ON CONNES AMENABILITY OF UPPER TRIANGULAR MATRIX ALGEBRAS

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In this paper, we study the notion of Connes amenability for a class of $I \times I$-upper triangular matrix algebra $UP(I, \mathcal{A})$, where $\mathcal{A}$ is a dual Banach algebra with a non-zero wk^*-continuous character and I is a totally ordered set. For this purpose, we characterize the ϕ-Connes amenability of a dual Banach algebra $\mathcal{A}$ through the existence of a specified net in $\mathcal{A}\hat{\otimes}\mathcal{A}$, where ϕ is a non-zero wk^*-continuous character. Using this, we show that $UP(I, \mathcal{A})$ is Connes amenable if and only if I is singleton and $\mathcal{A}$ is Connes amenable. In addition, some examples of ϕ-Connes amenable dual Banach algebras, which is not Connes amenable are given.

Keywords: Upper triangular Banach algebras, Connes amenability, ϕ-Connes amenability.

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1. Introduction

The concept of amenability for Banach algebras was first introduced by B. E. Johnson [1]. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \to E$ is called a derivation if for every $a, b \in \mathcal{A}$, $D(ab) = a \cdot D(b) + D(a) \cdot b$. A Banach algebra $\mathcal{A}$ is called amenable if every derivation from $\mathcal{A}$ into each dual Banach $\mathcal{A}$-bimodule E^* is inner, that is, there exists a $x \in E^*$ such that $D(a) = a \cdot x - x \cdot a$ $(a \in \mathcal{A})$. Let $\mathcal{A}$ be a Banach algebra. An $\mathcal{A}$-bimodule E is called dual if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. The Banach algebra $\mathcal{A}$ is called dual if it is dual as a Banach $\mathcal{A}$-bimodule. A dual Banach $\mathcal{A}$-bimodule E is normal, if for each $x \in E$ the module maps $\mathcal{A} \to E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are wk^*-wk^* continuous. The class of dual Banach algebras was introduced by Runde [5]. The measure algebras $M(G)$ of a locally compact group G, the algebra of bounded operators $\mathcal{B}(E)$, for a reflexive Banach space E and the second dual $\mathcal{A}^{**}$ of Arens regular Banach algebra $\mathcal{A}$ are examples of dual Banach algebras. A suitable concept of amenability for dual Banach algebras is the Connes amenability. This notion under different name, for the first time was introduced by Johnson, Kadison, and Ringrose for von Neumann algebras [1]. The concept of Connes amenability for the larger class of dual Banach algebras was later extended by Runde [5]. A dual Banach algebra $\mathcal{A}$ is called Connes amenable if for every normal dual Banach $\mathcal{A}$-bimodule E, every wk^*-continuous derivation $D : \mathcal{A} \longrightarrow E$ is inner.

Let $\mathcal{A}$ be a Banach algebra and let I be a totally ordered set. Sahami [8] studied the notions of amenability and its related homological notions for a class of $I \times I$-upper triangular matrix algebra $UP(I, \mathcal{A}) = \Big\{ \begin{bmatrix} a_{i,j} \end{bmatrix}_{i,j \in I} ; a_{i,j} \in A \text{ and } a_{i,j} = 0 \text{ for every } i > j \Big\}$.

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He showed that $UP(I,\mathcal{A})$ is pseudo-contractible (amenable) if and only if I is singleton and $\mathcal{A}$ is pseudo-contractible (amenable), respectively. He also studied the notions of pseudo-amenability and approximate biprojectivity of $UP(I,\mathcal{A})$.

In this paper, we investigate the notion of Connes amenability for a class of $I\times I$-upper triangular matrix $UP(I,\mathcal{A})$, where $\mathcal{A}$ is a dual Banach algebra and I is a totally ordered set. For this purpose, first in section 2 we study the duality of $UP(I,\mathcal{A})$ by considering the isometric-isomorphism $UP(I,\mathcal{A}) \cong \ell^1\text{-}\bigoplus_{\alpha\in J}\mathcal{A}_\alpha$ as Banach spaces, where J is a subset of $I\times I$ and for every $\alpha\in J$, $\mathcal{A}_\alpha = \mathcal{A}$. In section 3 by using the fact that every Connes amenable Banach algebra is ϕ-Connes amenable, we obtain a new characterization of ϕ-Connes amenability through the existence of a bounded net with a certain condition, where ϕ is a non-zero wk^*-continuous character. By applying latter characterization, we show that $UP(I,\mathcal{A})$ is Connes amenable if and only if $\mathcal{A}$ is Connes amenable and I is singleton. Finally in section 4 we provide some examples of ϕ-Connes amenable dual Banach algebras, which are not Connes amenable.

2. **Preliminaries**

For a given dual Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule E, $\sigma wc(E)$ denote the set of all elements $x\in E$ such that the module maps $\mathcal{A}\to E$; $a\mapsto a\cdot x$ and $a\mapsto x\cdot a$ are wk^*-wk-continuous, one can see that, it is a closed submodule of E. If $\theta: E\longrightarrow F$ is a bounded $\mathcal{A}$-bimodule homomorphism, where F is another Banach $\mathcal{A}$-bimodule, then $\theta(\sigma wc(E))\subseteq \sigma wc(F)$. Runde also showed that $E=\sigma wc(E)$ if and only if E^* is a normal dual Banach $\mathcal{A}$-bimodule [6, Proposition 4.4]. Let $\mathcal{A}$ be a Banach algebra. The projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule with the usual left and right operations with respect to $\mathcal{A}$. Then the map $\pi:\mathcal{A}\hat{\otimes}\mathcal{A}\longrightarrow\mathcal{A}$ defined by $\pi(a\otimes b)=ab$ is an $\mathcal{A}$-bimodule homomorphism. Since $\sigma wc(\mathcal{A}_*)=\mathcal{A}_*$, the adjoint of π maps $\mathcal{A}_*$ into $\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Therefore, π^{**} drops to an $\mathcal{A}$-bimodule homomorphism $\pi_{\sigma wc}: (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*\longrightarrow\mathcal{A}$. Any element $M\in(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ satisfying $a\cdot M = M\cdot a$ and $a\cdot\pi_{\sigma wc}M = a$ $(a\in\mathcal{A})$, is called a σwc-virtual diagonal for $\mathcal{A}$. Runde showed that a dual Banach algebra $\mathcal{A}$ is Connes amenable if and only if there is a σwc-virtual diagonal for $\mathcal{A}$ [6, Theorem 4.8].

3. **The duality of $UP(I,\mathcal{A})$**

Let $\mathcal{A}$ be a dual Banach algebra and let I be a totally ordered set. Then the set of all $I\times I$-upper triangular matrices with the usual matrix operations and the norm $\|[a_{i,j}]_{i,j\in I}\| = \sum_{i,j\in I}\|a_{i,j}\|<\infty$, becomes a Banach algebra. Before we study the duality of $UP(I,\mathcal{A})$, we state the following Lemma:

Lemma 3.1. *If $\mathcal{A}$ is a dual Banach algebra with the predual $\mathcal{A}_*$ and I is a non-empty set, then*
$(c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*})^*\cong\ell^1\text{-}\bigoplus_{i\in I}\mathcal{A}_i$, where for every $i\in I$, $\mathcal{A}_i=\mathcal{A}$ and $\mathcal{A}_{i_*}=\mathcal{A}_*$.

Proof. Let $g=(g_\alpha)_{\alpha\in I}\in\ell^1\text{-}\bigoplus_{i\in I}\mathcal{A}_i$. We define $\phi_g: c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}\longrightarrow\mathbb{C}$ by $\phi_g(f)=\sum_{\alpha\in I}g_\alpha(f_\alpha)$, where $f=(f_\alpha)_{\alpha\in I}\in c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$. We show that ϕ_g is bounded,

$$|\phi_g(f)|\le\sum_{\alpha\in I}|g_\alpha(f_\alpha)|\le\sum_{\alpha\in I}\|f_\alpha\|\|g_\alpha\|\le\|f\|_\infty\sum_{\alpha\in I}\|g_\alpha\|\le\|f\|_\infty\|g\|_1, \tag{1}$$

So $\|\phi_g\|\le\|g\|_1$. Now we define $T:\ell^1\text{-}\bigoplus_{i\in I}\mathcal{A}_i\longrightarrow(c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*})^*$ by $T(g)=\phi_g$ and we show that T is isometric-isomorphism. It is clear that the map T is linear. Let $\phi\in(c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*})^*$, we show that there exists $g\in\ell^1\text{-}\bigoplus_{i\in I}\mathcal{A}_i$ such that $\phi_g=\phi$. Fixed $\alpha_0\in I$ and $\lambda_0\in\mathcal{A}_*$, we define $\delta(\alpha_0;\lambda_0)\colon I \longrightarrow \mathcal{A}_*$ by $\delta(\alpha_0;\lambda_0)(\alpha) = \lambda_0$ whenever $\alpha = \alpha_0$ and $\delta(\alpha_0;\lambda_0)(\alpha) = 0$ for every $\alpha \neq \alpha_0$. It is obvious that $\delta(\alpha_0;\lambda_0) \in c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$ and $\|\delta(\alpha_0;\lambda_0)\|_\infty = \|\lambda_0\|$. Now we consider $g = (g_\alpha)_{\alpha\in I}$, where $g_\alpha(\lambda) = \phi(\delta(\alpha;\lambda))$ for every $\alpha \in I$ and $\lambda \in \mathcal{A}_*$. It is easy to see that $\|g_\alpha\| \leq \|\phi\|$ for every $\alpha \in I$, so g_α is a continuous linear functional on $\mathcal{A}_*$.

Consider $(f_\alpha)_{\alpha\in I} \in c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$, we define $\eta_\alpha := \dfrac{\overline{\phi(\delta(\alpha;f_\alpha))}}{|\phi(\delta(\alpha;f_\alpha))|}$, whenever $\phi(\delta(\alpha;f_\alpha)) \neq 0$ otherwise we define $\eta_\alpha = 1$. So for every $\alpha \in I$, $|\eta_\alpha| = 1$. Let $\mathcal{F}$ be the family of all finite subsets of I. Then for every $f = (f_\alpha)_{\alpha\in I} \in c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$ we have $\|f\|_\infty = \sup_{F\in\mathcal{F}}\sum_{\alpha\in F}(\|f_\alpha\|)$. So for every f in the unit ball of $c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$, we have

$$\begin{aligned}\sum_{\alpha\in F}|g_\alpha(f_\alpha)| &= \sum_{\alpha\in F}|\phi(\delta(\alpha;f_\alpha))| = \sum_{\alpha\in F}\eta_\alpha\phi(\delta(\alpha;f_\alpha))\\ &= \sum_{\alpha\in F}\phi(\eta_\alpha\delta(\alpha;f_\alpha)) = \sum_{\alpha\in F}\phi(\delta(\alpha;\eta_\alpha f_\alpha)) = \phi(\sum_{\alpha\in F}\delta(\alpha;\eta_\alpha f_\alpha)) \leq \|\phi\|\|f\|_\infty \leq \|\phi\|.\end{aligned}$$

So for every $F \in \mathcal{F}$ we have $\sum_{\alpha\in F}\|g_\alpha\| \leq \|\phi\|$ which implies that

$$\sum_{\alpha\in I}\|g_\alpha\| \leq \|\phi\|. \tag{2}$$

Thus $g = (g_\alpha)_{\alpha\in I} \in \ell^1\text{-}\bigoplus_{i\in I}\mathcal{A}_i$. Since $c_{00}\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$ is dense in $c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$, first we show that $\phi_g = \phi$ on $c_{00}\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$. Let $f = (f_\alpha)_{\alpha\in I} \in c_{00}\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$, so there exists a finite subset F of I such that for any $\alpha \in I - F$, $f_\alpha = 0$. We have

$$\phi_g(f) = \sum_{\alpha\in I} g_\alpha(f_\alpha) = \sum_{\alpha\in F}\phi(\delta(\alpha;f_\alpha)) = \phi(\sum_{\alpha\in F}\delta(\alpha;f_\alpha)) = \phi(f). \tag{3}$$

Now suppose that $f \in c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$, so there exists a net $f_\alpha \in c_{00}\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*}$ such that $f_\alpha \xrightarrow{\|\cdot\|_\infty} f$. By (3) we have $\phi(f) = \phi(\lim_\alpha f_\alpha) = \lim_\alpha \phi(f_\alpha) = \lim_\alpha \phi_g(f_\alpha) = \phi_g(\lim_\alpha f_\alpha) = \phi_g(f)$. Hence $\phi_g = \phi$. Now by (2) and (1) we have $\|g\|_1 \leq \|\phi_g\| = \|T(g)\| \leq \|g\|_1$. Therefore the map T is isometry and by applying the open mapping theorem, we have

$$\ell^1\text{-}\bigoplus_{i\in I}\mathcal{A}_i \cong (c_0\text{-}\bigoplus_{i\in I}\mathcal{A}_{i_*})^*.$$

□

Remark 3.1. *Let $\mathcal{A}$ be a dual Banach algebra and let I be a totally ordered set. Consider the subset J of $I \times I$ defined by $J = \{(i,j) \mid i,j \in I, i \leq j\}$. So we have an isometric-isomorphism $UP(I,\mathcal{A}) \cong \ell^1\text{-}\bigoplus_{\alpha\in J}\mathcal{A}_\alpha$ as Banach spaces, where for every $\alpha = (i,j) \in J$, $\mathcal{A}_\alpha = \mathcal{A}$.*

Theorem 3.1. *If $\mathcal{A}$ is a dual Banach algebra with the predual $\mathcal{A}_*$ and I is a totally ordered set, then $UP(I,\mathcal{A})$ is a dual Banach algebra.*

Proof. According to Remark 3.1 and by Lemma 3.1, it is sufficient to show that $c_0\text{-}\bigoplus_{\alpha\in J}\mathcal{A}_{\alpha_*}$ is a closed $UP(I,\mathcal{A})$-submodule of $\ell^\infty\text{-}\bigoplus_{\alpha\in J}\mathcal{A}^*_\alpha$, where for every $\alpha \in J$, $\mathcal{A}^*_\alpha = \mathcal{A}^*$ and $\mathcal{A}_{\alpha_*} = \mathcal{A}_*$. First we show that $c_0\text{-}\bigoplus_{\alpha\in J}\mathcal{A}_{\alpha_*}$ is a closed subspace of $\ell^\infty\text{-}\bigoplus_{\alpha\in J}\mathcal{A}^*_\alpha$. Let $x_n = (\xi^n_\alpha)_{\alpha\in J}$ be in $c_0\text{-}\bigoplus_{\alpha\in J}\mathcal{A}_{\alpha_*}$ and suppose that $x_n \longrightarrow x = (\xi_\alpha)_{\alpha\in J}$ in $\ell^\infty\text{-}\bigoplus_{\alpha\in J}\mathcal{A}^*_\alpha$. Fixed $\varepsilon > 0$. For sufficiently large N,

$$\sup_{\alpha\in J}\|\xi^N_\alpha - \xi_\alpha\| < \frac{\varepsilon}{2}.$$

Since (ξ_α^N) vanishes at infinity, for sufficiently large α, we have $\|\xi_\alpha^N\| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}\|\xi_\alpha\| = \|\xi_\alpha - \xi_\alpha^N + \xi_\alpha^N\| &\leq \|\xi_\alpha - \xi_\alpha^N\| + \|\xi_\alpha^N\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

for sufficiently large α. It follows that $x = (\xi_\alpha)_{\alpha\in J} \in c_0\text{-} \bigoplus_{\alpha\in J} \mathcal{A}_{\alpha_*}$. Now we show that $c_0\text{-} \bigoplus_{\alpha\in J} \mathcal{A}_{\alpha_*}$ is an $UP(I,\mathcal{A})$-module. Suppose that $X = (a_\alpha)_{\alpha\in J}$ and $\Lambda = (f_\alpha)_{\alpha\in J}$ are arbitrary elements in $\ell^1\text{-} \bigoplus_{\alpha\in J} \mathcal{A}_\alpha$ and $c_0\text{-} \bigoplus_{\alpha\in J} \mathcal{A}_{\alpha_*}$, respectively. For every $Y = (b_\alpha)_{\alpha\in J}$ in $\ell^1\text{-} \bigoplus_{\alpha\in J} \mathcal{A}_\alpha$, we have $X\cdot\Lambda(Y) = \Lambda(YX)$. Then by Lemma 3.1 and (1) we have

$$X\cdot\Lambda(Y) = \sum_{\alpha\in J} f_\alpha(c_\alpha), \tag{4}$$

where $(c_\alpha)_{\alpha\in J} = [c_{i,j}]_{i,j\in I} = YX$ with respect to matrix multiplication in $UP(I,\mathcal{A})$. Since $\ell^\infty\text{-} \bigoplus_{\alpha\in J} \mathcal{A}_\alpha^*$ is an $UP(I,\mathcal{A})$-bimodule with dual actions, then we have $X\cdot\Lambda \in \ell^\infty\text{-} \bigoplus_{\alpha\in J} \mathcal{A}_\alpha^*$. We claim that $X\cdot\Lambda$ belongs to $c_0\text{-} \bigoplus_{\alpha\in J} \mathcal{A}_{\alpha_*}$, that is, vanishes at infinity. By (4) we have

$$\begin{aligned}X\cdot\Lambda(Y) &= \sum_{i,j\in I}\langle c_{i,j}, f_{i,j}\rangle = \sum_{i,j\in I}\langle \sum_{k\in I} b_{i,k}a_{k,j}, f_{i,j}\rangle \\ &= \sum_{i,j\in I}\sum_{k\in I}\langle b_{i,k}a_{k,j}, f_{i,j}\rangle = \sum_{i,j\in I}\sum_{k\in I}\langle b_{i,k}, a_{k,j}\cdot f_{i,j}\rangle,\end{aligned} \tag{5}$$

where $i,j\in I$ and $i\leq j$. Since $\|Y\|_1 < \infty$, one can see that $\sup_{i,j\in I}\|b_{i,j}\| < \infty$. Let $M = \sup_{i,j\in I}\|b_{i,j}\|$. Take a finite subset F of I. We have

$$\begin{aligned}\sum_{i,j\in F}\sum_{k\in F}|\langle b_{i,k}, a_{k,j}\cdot f_{i,j}\rangle| &\leq \sum_{i,j\in F}\sum_{k\in F}\|a_{k,j}\|\|f_{i,j}\|\|b_{i,k}\| \\ &\leq \|\Lambda\|_\infty M\sum_{i,j\in F}\sum_{k\in F}\|a_{k,j}\| \\ &\leq \|\Lambda\|_\infty M\sum_{i,j\in F}\|a_{i,j}\| \leq \|\Lambda\|_\infty M\|X\|_1.\end{aligned}$$

So $\sum_{i,j\in I}\sum_{k\in I}|\langle b_{i,k}, a_{k,j}\cdot f_{i,j}\rangle| < \infty$. By rearrangement series in (5), we have

$$X\cdot\Lambda(Y) = \sum_{i,j\in I}\sum_{k\in I}\langle b_{i,j}, a_{j,k}\cdot f_{i,k}\rangle = \sum_{i,j\in I}\langle b_{i,j}, \sum_{k\in I} a_{j,k}\cdot f_{i,k}\rangle. \tag{6}$$

Suppose that $X\cdot\Lambda = (g_\alpha)_{\alpha\in J}$. By (6) for every $\alpha = (i,j)\in J$, we have $g_\alpha = g_{i,j} = \sum_{k\in I} a_{j,k}\cdot f_{i,k}$. Fixed $\varepsilon > 0$. Since Λ vanishes at infinity, there is a $\alpha_0 = (i_0, j_0)\in J$ such that for every $\alpha\geq\alpha_0$ we have $\|f_\alpha\| \leq \frac{\varepsilon}{\|X\|}$. Now for every $(i,j)\geq(i_0,j_0)$ in J with product ordering, we have

$$\|g_{i,j}\| \leq \sum_{k\in I}\|a_{j,k}\|\|f_{i,k}\| \leq \frac{\varepsilon}{\|X\|}\sum_{k\in I}\|a_{j,k}\| \leq \frac{\varepsilon}{\|X\|}\|X\| \leq \varepsilon,$$

note that in $UP(I,\mathcal{A})$, if $j > k$, then $a_{j,k} = 0$. Therefore $X\cdot\Lambda$ vanishes at infinity. $\square$

4. **Connes amenability of** $UP(I,\mathcal{A})$

Let $\mathcal{A}$ be a dual Banach algebra and let I be a totally ordered set. In this section we characterize the notion of Connes amenability of $UP(I,\mathcal{A})$. Throughout this section the set of all homomorphism from $\mathcal{A}$ into $\mathbb{C}$ is denoted by $\Delta(\mathcal{A})$ and the set of all wk^*-continuous homomorphism from $\mathcal{A}$ into $\mathbb{C}$ is denoted by $\Delta_{wk^*}(\mathcal{A})$. For every $\varphi \in \Delta(\mathcal{A})$, the notion of φ-amenability for a Banach algebra was introduced by Kaniuth, Lau and Pym [2]. Indeed $\mathcal{A}$ is φ-amenable if there exists a bounded linear functional m on $\mathcal{A}^*$ satisfying $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for every $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. They characterized φ-amenability in different ways:

- Through vanishing of the cohomology groups $\mathcal{H}^1(\mathcal{A}, X^*)$ for certain Banach $\mathcal{A}$-bimodule X [2, Theorem 1.1].
- Through the existence of a bounded net (u_α) in $\mathcal{A}$ such that $\|au_\alpha - \varphi(a)u_\alpha\| \to 0$ for all $a \in \mathcal{A}$ and $\varphi(u_\alpha) = 1$ for all α [2, Theorem 1.4].

By [2, Theorem 1.1], we conclude that every amenable Banach algebra is φ-amenable for any $\varphi \in \Delta(\mathcal{A})$.

In the sense of Connes amenability for a dual Banach algebra $\mathcal{A}$, the notion of φ-Connes amenability for $\varphi \in \Delta_{wk^*}(\mathcal{A})$, was introduced by Mahmoodi and some characterizations were given [4]. We say that $\mathcal{A}$ is φ-Connes amenable if there exists a bounded linear functional m on $\sigma wc(\mathcal{A}^*)$ satisfying $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for any $a \in \mathcal{A}$ and $f \in \sigma wc(\mathcal{A}^*)$. The concept of φ-Connes amenability was characterized through vanishing of the cohomology groups $\mathcal{H}^1_{wk^*}(\mathcal{A}, E)$ for certain normal dual Banach $\mathcal{A}$-bimodule E. By [4, Theorem 2.2], we conclude that every Connes amenable Banach algebra is φ-Connes amenable for any $\varphi \in \Delta_{wk^*}(\mathcal{A})$. If $\varphi \in \Delta_{wk^*}(\mathcal{A})$, then one may show that, $\varphi \otimes \varphi \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$, where $\varphi \otimes \varphi(a \otimes b) = \varphi(a)\varphi(b)$ for any $a, b \in \mathcal{A}$.

Now by inspiration of methods that used in [3, Proposition 3.2], we characterize the notion of φ-Connes amenability through the existence of a bounded net in $\mathcal{A}\hat{\otimes}\mathcal{A}$ with certain properties.

Proposition 4.1. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \Delta_{wk^*}(\mathcal{A})$. Then $\mathcal{A}$ is φ-Connes amenable if and only if there exists a bounded net $\{u_\alpha\}$ in $\mathcal{A}\hat{\otimes}\mathcal{A}$ such that*

(i) $a \cdot u_\alpha - \varphi(a)u_\alpha \xrightarrow{wk^*} 0$ *in* $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$.
(ii) $\langle u_\alpha, \varphi \otimes \varphi\rangle \longrightarrow 1$.

Proof. Let $\mathcal{A}$ be a φ-Connes amenable. Then by [4, Theorem 3.2], there exists an element M in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$such that for any $a \in \mathcal{A}$, $a \cdot M = \varphi(a)M$ and $\langle \varphi \otimes \varphi, M\rangle = 1$. Since $\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ is a closed subspace of $(\mathcal{A}\hat{\otimes}\mathcal{A})^*$, we have a quotient map $q : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. Composing the canonical inclusion map $\mathcal{A}\hat{\otimes}\mathcal{A} \hookrightarrow (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ with q, we obtain a continuous $\mathcal{A}$-bimodule map $\tau : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ which has a wk^*-dense range. So there exists a net $(u_\alpha)_{\alpha\in I}$ in $(\mathcal{A}\hat{\otimes}\mathcal{A})$ such that

$$M = wk^*\text{-}\lim_\alpha \tau(u_\alpha) = wk^*\text{-}\lim_\alpha(\hat{u}_\alpha)|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}. \tag{7}$$

By Goldstein's theorem, the net $(u_\alpha)_{\alpha\in I}$ can be chosen to be a bounded net. We know that for any $T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ and for any $a \in \mathcal{A}$,

$$T \cdot a - \varphi(a)T \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*.$$

So

$$\langle T \cdot a - \varphi(a)T, \hat{u}_\alpha\rangle \longrightarrow \langle T \cdot a - \varphi(a)T, M\rangle.$$

Thus we have

$$\langle T, a \cdot \hat{u}_\alpha\rangle - \langle T, \varphi(a)\hat{u}_\alpha\rangle \longrightarrow \langle T, a \cdot M\rangle - \langle T, \varphi(a)M\rangle.$$

This equation is equivalent with

$$\langle T, a\cdot \hat{u}_\alpha - \varphi(a)\hat{u}_\alpha\rangle \longrightarrow \langle T, a\cdot M - \varphi(a)M\rangle = 0. \tag{8}$$

Therefore, $a\cdot u_\alpha - \varphi(a)u_\alpha \xrightarrow{wk^*} 0 \quad \text{in} \quad (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$.

On the other hand, since $\varphi\otimes\varphi \in \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$, by (7) $\langle \varphi\otimes\varphi, \hat{u}_\alpha\rangle \longrightarrow \langle\varphi\otimes\varphi, M\rangle = 1$, that is,

$$\langle u_\alpha, \varphi\otimes\varphi\rangle \longrightarrow 1.$$

Conversely, regard (u_α) as a bounded net in $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. By Banach-Alaoglu theorem the bounded net $(\hat{u}_\alpha)|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}$ has a wk^*-limit point. Let

$$M = wk^*\text{-}\lim_\alpha ((\hat{u}_\alpha)|_{\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*}).$$

So $M \in (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. By the similar argument that we apply in (8), we have $a\cdot M - \varphi(a)M = 0$ and $\langle\varphi\otimes\varphi, M\rangle = 1$ as required. Hence by [4, Theorem 3.2] $\mathcal{A}$ is φ-Connes amenable. □

Now we deduce the main result of this paper.

Theorem 4.1. *Let I be a totally ordered set and let $\mathcal{A}$ be a unital dual Banach algebra with $\Delta_{wk^*}(\mathcal{A}) \neq \varnothing$. Then $UP(I,\mathcal{A})$ is Connes amenable if and only if I is singleton and $\mathcal{A}$ is Connes amenable.*

Proof. Let $UP(I,\mathcal{A})$ be Connes amenable. Then by [5, proposition 4.1], $UP(I,\mathcal{A})$ has an identity element. But every matrix algebra with unit must be finite dimensional. So in this case I is a finite set.

Assume that $I = \{i_1, ..., i_n\}$ and $\phi \in \Delta_{wk^*}(\mathcal{A})$. We define a map $\psi : UP(I,\mathcal{A}) \longrightarrow \mathbb{C}$ by $[a_{i,j}]_{i,j\in I} \longmapsto \phi(a_{i_n,i_n})$ for every $[a_{i,j}]_{i,j\in I} \in UP(I,\mathcal{A})$.

Since ϕ is wk^*-continuous, $\psi \in \Delta_{wk^*}(UP(I,\mathcal{A}))$. Now apply [4, Theorem 2.2], one can see that $UP(I,\mathcal{A})$ is ψ-Connes amenable. Using Proposition 4.1, there exists a bounded net $(u_\alpha) \subseteq UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A})$ such that

$$a\cdot \hat{u}_\alpha|_{\sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*} - \psi(a)\hat{u}_\alpha|_{\sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*} \xrightarrow{wk^*} 0 \qquad (a \in UP(I,\mathcal{A})) \tag{9}$$

and

$$\langle u_\alpha, \psi\otimes\psi\rangle \longrightarrow 1, \tag{10}$$

where $\psi\otimes\psi \in \sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*$ and $\psi\otimes\psi(a\otimes b) = \psi(a)\psi(b)$ for every $a, b \in UP(I,\mathcal{A})$.

It is well known that the map $\pi_{\sigma wc} : (\sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*)^* \longrightarrow UP(I,\mathcal{A})$ is wk^*-continuous. So by (9) we have

$$a\cdot \pi_{\sigma wc}(\hat{u}_\alpha|_{\sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*}) - \psi(a)\pi_{\sigma wc}(\hat{u}_\alpha|_{\sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*}) \xrightarrow{wk^*} 0,$$

for every $a \in UP(I,\mathcal{A})$. Let $\pi_{\sigma wc}(\hat{u}_\alpha|_{\sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*}) = m_\alpha$. Then (m_α) is a net in $UP(I,\mathcal{A})$ that satisfies $am_\alpha - \psi(a)m_\alpha \xrightarrow{wk^*} 0 \qquad (a \in UP(I,\mathcal{A}))$. On the other hand for every $f \in UP(I,\mathcal{A})_*$ we have

$$\begin{aligned} & \langle f, \pi_{\sigma wc}(\hat{u}_\alpha|_{\sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*})\rangle = \langle \pi^*|_{UP(I,\mathcal{A})_*}(f), \hat{u}_\alpha|_{\sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*}\rangle \\ = \quad & \langle \pi^*(f), \hat{u}_\alpha|_{\sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*}\rangle \\ = \quad & \langle \pi^*(f), \hat{u}_\alpha\rangle = \langle u_\alpha, \pi^*(f)\rangle = \langle \pi(u_\alpha), f\rangle, \end{aligned}$$

so

$$m_\alpha = \pi_{\sigma wc}(\hat{u}_\alpha|_{\sigma wc(UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A}))^*}) = \pi(u_\alpha). \tag{11}$$

Fixed α. Since $u_\alpha \in UP(I,\mathcal{A})\hat{\otimes}UP(I,\mathcal{A})$, there are b_k^α and c_k^α in $UP(I,\mathcal{A})$ such that $u_\alpha = \sum\limits_{k=1}^{\infty} b_k^\alpha \otimes c_k^\alpha$. So by (10), we have

$$\psi(\pi(u_\alpha)) = \psi(\pi(\sum_{k=1}^{\infty} b_k^\alpha \otimes c_k^\alpha)) = \psi(\sum_{k=1}^{\infty} b_k^\alpha c_k^\alpha) = \sum_{k=1}^{\infty} \psi(b_k^\alpha)\psi(c_k^\alpha) = \psi\otimes\psi(u_\alpha) \longrightarrow 1,$$

therefore by (11), $\psi(m_\alpha) \longrightarrow 1$. Let $L = \{[a_{i,j}] \in UP(I,\mathcal{A}) \mid a_{i,j} = 0, \forall j \neq i_n\}$. Since I is a finite set, it is easy to see that L is a wk^*-closed ideal in $UP(I,\mathcal{A})$. By definition of the map ψ, we have $\psi|_L \neq 0$. So there exists $\lambda \in L$ such that $\psi(\lambda) \neq 0$, by replacing $\frac{\lambda}{\psi(\lambda)}$ if necessary, we may assume that $\psi(\lambda) = 1$. Let $n_\alpha = m_\alpha\lambda$. Then n_α is a net in L. Since $lm_\alpha - \psi(l)m_\alpha \xrightarrow{wk^*} 0$ for any $l \in L$ and since the multiplication in $UP(I,\mathcal{A})$ is separately wk^*-continuous [7, Exercise 4.4.1], we have

$$ln_\alpha - \psi(l)n_\alpha = (lm_\alpha - \psi(l)m_\alpha)\lambda \xrightarrow{wk^*} 0, \tag{12}$$

for every $l \in L$ and also $\psi(n_\alpha) = \psi(m_\alpha)\psi(\lambda) = \psi(m_\alpha) \longrightarrow 1$. Now suppose that $|I| > 1$. Set $n_\alpha = \begin{pmatrix} 0 & \cdots & x_1^\alpha \\ : & \cdots & : \\ 0 & \cdots & x_n^\alpha \end{pmatrix}$, where $x_1^\alpha, \ldots, x_n^\alpha \in \mathcal{A}$. Consider $l = \begin{pmatrix} 0 & \cdots & l_1 \\ : & \cdots & : \\ 0 & \cdots & l_n \end{pmatrix}$, where $l_1, \ldots, l_n \in \mathcal{A}$ and $\phi(l_1) = \ldots = \phi(l_{n-1}) = 1$ but $\psi(l) = \phi(l_n) = 0$. So we have $ln_\alpha = \begin{pmatrix} 0 & \cdots & l_1x_n^\alpha \\ : & \cdots & : \\ 0 & \cdots & l_nx_n^\alpha \end{pmatrix}$. By (12), we have $ln_\alpha \xrightarrow{wk^*} 0$. Since I is a finite set, it is easy to see that $l_1x_n^\alpha \xrightarrow{wk^*} 0$. Since ϕ is wk^*-continuous, $\phi(l_1x_n^\alpha) \longrightarrow 0$. So $\phi(l_1)\phi(x_n^\alpha) \longrightarrow 0$. Since $\phi(l_1) = 1$, $\phi(x_n^\alpha) \longrightarrow 0$, which is a contradiction with $\phi(x_n^\alpha) = \psi(n_\alpha) \longrightarrow 1$. Thus $|I| = 1$.

Converse is clear. □

5. **Examples**

Here we give two examples of ϕ-Connes amenable dual Banach algebras, which are not Connes amenable.

Example 5.1. *Let $\mathcal{H}$ be a Hilbert space with $dim\mathcal{H} > 1$. Suppose that ϕ is a non-zero linear functional on $\mathcal{H}$ with $\|\phi\| \leq 1$. Define $a * b = \phi(a)b$ for every $a, b \in \mathcal{H}$. One can easily show that $(\mathcal{H}, *)$ is a Banach algebra and $\Delta(\mathcal{H}) = \{\phi\}$. We claim that $(\mathcal{H}, *)$ is a dual Banach algebra. By* [7, Exercise 4.4.1]*, it is sufficient to show that the multiplication $*$ is separately wk^*-continuous. Let $(a_\alpha)_{\alpha\in I}$ be a net in $\mathcal{H}$ such that $a_\alpha \xrightarrow{wk^*} a$ and let $b \in \mathcal{H}$. So $b * a_\alpha = \phi(b)a_\alpha \xrightarrow{wk^*} \phi(b)a = b * a$. Since $\mathcal{H}^{**} = \mathcal{H}$, $a_\alpha(\phi) \longrightarrow a(\phi)$. So $\phi(a_\alpha) \longrightarrow \phi(a)$. Hence $a_\alpha * b = \phi(a_\alpha)b \longrightarrow \phi(a)b = a * b$. So $a_\alpha * b \xrightarrow{wk^*} a * b$. Thus $(\mathcal{H}, *)$ is a dual Banach algebra. Already we have shown that ϕ is a wk^*-continuous character on $\mathcal{H}^*$. Pick a_0 in $\mathcal{H}$ such that $\phi(a_0) = 1$. So $a * a_0 = \phi(a)a_0$ and $\phi(a_0) = 1$ for every $a \in \mathcal{H}$. Thus $\mathcal{H}$ is ϕ-amenable. Since $\mathcal{H}^{**} = \mathcal{H}$ is a normal dual Banach $\mathcal{H}$-bimodule, by* [6, Proposition 4.4]*, $\sigma wc(\mathcal{H}^*) = \mathcal{H}^* = \mathcal{H}$. So $a_0 \in (\sigma wc(\mathcal{H}^*))^*$ such that $a_0(\phi) = 1$ and*

$$a_0(f \cdot a) = f \cdot a(a_0) = f(a * a_0) = f(\phi(a)a_0) = \phi(a)a_0(f),$$

for every $a \in \mathcal{H}$ and $f \in \sigma wc(\mathcal{H}^*)$. So $\mathcal{H}$ is ϕ-Connes amenable. We assume conversely that $\mathcal{H}$ is Connes amenable. Then $\mathcal{H}$ has an identity, say E. So for every $a \in \mathcal{H}$, $\phi(a)E = a * E = E * a = a$. It follows that $a = \phi(a)E$ for every $a \in \mathcal{H}$. So $dim\mathcal{H} = 1$, which is a contradiction.

Example 5.2. *Set* $\mathcal{A} = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{pmatrix}$. *With the usual matrix multiplication and* ℓ^1*-norm,* $\mathcal{A}$ *is a Banach algebra. Since* $\mathbb{C}$ *is a dual Banach algebra,* $\mathcal{A}$ *is a dual Banach algebra. We define a map* $\phi : \mathcal{A} \longrightarrow \mathbb{C}$ *by* $\phi\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = x$. *It is clear that* ϕ *is linear and multiplicative. Suppose that* $X_\alpha = \begin{pmatrix} x_\alpha & y_\alpha \\ 0 & 0 \end{pmatrix}$ *and* $X = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ *are elements in* $\mathcal{A}$ *such that* $X_\alpha \xrightarrow{wk^*} X$, *it is easy to see that* $x_\alpha \to x$, *thus* ϕ *is* wk^**-continuous and* $\phi \in \Delta_{wk^*}(\mathcal{A})$. *Now we show that* $\mathcal{A}$ *is not Connes amenable. If* $\mathcal{A}$ *is Connes amenable, then by applying* [5, Proposition 4.1], $\mathcal{A}$ *has an identity say* $E = \begin{pmatrix} x_0 & y_0 \\ 0 & 0 \end{pmatrix}$, *where* $x_0, y_0 \in \mathbb{C}$. *Since* ϕ *is a multiplicative functional,* $\phi(E) = 1$. *So* $x_0 = 1$. *For every* $a, b \in \mathbb{C}$, *we have*

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & y_0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & ay_0 \\ 0 & 0 \end{pmatrix}, \tag{13}$$

which implies that $ay_0 = b$ *for every* $a, b \in \mathbb{C}$, *which is a contradiction. Hence* $\mathcal{A}$ *is not Connes amenable. Next we show that* $\mathcal{A}$ *is* ϕ*-Connes amenable. Let* $u = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}\hat{\otimes}\mathcal{A}$. *Since* $\mathcal{A}\hat{\otimes}\mathcal{A}$ *embeds in* $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, *we may assume that* u *is in* $(\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. *Now for every* $a, b \in \mathbb{C}$, *we have*

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \cdot u &= \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ &= a\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \phi\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \phi\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} u. \end{aligned}$$

and also

$$\langle u, \phi \otimes \phi \rangle = \phi\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\phi\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 1 \times 1 = 1.$$

Now by Proposition 3.1, $\mathcal{A}$ *is* ϕ*-Connes amenable.*

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