# A NOTE ON JOHNSON PSEUDO-CONTRACTIBILITY OF TRIANGULAR BANACH ALGEBRAS

Amir SAHAMI[1]

*In this note, we characterize Johnson pseudo-contractibility of triangular Banach algebras. We show that a triangular Banach algebra* $T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ *is Johnson pseudo-contractible if and only if* $X = 0$ *and* $A, B$ *are Johnson pseudo-contractible.*

**Keywords:** Triangular Banach algebra, Johnson pseudo-contractibility
**MSC2020:** 46H05, 46H20

## 1. Introduction and preliminaries

A Banach algebra $A$ is called amenable, if for each Banach $A$-bimodule $X$, every continuous derivation $D$ from $A$ into $X^*$ has the form $D(a) = a \cdot x_0 - x_0 \cdot a$, for some $x_0 \in X^*$. Johnson showed that $A$ is amenable if and only if there exists a bounded net $(m_\alpha)$ in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_A(m_\alpha)a \to a$ for every $a \in A$, where $\pi_A$ is the product morphism from $A \otimes_p A$(the projective tensor product of $A$ with $A$) into $A$ given by $\pi_A(a \otimes b) = ab$, see [8].

By removing the boundedness from the definition of amenability, Ghahramani and Zhang gave the definitons of pseudo-amenable and pseudo-contractible for Banach algebras. In fact $A$ is pseudo-amenable (pseudo-contractible) if there exists a net $(m_\alpha)$ in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0 (a \cdot m_\alpha = m_\alpha \cdot a)$ and $\pi_A(m_\alpha)a \to a$, for each $a \in A$, respectively. For more information about these concepts, see [2] and [6].

Recently the author with A. Pourabbas defined the notion of Johnson pseudo-contractible for Banach algebras. A Banach algebra $A$ is called Johnson pseudo-contractible if there exists a net $(m_\alpha)$ in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a \to a$, for each $a \in A$. We characterized Johnson pseudo-contractibility of some semigroup algebras, Lipschitz algebras and matrix algebras, see [10], [9] and [1].

Triangular Banach algebras are important Banach algebras among matrix algebras. Indeed let $A$ and $B$ be Banach algebras and let $X$ be a Banach $(A, B)$-module, that is, $X$ is a Banach space, a left $A$-module and a right $B$-module with the compatible module action that satisfies $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ and $||a \cdot x \cdot b|| \leq ||a||||x||||b||$ for every $a \in A, x \in X, b \in B$. With the usual matrix operations and $|| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} || = ||a|| + ||x|| + ||b||$, $T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ becomes a Banach algebra which is called triangular Banach algebra. Cohomological properties and Hochschild cohomology group of triangular Banach algebras were studied in [4] and [5].

In this short note, we study Johnson psudo-contractibility of triangular Banach algebras. We characterize Johnson pseudo-contractibility of triangular Banach algebra $T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ through the annihilation of $X$ and Johnson pseudo-contractibility of $A$ and $B$.

Department of Mathematics, Faculty of Basic Sciences Ilam University P.O. Box 69315-516 Ilam, Iran. E-mail: a.sahami@ilam.ac.ir

## 2. **Main Results**

The following lemma has a key-role in characterizing Johnson pseudo-contractibility of triangular Banach algebras.

**Lemma 2.1.** *Let $A$ be a Johnson pseudo-contractible Banach algebra and let $E$ be a closed ideal of $A$. Suppose that $N$ is a closed complemented ideal in $A$ such that $E \subseteq N$ and $EN = (0)$. Then $E = (0)$.*

*Proof.* Inspired by the proof of [11, Lemma 2] we give our argument. We denote $i_N : N \to A$ for the inclusion map. Also $q : A \to \frac{A}{E}$ is denoted for the quotient map and $Id$ is given for the identity map. By hypothesis, we have $EN = (0)$, it follows that $p : \frac{A}{E} \otimes_p N \to N$ which is given by $p((a + E) \otimes c) = ac$ for each $a \in A$ and $c \in N$, becomes a well-defined bounded linear map. We assume in contradiction that $E \neq (0)$. So there exists a non-zero element $e \in E$. On the other hand, we know that $A$ is Johnson pseudo-contractible. Then there exists a net $(m_\alpha)$ in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a \to a$, for every $a \in A$. Applying [10, Corollary2.7], $A$ becomes pseudo-amenable. So $A$ has an approximate identity, say $(e_\beta)$. We denote $(\rho_\alpha)$ for a net of $A$-bimodule morphisms from $A$ into $(A \otimes_p A)^{**}$ such that $\rho_\alpha(a) = a \cdot m_\alpha$. It follows that $\pi_A^{**} \circ \rho_\alpha(a) \to a$, for every $a \in A$. For each $n \in N$, we define respectively, $R_n$ and $L_n$ for bounded linear maps from $A$ into $N$ given by $R_n(a) = an$ and $L_n(a) = na$. Define $d_\alpha^\beta = (q \otimes R_e)^{**} \circ \rho_\alpha(e_\beta)$. Since each $(\rho_\alpha)$ is a net of $A$-bimodule morphisms, we have

$$\lim_\alpha \lim_\beta \pi_A^{**} \circ \rho_\alpha(e_\beta)e = \lim_\alpha \lim_\beta \pi_A^{**} \circ \rho_\alpha(e_\beta e) = \lim_\alpha \pi_A^{**} \circ \rho_\alpha(e) = e.$$

Using iterated limit theorem [7, page 69], we can see that

$$\lim_\alpha \pi_A^{**} \circ \rho_\alpha(e_{\beta_\alpha})e = e \neq 0.$$

Hence

$$\begin{aligned}
\lim_\alpha p^{**}(d_\alpha^{\beta_\alpha}) &= \lim_\alpha p^{**} \circ (q \otimes R_e)^{**} \circ \rho_\alpha(e_{\beta_\alpha}) \\
&= \lim_\alpha R_e^{**} \circ (\pi_A^{**} \circ \rho_\alpha)(e_{\beta_\alpha}) \\
&= \lim_\alpha \pi_A^{**} \circ \rho_\alpha(e_{\beta_\alpha} e) = e \neq 0.
\end{aligned}$$

Thus it can be assumed that $p^{**}(d_\alpha^{\beta_\alpha}) \neq 0$ for each $\alpha$. It gives that $d_\alpha^{\beta_\alpha} \neq 0$, for every $\alpha$. Let us fix $\alpha$. Since $A$ has an approximate identity and $E$ is an ideal, $e_{\beta_\alpha} e \in AE \subseteq E = \overline{EA}$. Thus we can find sequences $a_n$ and $e_n$ in $A$ and $E$ such that $e_{\beta_\alpha} e = \lim e_n a_n$. Now using the following facts

$$\begin{aligned}
(Id_{\frac{A}{E}} \otimes i_N)^{**}(d_\alpha^{\beta_\alpha}) &= (Id_{\frac{A}{E}} \otimes i_N)^{**} \circ (q \otimes R_e)^{**} \circ \rho_\alpha(e_{\beta_\alpha}) \\
&= (Id_{\frac{A}{E}} \otimes i_N)^{**} \circ (q \otimes Id_N)^{**} \circ (Id_A \otimes R_e)^{**} \circ \rho_\alpha(e_{\beta_\alpha}),
\end{aligned}$$

and also

$$(Id_{\frac{A}{E}} \otimes i_N) \circ (q \otimes Id_N) = (q \otimes Id_A) \circ (Id_A \otimes i_N),$$

we have

$$\begin{aligned}(Id_{\frac{A}{E}} \otimes i_N)^{**}(d_\alpha^{\beta_\alpha}) &= [(Id_{\frac{A}{E}} \otimes i_N) \circ (q \otimes Id_N) \circ (Id_A \otimes R_e)]^{**} \circ \rho_\alpha(e_{\beta_\alpha}) \\ &= [(q \otimes Id_A) \circ (Id_A \otimes i_N) \circ (Id_A \otimes R_e)]^{**} \circ \rho_\alpha(e_{\beta_\alpha}) \\ &= [(q \otimes Id_A) \circ (Id_A \otimes R_e)]^{**} \circ \rho_\alpha(e_{\beta_\alpha}) \\ &= (q \otimes Id_A)^{**}(\rho_\alpha(e_{\beta_\alpha}) \cdot e) \\ &= (q \otimes Id_A)^{**}(\rho_\alpha(e_{\beta_\alpha} e)) \\ &= (q \otimes Id_A)^{**}(\lim e_n \cdot \rho_\alpha(a_n)) \\ &= \lim_n (q \otimes Id_A)^{**}(e_n \cdot \rho_\alpha(a_n)) \\ &= \lim_n ((q \otimes Id_A)^{**} \circ ((i_N \circ L_{e_n}) \otimes Id_A)^{**})(\rho_\alpha(a_n)) \\ &= \lim_n (q \otimes Id_A)^{**} \circ ((i_N \otimes Id_A) \circ (L_{e_n} \otimes Id_A))^{**}(\rho_\alpha(a_n)) \\ &= \lim_n ((q \otimes Id_A) \circ (i_N \otimes Id_A) \circ (L_{e_n} \otimes Id_A))^{**}(\rho_\alpha(a_n)) \\ &= \lim_n (q \circ i_N \circ L_{e_n})^{**} \circ (\rho_\alpha(a_n)) = 0,\end{aligned}$$

the last equality holds because $q \circ i_N \circ L_{e_n} = 0$.

Hence for each $\alpha$, we have $(Id_{\frac{A}{E}} \otimes i_N)^{**}(d_\alpha^{\beta_\alpha}) = 0$. On the other hand $N$ is a closed complemented ideal of $A$, then by [1, Lemma 1] $Id_{\frac{A}{E}} \otimes i_N$ is a closed map. Now applying [3, A.3.48] it follows that $(Id_{\frac{A}{E}} \otimes i_N)^{**}$ is injective. Thus $d_\alpha^{\beta_\alpha} = 0$ for each $\alpha$ which is a contradiction. □

**Theorem 2.1.** *Let $A$ and $B$ be Banach algebras and also let $X$ be a Banach $(A, B)$-module. Then $T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ is Johnson pseudo-contractible if and only if $X = 0$ and $A, B$ are Johnson pseudo-contractible.*

*Proof.* Suppose that $X = 0$ and $A, B$ are Johnson pseudo-contractible. Since $T$ is $\ell^1$-direct sum of $A$ and $B$, [9, Theorem 2.11] gives that $T$ is Johnson pseudo-contractible.
For converse, suppose that $T$ is Johnson pseudo-contractible. Define $I = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$. We can show that $I$ is a closed ideal of $T$. Put $E = N = I$ (in the previous Lemma). Since $EN = (0)$ and $E \subseteq N$, the previous lemma implies that $I = (0)$. Thus $X = 0$. Define $\xi_A : T \to A$ ($\xi_B : T \to B$) by $\xi_A \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a$ ($\xi_B \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = b$), for each $a \in A, b \in B$, respectively. One can easily see that $\xi_A$ and $\xi_B$ are continuous epimorphisms. Therefore [9, Proposition 2.9] implies that $A$ and $B$ are Johnson pseudo-contractible. □

As an easy consequence of the previous theorem we have the following result.

**Corollary 2.1.** *For a non-zero Banach algebra $A$, triangular Banach algebra $T = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ is never Johnson pseudo-contractible.*

**Acknowledgments.** The author would like to thank the referee for his/her valuable comments and suggestions. The author is thankful to Ilam university for its support.

## REFERENCES

[1] M. Askari-Sayah, A. Pourabbas and A. Sahami, *Johnson pseudo-contractibility of certain Banach algebras and its nilpotent ideals.* Anal Math **45**, 461-473 (2019).

[2] Y. Choi, F. Ghahramani and Y. Zhang, *Approximate and pseudo-amenability of various classes of Banach algebras,* J. Func. Analysis **256** (2009) 3158-3191.

[3] H. G. Dales, *Banach Algebras and Automatic Continuity* (Oxford, 2000).

[4] B. E. Forrest and L. E. Marcoux, *Derivations of triangular Banach algebras*, Indiana Univ. Math. J. **45** (1996), 441-462.

[5] B. E. Forrest and L. E. Marcoux, *Weak amenability of triangular Banach algebras*, Trans. Amer. Math. Math. J. **354** (2001), 1435-1452.

[6] F. Ghahramani and Y. Zhang, *Pseudo-amenable and pseudo-contractible Banach algebras*, Math. Proc. Camb. Phil. Soc. **142** (2007) 111-123.

[7] J. L. Kelley, *General topology*, D. Van Nostrand Company, Inc., Toronto-New York- London, 1955.

[8] V. Runde; *Lectures on Amenability*, (Springer, New York, 2002).

[9] A. Sahami and A. Pourabbas, *Johnson pseudo-contractibility of various classes of Banach algebras*, Bull. Belg. Math. Soc. Vol. **25**, (2018), 171-182.

[10] A. Sahami and A. Pourabbas, *Johnson pseudo-contractibility of certain semigroup algebras*, Semigroup Forum. Vol. **97**, Issue 2, (2018) 203-213.

[11] Y. Zhang; *Nilpotent ideals in a class of Banach algebras*, Proc. Amer. Math. Soc. **127** (11) (1999), 3237-3242.