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AVERAGE MALIČKY-RIEČAN'S ENTROPY OF DOUBLY STOCHASTIC OPERATORS

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We review the concept of Maličky-Riečan's entropy for doubly stochastic operators. Then we define a map which has similar structure of Maličky-Riečan's entropy in a local manner. It results in a type of entropy called average Maličky-Riečan's entropy. Then the relationship between this quantity and the Maličky-Riečan's entropy is studied.

Keywords: Doubly stochastic operator, Maličky-Riečan's entropy, Maličky-Riečan's entropy map.

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1. Introduction

There are many generalizations of the concept of entropy of dynamical systems introduced by Kolmogorov and Sinai [11, 24]. An effective generalization is based on the notion of partitions of unity, i.e., a finite collection of non negative measurable functions whose sum is 1. It obviously generalizes the classical partitions, if we replace any atom of the partition by the corresponding characteristic function. On the other hand, the definition of entropy depends on the operation defined between the partitions. According to these optional variations, we have various kinds of entropy of dynamical systems which are expected to have some logical properties generalizing the Kolmogorov entropy [4, 5, 8, 13, 21, 22].

Another aspect of the generalization of the concept of entropy is to work with operators on function spaces, instead of classical dynamical systems. This results in the definition of entropy for doubly stochastic operators [3, 10].

A general framework, based on an axiomatic approach, is given in [3]. It covers most of the known definitions of entropy and generalizes the Kolmogorov entropy as well.

Local approaches to the entropy of classical dynamical systems [1, 15, 16, 23] as well as generalized cases [17, 18, 19, 20] are discussed extensively.

The idea of Maličky-Riečan's entropy, for classical dynamical systems, was to resolve a defect in the definition of entropy of dynamical systems when replacing measurable partitions by partitions of unity [4]. In the definition of entropy in the sense of Dumitrescu [4], if the elements of partitions of unity come from a family containing all constant functions then the entropy $h_\mu(T)$ equals infinity. Thereafter, the Hudetz entropy $h^b_\mu(T)$ [9] and the Maličky-Riečan's entropy $h^{MR}_\mu(T)$ [12] were introduced to resolve this defect, since it is proved that

$$h^b_\mu(T) \leq h_\mu(T), \quad h^{MR}_\mu(T) \leq h_\mu(T).$$

So, the Maličky-Riečan's entropy, as well as the Hudetz entropy, is a modification of the concept of entropy defined via partitions of unity by giving a smaller quantity for entropy of dynamical systems.

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In this paper, we attempt to define a map which locally resembles the Maličky-Riečan's entropy [7, 12] of doubly stochastic operators. Note that, since the Maličky-Riečan's construction lacks a joining operation, it does not follow the axiomatic approach presented in [3]. So, the locally defined map in this paper, also does not contain any joining operation. Then, using this map, we define *average Maličky-Riečan's entropy* for doubly stochastic operators on the space of Borel measurable functions on compact meatric spaces. We state and prove some properties of the new quantity and present its relationship with the Maličky-Riečan's entropy.

In section 2, we recall some preliminary facts which are necessary for the rest of the paper. In section 3, we define the Maličky-Riečan's entropy map of doubly stochastic operators which is applied to define the average Maličky-Riečan's entropy. We prove some of the properties of the introduced map. Then we conclude this section by connecting the average Maličky-Riečan's entropy to the Maličky-Riečan's entropy of doubly stochastic operators. In section 4, we give a summary and discussion.

2. **Preliminary facts**

In this section, we review the concept of Maličky-Riečan's entropy [7] for doubly stochastic operators.

Let $(X, \mathcal{B})$ be a measurable space and μ be a probability measure on X. A doubly stochastic operator is a linear operator $T : L^1(\mu) \to L^1(\mu)$ (or $T : L \to L$ where L is a suitable sub space of $L^1(\mu)$) satisfying the following conditions [6]:

(1) If $f \geq 0$ then $Tf \geq 0$, for every $f \in L^1(\mu)$.
(2) $T1 = 1$ where $1(x) = 1$ is the constant function.
(3) $\int_X Tf d\mu = \int_X f d\mu$ for every $f \in L^1(\mu)$.

By a partition of unity, we mean a finite family $\Phi = \{\phi\}$ of measurable functions $\phi : X \to [0, 1]$ such that $\sum_{\phi\in\Phi} \phi = 1$. As a special case, if $\mathcal{A}$ is a finite measurable partition of X, then $1_{\mathcal{A}} = \{\chi_A : A \in \mathcal{A}\}$ is a partition of unity. We have the following order on the set of partitions of unity:

"$\Phi \prec \Psi$ if for each $\phi \in \Phi$ there exists a subset $\Psi_\phi \subseteq \Psi$ such that $\phi = \sum_{\psi\in\Psi_\phi} \psi$."

We also have the following joining operation on the set of partitions of unity:

$$\Phi \vee \Psi = \{\phi\psi \ : \ \phi \in \Phi, \ \psi \in \Psi\}.$$

The entropy of a partition of unity Φ is defined by

$$H_\mu(\Phi) := \sum_{\phi\in\Phi} \eta(m(\phi))$$

where $m(\phi) = \int_X \phi d\mu$, $\eta(s) = -s \log s$ $(s > 0)$ and $\eta(0) = 0$.

If $\Phi_1, \Phi_2, \cdots, \Phi_n$ are several partitions of unity, we set

$$H^{MR}_\mu(\Phi_1, \cdots, \Phi_n) := \inf\{H_\mu(\Gamma) : \Phi_1 \prec \Gamma, \cdots, \Phi_n \prec \Gamma\}.$$

Specially, if Φ is a partition of unity, then

$$H^{MR}_\mu(\Phi, n) := H^{MR}_\mu(\Phi, T\Phi, \cdots, T^{n-1}\Phi) = \inf\{H_\mu(\Gamma) : \Phi \prec \Gamma, \cdots, T^{n-1}\Phi \prec \Gamma\}.$$

The Maličky-Riečan's entropy of a doubly stochastic operator T with respect to Φ is defined by

$$h^{MR}_\mu(T, \Phi) := \lim_{n\to+\infty} \frac{1}{n} H^{MR}_\mu(\Phi, n).$$

Note that, the sequence $\{H^{MR}_\mu(\Phi, n)\}_{n\geq1}$ is sub additive [7, 12], so, the limit above exists.

Finally, the Maličky-Riečan's entropy of T is defined by

$$h^{MR}_\mu(T) := \sup_\Phi h^{MR}_\mu(T, \Phi)$$

where the supremum is taken over all measurable partitions of unity [7].

We recall that the L^1-distance of two partitions of unity $\Phi = \{\phi_1, \cdots, \phi_r\}$ and $\Psi = \{\psi_1, \cdots, \psi_{r'}\}$ $(r' \leq r)$ is defined as

$$\text{dist}(\Phi, \Psi) := \min_{\pi}\{\max_{1 \leq i \leq r} \int_X |\phi_i - \psi_{\pi(i)}| d\mu\}$$

where the minimum ranges over all permutations of the set $\{1, 2, \cdots, r\}$, and where Ψ is considered as a family of measurable functions with r elements by setting $\psi_i = 0$ for $r' < i \leq r$ [3].

Note that, since the function η on $[0, +\infty)$ is continuous and $H_\mu(\Gamma) = \sum_{\phi \in \Gamma} \eta(m(\phi))$, one can easily conclude the following lemma.

Lemma 2.1. *Given any $r \geq 1$ and $\epsilon > 0$, there exists $\delta = \delta_\epsilon > 0$ such that, for any partitions of unity Γ, $\tilde{\Gamma}$ and $\tilde{\tilde{\Gamma}}$ with cardinality at most r, if* $\text{dist}(\Gamma, \tilde{\Gamma}) < \delta$ *then* $\text{dist}(\Gamma \vee \tilde{\tilde{\Gamma}}, \tilde{\Gamma} \vee \tilde{\tilde{\Gamma}}) < \delta$ *and* $|H_\mu(\Gamma) - H_\mu(\tilde{\Gamma})| < \epsilon$.

The following proposition shows the continuity of Maličky-Riečan's entropy.

Proposition 2.1. *The following continuity laws hold:*

(1) *For every $\epsilon > 0$ and $r \in \mathbb{N}$ there exists $\delta > 0$ such that if Φ and Ψ have at most r elements and satisfy* $\text{dist}(\Phi, \Psi) < \delta$ *then for every $n \in \mathbb{N}$,*

$$\left|H_\mu^{MR}(\Phi, n) - H_\mu^{MR}(\Psi, n)\right| < n\epsilon.$$

(2) *For every $\epsilon > 0$ and $r \in \mathbb{N}$ there exists $\delta > 0$ such that if Φ and Ψ have at most r elements and* $\text{dist}(\Phi, \Psi) < \delta$ *then*

$$\left|h_\mu^{MR}(T, \Phi) - h_\mu^{MR}(T, \Psi)\right| < \epsilon.$$

See [7], Proposition 3.6 for a proof of Proposition 2.1.

3. Average Maličky-Riečan's entropy

In this section, we present a local view to Maličky-Riečan's entropy of a doubly stochastic operator. Then we apply it to define another version of the Maličky-Riečan's entropy. Let X be a compact metric space and $\mathcal{B} = \mathcal{B}_X$ be the Borel σ-algebra. Let also, L be the collection of all Borel measurable functions with range in $[0, 1]$ and $T : L \to L$ be a stochastic operator. Denote the set of all Borel probability measures on X by $\mathcal{M}(X)$ and set

$$\mathcal{M}_T(X) = \{\mu \in \mathcal{M}(X) \ : \ \int_X Tf d\mu = \int_X f d\mu \quad \forall f \in L\}$$

and

$$\mathcal{E}_T(X) = \{\mu \in \mathcal{M}_T(X) \ : \ Tf = f \text{ implies } f \text{ is constant}\}.$$

Obviously, $\mathcal{M}_T(X)$ is a compact convex set in the weak* topology and $\mathcal{E}_T(X)$ is the set of extreme points of $\mathcal{M}_T(X)$.

3.1. Local Approach

We first define an entropy like map of a doubly stochastic operator which locally resembles the Maličky-Riečan's entropy of doubly stochastic operators. Since X is a compact metric space then $C(X)$, the space of all continuous real valued functions on X, is separable [25]. Let $\mathcal{K}$ be a countable dense subset of $C(X)$ and let $\mathcal{D}$ be the collection of all rational linear combinations of elements of $\{1\} \cup \mathcal{K}$, which will be fixed in the rest of the paper. Let also, $\mathcal{P}$ be the family of all partitions of unity and $\mathcal{P}_\mathcal{D}$ be the family of all partitions of unity with elements in $\mathcal{D}$.

For $x \in X$ and $\phi \in L$, set

$$\omega(x,\phi) := \limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} T^j(\phi)(x).$$

If Φ is a partition of unity then we set

$$\Omega(x,\Phi) := \sum_{\phi\in\Phi} \eta(\omega(x,\phi)).$$

Now, for $n \in \mathbb{N}$, set

$$\mathcal{H}_n^{MR}(x,\Phi) := \inf\{\Omega(x,\Gamma) \ : \ T^k\Phi \prec \Gamma,\ k = 0,1,2,\cdots,n-1\}. \tag{1}$$

Finally, for a partition of unity Φ, we define the *Maličky-Riečan's entropy map* $\mathcal{J}^{MR}(\cdot,\Phi) : X \to [0,+\infty]$ by

$$\mathcal{J}^{MR}(x,\Phi) := \limsup_{n\to+\infty} \frac{1}{n}\mathcal{H}_n^{MR}(x,\Phi).$$

For two partitions of unity Φ and Ψ, we define the following conditional version of $\Omega(x,\Phi)$:

$$\Omega(x,\Phi|\Psi) := -\sum_{\phi\in\Phi,\psi\in\Psi} \omega(x,\phi\psi)\log\frac{\omega(x,\phi\psi)}{\omega(x,\psi)}.$$

Note that, the previous quantity will be set to be 0 if any of the factors in the summation of the right hand side equals to 0.

3.2. Primary properties

We have some properties of the local concepts defined above in the following.

Proposition 3.1. *Let $x \in X$, and Φ, Ψ and Λ be three partitions of unity of X. Then,*

(1) $\Omega(x,\Phi\vee\Psi|\Lambda) \geq \Omega(x,\Phi|\Lambda) + \Omega(x,\Psi|\Lambda\vee\Phi)$.
(2) $\Omega(x,\Phi\vee\Psi) \geq \Omega(x,\Phi) + \Omega(x,\Psi|\Phi)$.

Proof.

(1) Let $\Phi = \{\phi\}$, $\Psi = \{\psi\}$ and $\Lambda = \{\lambda\}$ be three partitions of unity. Assume, without loss of generality, that all the functions have the property that $\omega(x,f) \neq 0$. We have

$$\begin{aligned}
\Omega(x,\Phi\vee\Psi|\Lambda) &= -\sum_{\phi,\psi,\lambda}\omega(x,\phi\psi\lambda)\log\frac{\omega(x,\phi\psi\lambda)}{\omega(x,\lambda)}\\
&= -\sum_{\phi,\psi,\lambda}\omega(x,\phi\psi\lambda)\log\frac{\omega(x,\phi\lambda)}{\omega(x,\lambda)} - \sum_{\phi,\psi,\lambda}\omega(x,\phi\psi\lambda)\log\frac{\omega(x,\phi\psi\lambda)}{\omega(x,\phi\lambda)}\\
&= -\sum_{\phi,\psi,\lambda}\omega(x,\phi\psi\lambda)\log\frac{\omega(x,\phi\lambda)}{\omega(x,\lambda)} + \Omega(x,\Psi|\Phi\vee\Lambda). \qquad (2)
\end{aligned}$$

Also, the next inequality follows from $\sum_\psi \omega(x,\phi\psi\lambda) \geq \omega(x,\phi\lambda)$:

$$-\sum_{\phi,\psi,\lambda}\omega(x,\phi\psi\lambda)\log\frac{\omega(x,\phi\lambda)}{\omega(x,\lambda)} \geq \Omega(x,\Phi|\Lambda). \tag{3}$$

Combining (2) and (3) results in part 1.

(2) It is concluded by setting $\Lambda = \{1_X\}$ in part 1. □

We recall the following definition from [3].

Definition 3.1. *Let T_1 and T_2 be doubly stochastic operators acting on $L^\infty(X_1,\mu_1)$ and $L^\infty(X_2,\mu_2)$ respectively. We say that T_1 and T_2 are isomorphic if there exists a measure-preserving bijective map $\pi : X_2 \to X_1$ such that $\pi^{-1} : X_1 \to X_2$ is also measure-preserving and*

$$(T_1 f)\circ\pi = T_2(f\circ\pi) \qquad \forall f \in L^\infty(X_1,\mu_1).$$

Proposition 3.2. *If T_1 and T_2 are isomorphic doubly stochastic operators via $\pi : X_2 \to X_1$, then for any partition of unity Φ of X_1 we have:*

$$\mathfrak{J}_1^{MR}(\pi(x),\Phi) = \mathfrak{J}_2^{MR}(x,\pi\Phi) \qquad \forall x \in X_1,$$

where $\mathfrak{J}_k^{MR}$ $(k=1,2)$ is corresponding to T_k.

Proof. Let $x \in X_2$ and $\phi : X_1 \to [0,1]$ be a measurable function. Then, by definition, $\omega_1(\pi(x),\phi) = \omega_2(x,\phi\circ\pi)$. So, if $\Phi = \{\phi\}$ is a finite partition of X_1 then $\Omega_1(\pi(x),\Phi) = \Omega_2(x,\pi\Phi)$. Let $\mathcal{H}_n^{MR}(\pi(x),\Phi,T_i)$ $(i=1,2)$ be the quantity as in (1) corresponding to T_i. For $n \geq 1$, we have

$$\begin{aligned}
\mathcal{H}_n^{MR}(\pi(x),\Phi,T_1) &= \inf\{\Omega_1(\pi(x),\Gamma) : \Phi \prec \Gamma,\ T_1\Phi \prec \Gamma,\cdots,T_1^{n-1}\Phi \prec \Gamma\} \\
&= \inf\{\Omega_2(x,\pi\Gamma) : \pi\Phi \prec \pi\Gamma,\ \pi T_1\Phi \prec \pi\Gamma,\cdots,\pi T_1^{n-1}\Phi \prec \pi\Gamma\} \\
&= \inf\{\Omega_2(x,\pi\Gamma) : \pi\Phi \prec \pi\Gamma,\ T_2\pi\Phi \prec \pi\Gamma,\cdots,T_2^{n-1}\pi\Phi \prec \pi\Gamma\} \\
&= \inf\{\Omega_2(x,\Lambda) : \pi\Phi \prec \Lambda,\ T_2\pi\Phi \prec \Lambda,\cdots,T_2^{n-1}\pi\Phi \prec \Lambda\} \\
&= \mathcal{H}_n^{MR}(x,\pi\Phi,T_2).
\end{aligned}$$

Dividing both sides of the previous inequality by n and letting $n \to +\infty$, we will have

$$\mathfrak{J}_2^{MR}(x,\pi\Phi) = \mathfrak{J}_1^{MR}(\pi(x),\Phi) \tag{4}$$

which completes the proof. □

Before stating our main result, we first present the following lemmas.

Lemma 3.1. *If Φ is a partition of unity with continuous elements then, for each $n \in \mathbb{N}$, $\varepsilon > 0$, and any partition of unity Γ with continuous elements such that $\Phi \prec \Gamma, T\Phi \prec \Gamma \cdots, T^{n-1}\Phi \prec \Gamma$ there exists a partition of unity $\Gamma' \in \mathcal{P}_\mathcal{D}$ such that*

$$|H_\mu(\Gamma) - H_\mu(\Gamma')| < \varepsilon$$

and

$$|\Omega(x,\Gamma) - \Omega(x,\Gamma')| < \varepsilon \quad (x \in X).$$

Proof. Let $\Gamma = \{\gamma_1,\gamma_2,\cdots,\gamma_r\}$ be a partition of unity with continuous elements such that $T^k\Phi \prec \Gamma$, $0 \leq k \leq n-1$. Fix $x \in X$. Given $\varepsilon > 0$, let $\delta_1 > 0$ be the positive number corresponding to $\frac{\varepsilon}{r}$ due to uniform continuity of η on $[0,1]$. By L^1- continuity property, for partitions of cardinality r, there exists $\delta_2 > 0$ such that dist $(\Gamma,\Gamma') < \delta_2$ implies $|H_\mu(\Gamma) - H_\mu(\Gamma')| < \varepsilon$. Let $\delta = \min\{\delta_1,\delta_2\}$. Since $\mathcal{D}$ is dense in $C(X)$, one may choose $\Gamma' = \{\gamma_1',\gamma_2',\cdots,\gamma_r'\} \in \mathcal{P}_\mathcal{D}$ such that $||\gamma_i - \gamma_i'||_\infty < \delta$ $(1 \leq i \leq r)$ and so dist $(\Gamma,\Gamma') < \delta < \delta_2$. Hence $|H_\mu(\Gamma) - H_\mu(\Gamma')| < \varepsilon$. On the other hand for each i, since $||T|| = 1$, we have

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} T^k\gamma_i(x) - \frac{1}{n}\sum_{k=0}^{n-1} T^k\gamma_i'(x)\right| \leq \frac{1}{n}\sum_{k=0}^{n-1}\left|T^k(\gamma_i - \gamma_i')(x)\right| \leq ||\gamma_i - \gamma_i'||_\infty < \delta,$$

for all $x \in X$. Letting $n \to +\infty$ we conclude that

$$|\omega(x,\gamma_i) - \omega(x,\gamma_i')| < \delta.$$

Therefore,

$$|\eta(\omega(x,\gamma_i)) - \eta(\omega(x,\gamma_i'))| < \frac{\epsilon}{r} \quad (1 \leq i \leq l)$$

and consequently $|\Omega(x,\Gamma)-\Omega(x,\Gamma')|<\epsilon$, $(x\in X)$. This completes the proof. $\square$

Notation: For $\varepsilon>0$, $n\in\mathbb{N}$, and a partition of unity Φ, let $\Lambda_n(\mathit{\Phi},\epsilon)$ be the collection of partitions $\Gamma\in\mathcal{P}_{\mathcal{D}}$ such that there exists a measurable partition Γ' with $T^k\Phi\prec\Gamma'$, $0\le k\le n-1$,

$$|H_\mu(\Gamma)-H_\mu(\Gamma')|<\epsilon$$

and

$$|\Omega(y,\Gamma)-\Omega(y,\Gamma')|<\varepsilon \qquad (y\in X).$$

Lemma 3.2. *If Φ is a partition of unity with continuous elements then, for each $n\in\mathbb{N}$ we have*

$$H_\mu^{MR}(\Phi,n)=\lim_{m\to\infty}(\inf A_m),$$

where $A_m=\{H_\mu(\Gamma),\ \Gamma\ \in\Lambda_n(\mathit{\Phi},\frac{1}{m})\}$ and

$$\mathcal{H}_n^{MR}(x,\Phi)=\lim_{m\to\infty}(\inf B_m).$$

where $B_m=\{\Omega(x,\Gamma),\ \Gamma\ \in\Lambda_n(\mathit{\Phi},\frac{1}{m})\}$.

Proof. Since continuous functions are also dense in L^1- norm, by L^1- continuity property we have

$$H_\mu^{MR}(\Phi,n)=\inf_{\Gamma\in\mathcal{D}_{\mathcal{C}}^n(\Phi)}H_\mu(\Gamma) \tag{5}$$

where

$$\mathcal{D}_{\mathcal{C}}^n(\Phi)=\{\Gamma:\Gamma\in\mathcal{C}\text{ and }T^k\Phi\prec\Gamma,\ k=0,1,\cdots,n-1\}.$$

Let

$$C_m=\left(-\frac{1}{m},\frac{1}{m}\right).$$

Then we have

$$A_m\subseteq C_m+B$$

and

$$B\subseteq A_m+C_m,$$

where $B=\{H_\mu(\Gamma):\ \Gamma\in\mathcal{D}_{\mathcal{C}}^n(\Phi)\}$. Thus

$$\inf(A_m)\ge\inf(C_m+B)=\inf(B)-\frac{1}{m}$$

and

$$\inf(B)\ge\inf(C_m+A_m)=\inf(A_m)-\frac{1}{m}.$$

So,

$$|\inf(A_m)-\inf(B)|\le\frac{1}{m}.$$

Letting $m\to\infty$, we have

$$H_\mu^{MR}(\Phi,n)=\lim_{m\to\infty}(\inf A_m).$$

The proof of

$$\mathcal{H}_n^{MR}(x,\Phi)=\lim_{m\to\infty}(\inf B_m)$$

is similar. $\square$

Remark 3.1. *In light of Lemma 3.2, for $\Phi\in\mathcal{C}$, $\mathcal{J}^{MR}(\cdot,\Phi):X\to[0,+\infty]$ is a measurable map and so its Lebesgue integral is well-defined.*

3.3. Average entropy

Now we are ready to give the following definition.

Definition 3.2. *Let $T : L \to L$ be a stochastic operator as in section 3, and $\mu \in \mathcal{M}_T(X)$. The average Maličky-Riečan's entropy of T with respect to μ is defined as*

$$h_\mu^{AMR}(T) := \sup_{\Phi\in\mathcal{C}} h_\mu^{AMR}(T,\Phi)$$

where $\mathcal{C}$ is the family of partitions of unity of X with continuous elements and

$$h_\mu^{AMR}(T,\Phi) := \int_X \mathcal{J}^{MR}(x,\Phi)d\mu(x).$$

The following theorem states the relationship between the average Maličky-Riečan's entropy and the Maličky-Riečan's entropy.

Theorem 3.1. *Let $T : L \to L$ be a stochastic operator on the collection of Borel measurable functions with range in $[0,1]$, and $\mu \in \mathcal{M}_T(X)$. Then we have*

$$h_\mu^{AMR}(T) \leq h_\mu^{MR}(T). \tag{6}$$

Moreover, if $\mu \in \mathcal{E}_T(X)$ then we have equality in (6).

Proof. First let $\mu \in \mathcal{E}_T(X)$. Let $\Phi \in \mathcal{C}$ be given and $m,n \in \mathbb{N}$ be fixed. Let also $\Gamma \in \Lambda_n(\Phi, \frac{1}{m})$ be given. Applying Chacon-Ornstein's theorem (Theorem 1.3 in [2]), there exists a Borel measurable set $B_\Gamma \subset X$ such that $\mu(B_\Gamma) = 1$ and

$$\Omega(x,\Gamma) = H_\mu(\Gamma) \qquad \forall x \in B_\Gamma.$$

Set $B := \bigcap_{m,n\geq 1}\bigcap_{\Gamma\in\Lambda_n(\Phi,\frac{1}{m})} B_\Gamma$. Since $\Lambda_n(\Phi,\frac{1}{m})$ is countable, we have $\mu(B) = 1$. Now, for $x \in B$, we have $\Omega(x,\Gamma) = H_\mu(\Gamma)$ for every $\Gamma \in \Lambda_n(\Phi,\frac{1}{m})$. Taking infimum over all $\Gamma \in \Lambda_n(\Phi,\frac{1}{m})$, letting $m \to +\infty$ and applying Lemma 3.2 we will have

$$\mathcal{H}_n^{MR}(x,\Phi) = H_\mu^{MR}(\Phi,n) \qquad \forall x \in B. \tag{7}$$

Dividing (7) by n and letting $n \to +\infty$, we will have

$$\mathcal{J}^{MR}(x,\Phi) = h_\mu^{MR}(T,\Phi) \qquad \forall x \in B. \tag{8}$$

Integrating both sides of (8) we obtain that

$$\int_X \mathcal{J}^{MR}(x,\Phi)d\mu(x) = \int_B \mathcal{J}^{MR}(x,\Phi)d\mu(x) = h_\mu^{MR}(T,\Phi). \tag{9}$$

Now, let in general $\mu \in \mathcal{M}_T(X)$. Let $\mu = \int_{\mathcal{E}_T(X)} m d\tau(m)$ be the ergodic decomposition of μ. Let Φ be a partition of unity with continuous elements. Then we have

$$\begin{aligned}\int_X \mathcal{J}^{MR}(x,\Phi)d\mu(x) &= \int_{\mathcal{E}_T(X)}\left(\int_X \mathcal{J}^{MR}(x,\Phi)dm(x)\right)d\tau(m)\\ &= \int_{\mathcal{E}_T(X)} h_m^{MR}(T,\Phi)d\tau(m).\end{aligned} \tag{10}$$

Now, for any $\Gamma \in \mathcal{D}_{\mathcal{C}}^n(\Phi)$, the mapping $\mu \longmapsto H_\mu(\Gamma)$ is weak* continuous. So, by (5), the mapping $\mu \longmapsto H_\mu^{MR}(\Phi,n)$ is upper semi-continuous. Therefore, since the sequence $\{H_\mu^{MR}(\Phi,n)\}_{n\geq 1}$ is subadditive, the mapping

$$\mu \longmapsto h_\mu^{MR}(T,\Phi) := \lim_{n\to+\infty}\frac{1}{n}H_\mu^{MR}(\Phi,n) = \inf_{n\geq 1}\frac{1}{n}H_\mu^{MR}(\Phi,n)$$

is also upper semi-continuous. On the other hand, For any partition of unity Γ, and invariant measures μ and m, since the function $\phi(x) = -x \log x$ is concave and $\psi(x) = \log x$ is increasing, we have

$$0 \leq H_{\lambda\mu+(1-\lambda)m}(\Gamma) - \lambda H_{\mu}(\Gamma) - (1-\lambda)H_m(\Gamma) \leq \log 2, \tag{11}$$

where $\lambda \in [0,1]$ (See [27], the proof of Theorem 8.1.). Since the infimum of functions has the property of super additivity, one may easily conclude from (11) that

$$h^{MR}_{\lambda\mu+(1-\lambda)m}(T,\Phi) \geq \lambda h^{MR}_{\mu}(T,\Phi) + (1-\lambda)h^{MR}_{m}(T,\Phi). \tag{12}$$

So, the mapping $\mu \longmapsto h^{MR}_{\mu}(T,\Phi)$ is concave and upper semi-continuous. Therefore,

$$\int_{\mathcal{E}_T(X)} h^{MR}_{m}(T,\Phi)d\tau(m) \leq h^{MR}_{\mu}(T,\Phi). \tag{13}$$

Combining (10) and (13) we will have

$$\begin{aligned} \int_X \mathcal{J}^{MR}(x,\Phi)d\mu(x) &= \int_{\mathcal{E}_T(X)} \left(\int_X \mathcal{J}^{MR}(x,\Phi)dm(x)\right) d\tau(m) \\ &= \int_{\mathcal{E}_T(X)} h^{MR}_{m}(T,\Phi)d\tau(m) \\ &\leq h^{MR}_{\mu}(T,\Phi). \end{aligned} \tag{14}$$

Now, let $\Psi = \{\psi_1, \psi_2, \cdots, \psi_r\}$ be a partition of unity of X. Given $\epsilon > 0$, let δ be the positive number corresponding to ϵ and $r = \mathrm{card}(\Psi)$ as in Proposition 2.1. Applying Lusin's theorem, one may choose $\Phi = \{\phi_1, \phi_2, \cdots, \phi_r\} \in \mathcal{C}$ such that

$$\mu(\{x \in X : \ \psi_i(x) \neq \phi_i(x)\}) < \frac{\delta}{2} \qquad (i = 1, 2, \cdots, r).$$

For $1 \leq i \leq r$, we have

$$\int_X |\phi_i - \psi_i| d\mu = \int_{\{x\in X: \ \psi_i(x)\neq\phi_i(x)\}} |\phi_i - \psi_i| d\mu \leq 2\mu(\{x \in X : \ \psi_i(x) \neq \phi_i(x)\}) < \delta.$$

Therefore, $\mathrm{dist}(\Phi,\Psi) < \delta$. Then, by Proposition 2.1,

$$|h^{MR}_{\mu}(T,\Phi) - h^{MR}_{\mu}(T,\Psi)| < \epsilon.$$

In particular,

$$h^{MR}_{\mu}(T,\Psi) < h^{MR}_{\mu}(T,\Phi) + \epsilon \leq \sup_{\Phi\in\mathcal{C}} h^{MR}_{\mu}(T,\Phi) + \epsilon.$$

Since ϵ and Ψ are given, we conclude that

$$\sup_{\Phi\in\mathcal{P}} h^{MR}_{\mu}(T,\Phi) = \sup_{\Phi\in\mathcal{C}} h^{MR}_{\mu}(T,\Phi). \tag{15}$$

Finally, the result follows by taking supremum from equation (14) over all $\Phi \in \mathcal{C}$ and applying (15). Note that, when $\mu \in \mathcal{E}_T(X)$, combining (9) and (15) we will have equality in (6). □

4. **Summary and discussion**

The concept of Maličky-Riečan's entropy was first suggested by Maličky and Riečan in [12] for classical dynamics. This was extended for more general settings by replacing classical partitions by partitions of unity. It was then discussed for doubly stochastic operators in [7]. One should note that, this quantity coincides to Kolmogorov entropy when we work with Koopman operators and classical partitions [12, 14].

In the present paper, we introduced a local version of Maličky-Riečan's entropy for doubly stochastic operators. We defined a map $\mathcal{J}^{MR}$, corresponding to stochastic operators, which locally resembles the Maličky-Riečan's entropy. We proved that the integral of the introduced map with respect to any invariant measure is controlled by the corresponding Maličky-Riečan's entropy. Indeed we have equality when the invariant measure is also ergodic. So, using the map $\mathcal{J}^{MR}$, the average Maličky-Riečan's entropy for doubly stochastic operators is defined. In light of Theorem 3.1, we have

$$h_{\mu}^{AMR}(T) \leq h_{\mu}^{MR}(T),$$

with equality when μ is an ergodic measure. So, $h_{\mu}^{AMR}(T)$ is a modification of the Maličky-Riečan's entropy. Finally, note that, given any partition of unity Φ, the map $\mu \longmapsto h_{\mu}^{AMR}(T, \Phi)$ is affine while its corresponding map $\mu \longmapsto h_{\mu}^{MR}(T, \Phi)$ is only concave.

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