_i) < \infty$$

Therefore

$$\begin{aligned} q_{\alpha,l}(h - \sum_{i=1}^{\infty} \phi_i o g_i) =& q_{\alpha,l} \sum_{i=1}^{\infty} (f_i \cdot a_i - g_i \cdot a_i) \\ &= \sup_{x \in X} p_l \Big(\sum_{i=1}^{\infty} f_i(x) \cdot a_i - g_i(x) \cdot a_i \Big) \\ &\leq \sup_{x \in X} \sum_{i=1}^{\infty} |f_i(x) - g_i(x)| p_l(a_i) \\ &\leq \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i p_l(a_i)} \cdot p_l(a_i) = \varepsilon \end{aligned}$$

□

4. **Ideal amenability of** $\mathrm{Lip}_\alpha(X, A)$

In this section, we study the ideal amenability of vector-valued Fréchet lipschitz algebra $\mathrm{Lip}_\alpha(X, A)$.

Let (A, p_l) be a Fréchet algebra and I be a closed (two-sided) ideal in A. Similar to the Banach algebra case, we say that A is I-weakly amenable if every continuous derivation $D : A \to I^*$ is inner. Moreover we introduce the concept of ideal amenability for Fréchet algebras as the following:

A Fréchet algebra A is called ideally amenable if it is I-weakly amenable, for every closed (two-sided) ideal I in A.

Let A be a Fréchet algebra and $\phi \in \Delta(A)$, the set consisting of all non-zero continuous characters on A. A point derivation d at ϕ is a linear functional satisfying $d(xy) = d(x)\phi(y) + \phi(x)d(y)$, where $x, y \in A$, i.e. d is a derivation into the A-bimodule $\mathbb{C}$, where the module actions is defined by $x \cdot \lambda = \lambda \cdot x = \lambda\phi(x), x \in A$ and $\lambda \in \mathbb{C}$.

The following lemma is immediate.

Lemma 4.1. *Let (X, d) be a metric space with at least two elements, (A, p_l) be a Fréchet algebra and $\alpha > 0$. For each non-isolated point $x \in X$ and $\sigma \in A^*$. Let $\phi : \mathrm{Lip}_\alpha(X, A) \to \mathbb{C}$ is given by $\phi(f) = (\sigma o f)(x)$, ($f \in \mathrm{Lip}_\alpha(X, A)$) then $\phi \in \Delta(\mathrm{Lip}_\alpha(X, A))$.*

Let $\mathbb{C}^{\mathbb{N}} := \{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{C}$ and $n \in \mathbb{N}\}$ be the set of all complex sequences. Then $\mathbb{C}^{\mathbb{N}}$ becomes an algebra by defining algebraic operations coordinatewise. For each $n \in \mathbb{N}$, the function $p_n : \mathbb{C}^{\mathbb{N}} \to \mathbb{R}$ define by $p_n(x) = \max_{1\leq m\leq n} |x_m|$ defines a submultiplicativ seminorm on $\mathbb{C}^{\mathbb{N}}$.

Clearly $\{p_n\}$ defines the product topology on $\mathbb{C}^{\mathbb{N}}$, under which $\mathbb{C}^{\mathbb{N}}$ is complete [13]. Thus, $(\mathbb{C}^{\mathbb{N}}, p_n)$ becomes a Hausdorff Fréchet algebra.

Similar to the Banach algebra case in [24], we say that a bounded linear functional LIM: $(\mathbb{C}^{\mathbb{N}}, p_n) \to \mathbb{C}$ is is Fréchet limit if for every $x = (x_n) \in \mathbb{C}^{\mathbb{N}}$, $y = (y_n) \in \mathbb{C}^{\mathbb{N}}$ and $\alpha \in \mathbb{C}$ satisfying

(1) LIM is linear: $\mathrm{LIM}(x + y) = \mathrm{LIM}(x) + \mathrm{LIM}(y)$ and $\mathrm{LIM}(\alpha x) = \alpha\,\mathrm{LIM}(x)$.
(2) LIM is positive: $\mathrm{LIM}(x) \geq 0$, for every $x_n \geq 0$.
(3) LIM is normalized: $\mathrm{LIM}(\mathbf{1}) = 1$, where $\mathbf{1} = (1, 1, \cdots)$.
(4) LIM is shift invariant: $\mathrm{LIM}(sx) = \mathrm{LIM}(x)$.

The above properties on the functional LIM imply the following:

(5) LIM extends lim on the subspace of convergent sequences:

$$\lim_{n\to\infty} x_n = c \Rightarrow \mathrm{LIM}(x) = c$$

(6) If $\lim_{n\to\infty} x_n = c$ and $y = (y_n) \in \mathbb{C}^{\mathbb{N}}$, then $\mathrm{LIM}(x_n y_n) = c\,\mathrm{LIM}(y_n)$.
Let (X,d) be a fixed non-empty compact metric space, set

$$\Delta := \{(x,y) \in X \times X : x = y\} \text{ and } W := X \times X - \Delta$$

The following result is a generalization of [26, Theorem 3.4] for the Fréchet case.

Theorem 4.1. *Let (X,d) be an infinite compact metric space, (A,p_l) be a Fréchet algebra and $0 < \alpha \le 1$. Then $\mathrm{Lip}_\alpha(X,A)$ is not ideally amenable.*

Proof. Let x be a non-isolated point in K. We define

$$W_x := \{(x_n,y_n) : (x_n,y_n) \in W,\ \lim_{n\to\infty}(x_n,y_n) = (x,x)\}$$

For the net $W = \{(x_n,y_n)\}$ in W_x and $\sigma \in A$, we put

$$W_n : \mathrm{Lip}_\alpha(X,A) \to \mathbb{C}$$

defined by

$$W_n(f) = \frac{(\sigma o f)(x_n) - (\sigma o f)(y_n)}{d(x_n,y_n)^\alpha} \quad \Big(f \in \mathrm{Lip}_\alpha(X,A)\Big)$$

Since $f \in \mathrm{Lip}_\alpha(X,A)$ and $\sigma \in A^*$, so $\sigma o f : X \to \mathbb{C}$ is continues. Hence

$$\begin{aligned} \text{There exists } c > 0 \text{ and } l \in \mathbb{N} \text{ such that } &|(\sigma o f)(x_n) - (\sigma o f)(y_n)| = |\sigma(f(x_n) - f(y_n))| \\ &\le c p_l(f(x_n) - f(y_n)) \le c p_{\alpha,l}(f) d(x_n,y_n)^\alpha \end{aligned}$$

Thus

$$|\langle f, W_n\rangle| = |\frac{(\sigma o f)(x_n) - (\sigma o f)(y_n)}{d(x_n,y_n)^\alpha}| \le c p_{\alpha,l}(f)$$

Therefore, W_n is continues. Now set

$$D : \mathrm{Lip}_\alpha(X,A) \to \mathbb{C}, \text{ by } D(f) = \mathrm{LIM}(W_n(f))$$

We show that the linear map D is a non-zero point derivation at ϕ, for which ϕ is given by $\phi(f) = (\sigma o f)(x), (f \in \mathrm{Lip}_\alpha(X,A))$.
We have

$$\begin{aligned} D(fg) = \mathrm{LIM}(W_n(fg)) &= \mathrm{LIM}\, \frac{(\sigma o f g)(x_n) - (\sigma o f g)(y_n)}{d(x_n,y_n)^\alpha} \\ &= \mathrm{LIM}\, \frac{1}{d(x,y)^\alpha}(\sigma o(f(x_n)g(x_n) - f(y_n)g(y_n)) \\ &= \mathrm{LIM}\, \frac{1}{d(x,y)^\alpha}\Bigg(\sigma o\Big(f(x_n)\big(g(x_n) - g(y_n)\big) + \big(f(x_n) - f(y_n)\big)g(y_n)\Big)\Bigg) \\ &= (\sigma o f)(x)\,\mathrm{LIM}(W_n(g)) + (\sigma o g)(x)\,\mathrm{LIM}(W_n(f)) \\ &= \phi(f)D(g) - \phi(g)D(f) \end{aligned}$$

Therefore, by the continuity f, g and properties of Fréchet limit, we conclude D is a non-zero, continues point derivation at ϕ on $\mathrm{Lip}_\alpha(X,A)$. Hence by using [20, Proposition 3.1] $\mathrm{Lip}_\alpha(X,A)$ is not ideally amenable. □

Example 4.1. *Let (A,p_l) be a commutative Fréchet algebra, $T = \{z \in \mathbb{C} : |z| = 1\}$ be the group of complex numbers of modulus one, and $\alpha > \frac{1}{2}$. Then by using* [16, Proposition 3.5] *and* [20, Theorem 4.2] *the proof is immediate. Also $\mathrm{lip}_\alpha(T)\hat{\otimes}A$ is not ideally amenable.*

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