ON BIFLATNESS AND ϕ-BIFLATNESS OF SOME BANACH ALGEBRAS

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In this paper we continue our work in [20]. *For a Banach algebra A with a character $\phi \in \Delta(A)$, we discuss the relation of ϕ-biflatness and left ϕ-amenability. We show that if a Segal algebra $S(G)$ ($S(G)^{**}$) is ϕ-biflat, then G is an amenable group. Also we show that ϕ-biflatness of a symmetric Segal algebra $S(G)$ is equivalent with amenability of G. We give the notion of bounded character biflat Banach algebras and study its character spaces. We show that for a non-empty totally ordered set I with a smallest element, upper triangular $I \times I$-matrix algebra, say $UP_I(A)$ is biflat if and only if A is biflat and I is singleton, provided that $\Delta(A)$ is non-empty and A has a right identity. Also we give a class of non biflat Banach algebras.*

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1. Introduction and Preliminaries

A Banach algebra A is amenable if for every bounded derivation $D : A \to X^*$ there exists an element x_0 in X^* such that

$$D(a) = a \cdot x_0 - x_0 \cdot a \quad (a \in A),$$

for every Banach $A-$bimodule X, see [12]. A. Ya. Helemskii studied Banach algebras through its homological properties. He introduced the concepts of biflat and biprojective Banach algebras. Indeed, a Banach algebra A is called biflat(biprojective), if there exists a bounded A-bimodule morphism $\rho : A \to (A\otimes_p A)^{**}(\rho : A \to A\otimes_p A)$ such that $\pi_A^{**}\circ\rho$ $(\pi_A\circ\rho)$ is the canonical embedding of A into A^{**}(is the identity map on A), respectively, where $\pi_A : A \otimes_p A \to A$ is denoted for product morphism given by $\pi_A(a \otimes b) = ab \quad (a, b \in A)$. In fact a Banach algebra A with a bounded approximate identity is biflat if and only if A is amenable. Using this fact he showed that for a locally compact group G, $L^1(G)$ is biflat(biprojective) if and only if G is an amenable (compact) group, respectively, see [7].

Recently a new notion of the amenability of Banach algebras related to its character space has been introduced. Suppose that A is a Banach algebra and $\phi \in \Delta(A)$. A is called left ϕ-amenable, if for each continuous derivation $D : A \to X^*$ there exists x_0 in X^* such that

$$D(a) = a \cdot x_0 - \phi(a)x_0 \quad (a \in A),$$

for every Banach $A-$bimodule X with a left action $a \cdot x = \phi(a)x$ which $a \in A$ and $x \in X$. Alaghmandan *et. al.* in [2] showed that a Segal algebra $S(G)$ is left $\phi-$amenable if and only if G is an amenable group. For more information about left $\phi-$amenability see [13], [10], [15] and [16].

Motivated by these considerations, author with A. Pourabbas introduced some generalizations of Helemskii's concepts like ϕ-biflatness and ϕ-biprojectivity, where ϕ is a multiplicative linear functional on A. Indeed a Banach algebra A is called ϕ-biflat (ϕ-biprojective)

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if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**} (\rho : A \to A \otimes_p A)$ such that

$$\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a) \quad (\phi \circ \pi_A \circ \rho(a) = \phi(a)) \qquad (a \in A),$$

respectively. We showed for a locally compact group G, $L^1(G)$ is ϕ-biflat if and only if G is amenable. We also showed that for every locally compact group G, the Fourier algebra $A(G)$ is ϕ-biprojective if and only if G is discrete, see [20].

In this paper we give criterions to study the relation of left ϕ-amenability and ϕ-biflatness. We show that a symmetric Segal algebra $S(G)$ is ϕ-biflat if and only if G is amenable. We study ϕ-biflatness of A^{**} and we show that if $S(G)^{**}$ is biflat, then G is an amenable group. We introduce the new class of character biflat Banach algebras and study its maximal ideal space. Finally we investigate Helemskii-notion of biflatness for a class of matrix algebras using ϕ-biflatness and left $\phi-$amenability and we give a class of non-biflat Banach algebras.

We remark some standard notations and definitions that we shall need in this paper. Let A be a Banach algebra. If X is a Banach A-bimodule, then X^* is also a Banach A-bimodule via the following actions

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \qquad (a \in A, x \in X, f \in X^*).$$

Throughout, the character space of A is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on A. Let $\phi \in \Delta(A)$. Then ϕ has a unique extension $\tilde{\phi} \in \Delta(A^{**})$ which is defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$.

Let A be a Banach algebra. The projective tensor product $A \otimes_p A$ is a Banach A-bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \ (b \otimes c) \cdot a = b \otimes ca \qquad (a, b, c \in A).$$

2. **General results about ϕ-biflatness of Banach algebras**

A Banach algebra A is left(right) ϕ-amenable if and only if there exists an element $m \in A^{**}$ such that $am = \phi(a)m \quad (ma = \phi(a)m)$ and $\tilde{\phi}(m) = 1$ for every $a \in A$, respectively, see [13, Theorem 1.1]. At the following Theorem we study the relation of ϕ-biflatness and left (right) ϕ-amenability.

Theorem 2.1. *Let A be a Banach algebra with a left(right) approximate identity and let $\phi \in \Delta(A)$. If A is ϕ-biflat, then A is left(right) ϕ-amenable, respectively.*

Proof. Let A be a ϕ-biflat Banach algebra. Then there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$ for every $a \in A$. Put $L = \ker \phi$. Set $g = (id_A \otimes \overline{\phi})^{**} \circ (id_A \otimes q)^{**} \circ \rho : A \to (A \otimes_p \mathbb{C})^{**}$, where $q : A \to \frac{A}{L}$ is the quotient map and $\overline{\phi} : \frac{A}{L} \to \mathbb{C}$ is a character defined by $\overline{\phi}(a + L) = \phi(a)$ for every $a \in A$. We see that g is a bounded left A-module morphism. We show that $g(l) = 0$ for every $l \in L$. Since A has a left approximate identity, $\overline{AL} = L$. Then for each $l \in L$ there exist sequences $(a_n) \subseteq A$ and $(l_n) \subseteq L$ such that $a_n l_n \to l$. For $b \in L$, define a map $R_b : A \to L$ by $R_b(a) = ab$ for every

$a \in A$. Since $q \circ R_{l_n} = 0$, we have

$$\begin{aligned} g(l) &= (id_A \otimes \overline{\phi})^{**} \circ (id_A \otimes q)^{**}(\rho(l)) \\ &= \lim_n (id_A \otimes \overline{\phi})^{**} \circ (id_A \otimes q)^{**}(\rho(a_n l_n)) \\ &= \lim_n (id_A \otimes \overline{\phi})^{**} \circ (id_A \otimes q)^{**}(\rho(a_n) \cdot l_n) \\ &= \lim_n (id_A \otimes \overline{\phi})^{**} \circ (id_A \otimes q)^{**} \circ (id_A \otimes R_{l_n})^{**}(\rho(a_n)) \\ &= \lim_n ((id_A \otimes \overline{\phi}) \circ (id_A \otimes q) \circ (id_A \otimes R_{l_n}))^{**}(\rho(a_n)) \\ &= \lim_n ((id_A \otimes \overline{\phi}) \circ (id_A \otimes (q \circ R_{l_n}))^{**}(\rho(a_n)) = 0. \end{aligned}$$

Therefore g induces a map $\overline{g} : \frac{A}{L} \to (A \otimes_p \mathbb{C})^{**}$ which is defined by $\overline{g}(a + L) = g(a)$ for all $a \in A$. It is easy to see that $\overline{g}$ is a bounded left A-module morphism. Pick a_0 in A such that $\phi(a_0) = 1$. We denote $\lambda : A \otimes_p \mathbb{C} \to A$ for a map which is specified by $\lambda(a \otimes z) = az$ for every $a \in A$ and $z \in \mathbb{C}$. Set $m = \lambda^{**} \circ \overline{g}(a_0 + L) \in A^{**}$, we claim that $am = \phi(a)m$ and $\tilde{\phi}(m) = 1$ for every $a \in A$. Since λ^{**} is a left A-module morphism and also since $aa_0 + L = \phi(a)a_0 + L$, we have

$$\begin{aligned} am = a\lambda^{**} \circ \overline{g}(a_0 + L) = \lambda^{**} \circ \overline{g}(aa_0 + L) &= \lambda^{**} \circ \overline{g}(\phi(a)a_0 + L) \\ &= \phi(a)\lambda^{**} \circ \overline{g}(a_0 + L) \\ &= \phi(a)m \end{aligned} \tag{1}$$

for every $a \in A$. Since $\rho(a_0) \in (A \otimes_p A)^{**}$, by Goldestine's theorem there exists a net (a_α) in $A \otimes_p A$ such that $a_\alpha \xrightarrow{w^*} \rho(a_0)$. So

$$\begin{aligned} \tilde{\phi}(m) = m(\phi) &= [\lambda^{**} \circ \overline{g}(a_0 + L)](\phi) \\ &= [\lambda^{**} \circ g(a_0)](\phi) \\ &= [\lambda^{**} \circ (id_A \otimes \overline{\phi})^{**} \circ (id_A \otimes q)^{**}(\rho(a_0))](\phi) \\ &= [(\lambda \circ (id_A \otimes \overline{\phi}) \circ (id_A \otimes q))^{**}(\rho(a_0))](\phi) \\ &= [w^* - \lim(\lambda \circ (id_A \otimes \overline{\phi}) \circ (id_A \otimes q))^{**}(a_\alpha))](\phi) \\ &= \lim(\lambda \circ (id_A \otimes \overline{\phi}) \circ (id_A \otimes q))^{**}(a_\alpha)(\phi) \\ &= \lim(\lambda \circ (id_A \otimes \overline{\phi}) \circ (id_A \otimes q)(a_\alpha)(\phi) \\ &= \lim \phi \circ \lambda \circ (id_A \otimes \overline{\phi}) \circ (id_A \otimes q)(a_\alpha) \\ &= \lim \phi \circ \pi_A(a_\alpha). \end{aligned} \tag{2}$$

On the other hand since $a_\alpha \xrightarrow{w^*} \rho(a_0)$, the w^*-continuity of π_A^{**} implies that

$$\pi_A(a_\alpha) = \pi_A^{**}(a_\alpha) \xrightarrow{w^*} \pi_A^{**}(\rho(a_0)).$$

Thus

$$\phi(\pi_A(a_\alpha)) = \pi_A(a_\alpha)(\phi) = \pi_A^{**}(a_\alpha)(\phi) \to \pi_A^{**}(\rho(a_0))(\phi) = \tilde{\phi} \circ \pi_A^{**}(a_\alpha) = 1. \tag{3}$$

We see that from (2) and (3), $\tilde{\phi}(m) = 1$. Combine this result with (1) implies that A is left ϕ-amenable. Right case is similar to the left one. □

Example 2.1. Let A be a Banach algebra with $\dim(A) > 1$ such that $ab = \phi(b)a$ for every $a, b \in A$, where $\phi \in \Delta(A)$. Suppose conversely that A has a left approximate identity, say $(e_\alpha)_\alpha$. Suppose that a_0 is an element in A such that $\phi(a_0) = 1$. We claim that $\lim e_\alpha = a_0$. To see this

$$a_0 = \lim e_\alpha a_0 = \lim \phi(a_0)e_\alpha = \lim e_\alpha.$$

It follows that a_0 is a left unit of A. Suppose that a is an arbitrary element of A. Then $a = a_0 a = \phi(a)a_0$, for every $a \in A$. It means that $\dim A = 1$ which is a contradiction.

We claim that A is ϕ-biflat. To see this let a_0 be an element in A such that $\phi(a_0) = 1$. Define $\rho : A \to (A \otimes_p A)^{**}$ by $\rho(a) = a \otimes a_0$ for each $a \in A$. One can easily see that ρ is a bounded A-bimodule morphism and $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$ for each $a \in A$. Hence A is ϕ-biflat.

We claim that A is not left ϕ-amenable. Suppose conversely that A is left ϕ-amenable. Then by [13, Theorem 1.4] there exists a net (a_α) in A such that

$$aa_\alpha - \phi(a)a_\alpha \to 0 \quad \phi(a_\alpha) = 1, \qquad (a \in A). \tag{4}$$

Suppose that a_0 is an element in A such that $\phi(a_0) = 1$. Put a_0 in equation (4) one can see that $\lim a_\alpha = a_0$. Using (4) again follows that $a = \phi(a)a_0$ for every $a \in A$. It implies that $\dim A = 1$ which is a contradiction.

Theorem 2.2. *Let A be a Banach algebra with a left approximate identity. If A^{**} is $\tilde{\phi}$-biflat then A is left ϕ-amenable.*

Proof. The proof is similar to the proof of Theorem 2.1 which for the sake of completeness we give it here. Suppose that A^{**} is $\tilde{\phi}$-biflat. Then there exists a A^{**}-bimodule morphism $\rho : A^{**} \to (A^{**} \otimes_p A^{**})^{**}$ such that $\tilde{\tilde{\phi}} \circ \pi_{A^{**}}^{**} \circ \rho(a) = \tilde{\phi}(a) \quad a \in A^{**}$. By restricting ρ on A, we can assume that $\rho : A \to (A^{**} \otimes_p A^{**})^{**}$. There exists a bounded linear map $\psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**}$ such that for $a, b \in A$ and $m \in A^{**} \otimes_p A^{**}$, the following holds;

(i) $\psi(a \otimes b) = a \otimes b$,
(ii) $\psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot m)$,
(iii) $\pi_A^{**}(\psi(m)) = \pi_{A^{**}}(m)$,

see [8, Lemma 1.7]. Define

$$g = \lambda^{****} \circ (id_A \otimes \overline{\phi})^{****} \circ (id_A \otimes q)^{****} \circ \psi^{**} \circ \rho : A \to A^{****},$$

where id_A, q, λ and $\tilde{\phi}$ are same as in the proof of Theorem 2.1. It is easy to see that g is a left A-module morphism and the restriction of g on $L = \ker \phi$ is 0. Thus g induces a left A-module morphism $\overline{g} : \frac{A}{L} \to A^{****}$. Pick $a_0 \in A$ such that $\phi(0) = 1$. Set $m = \overline{g}(a_0 + L)$. It is easy to see that $\tilde{\tilde{\phi}}(m) = 1$ and $am = \phi(a)m$ for every $a \in A$. Suppose that $\epsilon > 0$ and $F = \{a_1, ..., a_r\} \subseteq A^{**}$. Set

$$\begin{aligned} V =& \{(a_1 n - \phi(a_1)n, ..., a_r n - \phi(a_r)n, \ \tilde{\tilde{\phi}}(n) - 1) | n \in A^{**}, ||n|| \leq ||m||\} \\ &\subseteq (\prod_{i=1}^{r} A^{**}) \oplus_1 \mathbb{C}. \end{aligned}$$

By Goldestine's theorem there exists a net (n_α) in A^{**} such that $n_\alpha \xrightarrow{w^*} m$ and $||n_\alpha|| \leq ||m||$. Thus $(0, 0, ..., 0)$ is a w^*-limit point of V. On the other hand since V is a convex set, the weak topology and the norm topology are coincide on V. So $(0, 0, ..., 0)$ is a $|| \cdot ||$-limit point of V. Therefore there exists an element $n_{(F,\epsilon)}$ in A^{**} which satisfies

$$||a_i n_{(F,\epsilon)} - \phi(a_i)n_{(F,\epsilon)}|| < \epsilon, \quad |\tilde{\tilde{\phi}}(n_{(F,\epsilon)}) - 1| < \epsilon \tag{5}$$

for every $i \in \{1, 2, \ldots, r\}$. Observe that

$$\Delta = \{(F, \epsilon) : F \text{ is a finite subset of } A, \epsilon > 0\},$$

with the following order

$$(F, \epsilon) \leq (F', \epsilon') \Longrightarrow F \subseteq F', \quad \epsilon \geq \epsilon'$$

is a directed set. Equation 5 follows that there exists a net bounded net $(n_{(F,\epsilon)})_{(F,\epsilon)\in\Delta}$ in A^{**} such that

$$an_{(F,\epsilon)}-\phi(a)n_{(F,\epsilon)}\rightarrow 0,\quad \tilde{\phi}(n_{(F,\epsilon)})\rightarrow 1$$

for every $a\in A$. By Alauglu's theorem suppose that $n=w^{*}-\lim n_{(F,\epsilon)}\in A^{**}$. It is easy to see that $an=\phi(a)n$ and $\tilde{\phi}(n)=1$, for every $a\in A$. It means that A is left ϕ-amenable. □

Suppose that A is a Banach algebra and $\phi\in\Delta(A)$. A is called (approximately) ϕ-inner amenable, if there exists a bounded (not necessarily bounded) net $(a_\alpha)_\alpha$ in A such that $aa_\alpha-a_\alpha a\rightarrow 0$ and $\phi(a_\alpha)\rightarrow 1$, for every $a\in A$, respectively. For more information about ϕ-inner amenability see [11].

Corollary 2.1. *Let A be a Banach algebra with an approximate identity and $\phi\in\Delta(A)$. If A is ϕ-biflat then A is ϕ-inner amenable.*

Proof. Since A is ϕ-biflat with an approximate identity, Theorem 2.1 implies that A is left and right ϕ-amenable. Thus there exist bounded nets $(m_\alpha)_{\alpha\in I}$ and $(n_\beta)_{\beta\in J}$ in A such that

$$am_\alpha-\phi(a)m_\alpha\rightarrow 0,\quad n_\beta a-\phi(a)n_\beta\rightarrow 0,\quad \phi(m_\alpha)=\phi(n_\beta)=1,\qquad (a\in A).$$

Define $a^{\beta}_{\alpha}=m_\alpha n_\beta$, it is easy to see that

$$aa^{\beta}_{\alpha}-a^{\beta}_{\alpha}a\rightarrow 0,\quad \phi(a^{\beta}_{\alpha})=1,\qquad (a\in A).$$

Since $(a^{\beta}_{\alpha})_{\alpha\in I,\beta\in J}$ is a bounded net, A is ϕ-inner amenable. □

Remark 2.1. For the previous Corollary, the existence of an approximate identity is necessary which we can not remove it. To see this let A be the Banach algebra as in Example 2.1. We showed that A is ϕ-biflat, for some $\phi\in\Delta(A)$. Using the similar method which we used in Example 2.1, one can show that A has an approximate identity if and only if $\dim A=1$ and also A is ϕ-inner amenable if and only if $\dim A=1$. So if $\dim A>1$, then A is ϕ-biflat but A does'nt have an approximate identity and A is not ϕ-inner amenable.

We recall that a Banach algebra is approximately left(right) ϕ-amenable if there exists a not necessarily bounded net $(m_\alpha)_\alpha$ in A such that

$$am_\alpha-\phi(a)m_\alpha\rightarrow 0,\quad (m_\alpha a-\phi(a)m_\alpha\rightarrow 0),\quad \phi(m_\alpha)\rightarrow 1,$$

and for each $a\in A$, respectively. For more details see [1].

Proposition 2.1. *Let A be a Banach algebra and $\phi\in\Delta(A)$. Suppose that A is a ϕ-biflat Banach algebra which is approximately ϕ-inner amenable. Then A is approximately left and right ϕ-amenable.*

Proof. Since A is ϕ-biflat, there exists a bounded A-bimodule morphism $\rho:A\rightarrow(A\otimes_p A)^{**}$ such that $\tilde{\phi}\circ\pi^{**}_A\circ\rho(a)=\phi(a)$ for each $a\in A$. Suppose that $(a_\alpha)_{\alpha\in I}$ is a net in A which satisfies $aa_\alpha-a_\alpha a\rightarrow 0$ and $\phi(a_\alpha)\rightarrow 1$ for each $a\in A$. Set $n_\alpha=\rho(e_\alpha)$. Since ρ is a bounded A-bimodule morphism, we have

$$a\cdot n_\alpha-n_\alpha\cdot a=a\cdot\rho(e_\alpha)-\rho(e_\alpha)\cdot a=\rho(aa_\alpha-a_\alpha a)\rightarrow 0\qquad (a\in A)$$

and

$$\tilde{\phi}\circ\pi^{**}_A(n_\alpha)=\tilde{\phi}\circ\pi^{**}_A\circ\rho(e_\alpha)=\phi(a_\alpha)\rightarrow 1.$$

Let F and Γ be finite subsets of A and $(A\otimes_p A)^{*}$, respectively and also let $\epsilon>0$ be an arbitrary element. Take an element $\alpha(\Gamma,F,\epsilon)$ in I such that

$$||a\cdot n_\alpha-n_\alpha\cdot a||\leq\frac{\epsilon}{3K}\qquad\text{and}\qquad|\tilde{\phi}\circ\pi^{**}_A(n_\alpha)-1|<\frac{\epsilon}{2}\quad(a\in F,\alpha\geq\alpha(\Gamma,F,\epsilon)),$$

where $K=\max\{||f|||f\in\Gamma\}$. Since A is w^{*}-dense in A^{**}, there exists a net $(m^{\alpha(\Gamma,F,\epsilon)}_{\beta})_{\beta\in J}$ in $A\otimes_p A$ such that $m^{\alpha(\Gamma,F,\epsilon)}_{\beta}\xrightarrow{w^{*}}n_{\alpha(\Gamma,F,\epsilon)}$. Therefore $a\cdot m^{\alpha(\Gamma,F,\epsilon)}_{\beta}\xrightarrow{w^{*}}a\cdot n_{\alpha(\Gamma,F,\epsilon)},\quad m^{\alpha(\Gamma,F,\epsilon)}_{\beta}\cdot$

$a \xrightarrow{w^*} n_{\alpha(\Gamma,F,\epsilon)} \cdot a$ for each $a \in F$. Since π_A^{**} is a w^*-continuous map, $\pi_A(m_\beta^{\alpha(\Gamma,F,\epsilon)}) \xrightarrow{w^*} \pi_A^{**}(n_{\alpha(\Gamma,F,\epsilon)})$. Thus for each $a \in F$ and $f \in \Gamma$, there exists $\beta(\Gamma, F, \epsilon)$ in J such that for every $\beta \geq \beta(\Gamma, F, \epsilon)$ we have

$$|a \cdot m_\beta^{\alpha(\Gamma,F,\epsilon)}(f) - a \cdot n_{\alpha(\Gamma,F,\epsilon)}(f)| \leq \frac{\epsilon}{3}, \qquad |m_\beta^{\alpha(\Gamma,F,\epsilon)} \cdot a(f) - a \cdot n_{\alpha(\Gamma,F,\epsilon)}(f)| \leq \frac{\epsilon}{3},$$

and also

$$|\phi \circ \pi_A(m_\beta^{\alpha(\Gamma,F,\epsilon)}) - \tilde{\phi} \circ \pi_A^{**}(n_{\alpha(\Gamma,F,\epsilon)})| < \frac{\epsilon}{3}.$$

It follows that

$$\begin{aligned}
&|a \cdot m_\beta^{\alpha(\Gamma,F,\epsilon)}(f) - m_\beta^{\alpha(\Gamma,F,\epsilon)} \cdot a(f)| = \\
&|a \cdot m_\beta^{\alpha(\Gamma,F,\epsilon)}(f) - a \cdot n_{\alpha(\Gamma,F,\epsilon)}(f) + a \cdot n_{\alpha(\Gamma,F,\epsilon)}(f) - n_{\alpha(\Gamma,F,\epsilon)} \cdot a(f) \\
&+ n_{\alpha(\Gamma,F,\epsilon)} \cdot a(f) - m_\beta^{\alpha(\Gamma,F,\epsilon)} \cdot a(f)| \\
&\leq |a \cdot m_\beta^{\alpha(\Gamma,F,\epsilon)}(f) - a \cdot n_{\alpha(\Gamma,F,\epsilon)}(f)| + ||a \cdot n_{\alpha(\Gamma,F,\epsilon)} - n_{\alpha(\Gamma,F,\epsilon)} \cdot a|| ||f|| \\
&+ |n_{\alpha(\Gamma,F,\epsilon)} \cdot a(f) - m_\beta^{\alpha(\Gamma,F,\epsilon)} \cdot a(f)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned} \tag{6}$$

Also

$$|\phi \circ \pi_A(m_\beta^{\alpha(\Gamma,F,\epsilon)}) - 1| = |\phi \circ \pi_A(m_\beta^{\alpha(\Gamma,F,\epsilon)}) - \tilde{\phi} \circ \pi_A^{**} n_{\alpha(\Gamma,F,\epsilon)} + \tilde{\phi} \circ \pi_A^{**} n_{\alpha(\Gamma,F,\epsilon)} - 1| < \epsilon.$$

Set $m_{(\Gamma,F,\epsilon)} = m_{\beta_{(\Gamma,F,\epsilon)}}^{\alpha(\Gamma,F,\epsilon)}$. Using the partial order

$$(\Gamma, F, \epsilon) \leq (\Gamma^{'}, F^{'}, \epsilon^{'}) \Leftrightarrow \Gamma \subseteq \Gamma^{'}, F \subseteq F^{'}, \epsilon \geq \epsilon^{'}$$

one can show that $\{(\Gamma, F, \epsilon)\}$ is a directed set, where Γ and F are finite subsets of $(A \otimes_p A)^*$ and A, respectively and also $\epsilon > 0$. So for the net $(m_{(\Gamma,F,\epsilon)})_{(\Gamma,F,\epsilon)}$, we have

$$a \cdot m_{(\Gamma,F,\epsilon)} - m_{(\Gamma,F,\epsilon)} \cdot a \xrightarrow{w^*} 0, \quad (a \in A)$$

and

$$\phi \circ \pi_A(m_{(\Gamma,F,\epsilon)}) \to 1.$$

Using Mazur's Lemma we can assume that

$$a \cdot m_{(\Gamma,F,\epsilon)} - m_{(\Gamma,F,\epsilon)} \cdot a \xrightarrow{||\cdot||} 0, \quad (a \in A).$$

Suppose that $L : A \otimes_p A \to A$ is a map given by $L(a \otimes b) = \phi(b)a \quad (a, b \in A)$. Clearly L is a bounded linear map which satisfies

$$aL(x) = L(a \cdot x), \quad L(x \cdot a) = \phi(a)L(x), \quad \phi(L(x)) = \phi \circ \pi_A(x),$$

for every $a \in A, x \in A \otimes_p A$. It follows that

$$||aL(m_{(\Gamma,F,\epsilon)}) - \phi(a)L(m_{(\Gamma,F,\epsilon)}|| \leq ||L(a \cdot m_{(\Gamma,F,\epsilon)} - m_{(\Gamma,F,\epsilon)} \cdot a)|| \to 0 \qquad (a \in A)$$

and

$$\phi(L(m_{(\Gamma,F,\epsilon)})) = \phi \circ \pi_A(m_{(\Gamma,F,\epsilon)}) \to 1.$$

It means that A is approximately left ϕ-amenable. Similarly we can show that A is approximately right ϕ-amenable. $\square$

3. Application to Segal algebras

Throughout this section G is a locally compact group. A linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra on G if it satisfies the following conditions

(i) $S(G)$ is dense in $L^1(G)$,
(ii) $S(G)$ with a norm $||\cdot||_{S(G)}$ is a Banach space and $||f||_{L^1(G)} \leq ||f||_{S(G)}$ for every $f \in S(G)$,
(iii) for $f \in S(G)$ and $y \in G$, we have $L_y(f) \in S(G)$ the map $y \mapsto L_y(f)$ from G into $S(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$,
(iv) $||L_y(f)||_{S(G)} = ||f||_{S(G)}$ for every $f \in S(G)$ and $y \in G$.

It is well-known that $S(G)$ always has a left approximate identity. A Segal algebra $S(G)$ is called symmetric, if for every $f \in S(G)$ and $y \in G$, $R_y(f) \in S(G)$ and the map $y \mapsto R_y(f)$ is continuous. Also $||R_y(f)||_S = ||f||_S$, for $f \in S(G)$ and $y \in G$. We remind that a symmetric Segal algebra is an ideal of $L^1(G)$, for more information see [18].

For a Segal algebra $S(G)$ it has been shown that

$$\Delta(S(G)) = \{\phi_{|_{S(G)}} | \phi \in \Delta(L^1(G))\},$$

see [2, Lemma 2.2]. They showed for a locally compact group G, $S(G)$ is left ϕ-amenable if and only if G is amenable [2, Corollary 3.4]. We will show that for a symmetric Segal algebra $S(G)$, ϕ-biflatness is equivalent with amenability of G.

Corollary 3.1. *If $S(G)$ is ϕ-biflat. Then G is amenable*

Proof. Since every Segal algebra has a left approximate identity, by the Theorem 2.1, $S(G)$ is left ϕ-amenable. Then [2, Corollary 3.4] implies that G is amenable. □

We show that the converse of Corollary 3.1 is valid for symmetric Segal algebras.

Proposition 3.1. *Let G be a locally compact group, and $S(G)$ be a symmetric Segal algebra on G. Then for every $\phi \in \Delta(S(G))$ the followings are equivalent*

(i) *G is amenable,*
(ii) *$S(G)$ is ϕ-biflat,*
(iii) *$S(G)$ is left ϕ-amenable.*

Proof. (i)⇒(ii) Let G be an amenable group. Then $L^1(G)$ is amenable. So there exists a bounded net (m_α) in $L^1(G) \otimes_p L^1(G)$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_{L^1(G)}(m_\alpha)a \to a$ for every $a \in L^1(G)$. It is easy to see that $\phi \circ \pi_{L^1(G)}(m_\alpha) \to 1$ for every $\phi \in \Delta(L^1(G))$. Fix $\phi \in \Delta(L^1(G))$. Define a map $R : L^1(G) \otimes_p L^1(G) \to L^1(G)$ by $R(a \otimes b) = \phi(b)a$ and set $L : L^1(G) \otimes_p L^1(G) \to L^1(G)$ for a map which is specified by $L(a \otimes b) = \phi(a)b$ for every $a, b \in L^1(G)$. It is easy to see that L and R are bounded linear maps which satisfy

$$L(m \cdot a) = L(m) * a, \quad L(a \cdot m) = \phi(a)L(m) \quad (a \in L^1(G), m \in L^1(G) \otimes_p L^1(G))$$

and

$$R(a \cdot m) = a * R(m) \quad R(m \cdot a) = \phi(a)R(m) \quad (a \in L^1(G), m \in L^1(G) \otimes_p L^1(G)).$$

Thus

$$L(m_\alpha) * a - \phi(a)L(m_\alpha) = L(m_\alpha \cdot a - a \cdot m_\alpha) \to 0,$$

similarly we have $a * R(m_\alpha) - \phi(a)R(m_\alpha) \to 0$ for every $a \in L^1(G)$. Since

$$\phi \circ L = \phi \circ R = \phi \circ \pi_{L^1(G)},$$

it is easy to see that

$$\phi \circ L(m_\alpha) = \phi \circ R(m_\alpha) = \phi \circ \pi_{L^1(G)}(m_\alpha) \to 1.$$

Pick an element i_0 in $S(G)$ such that $\phi(i_0) = 1$. Set $n_\alpha = R(m_\alpha)i_0 \otimes i_0 L(m_\alpha)$ for every α. Since $L(m_\alpha)$ and $R(m_\alpha)$ are bounded nets in $L^1(G)$ and since $S(G)$ is an ideal of $L^1(G)$, we see that (n_α) is a bounded net in $S(G) \otimes_p S(G)$. Also

$$\begin{aligned} ||a \cdot n_\alpha - n_\alpha \cdot a||_{S\otimes_p S} &= ||a \cdot n_\alpha - \phi(a)n_\alpha + \phi(a)n_\alpha - n_\alpha \cdot a||_{S\otimes_p S} \\ &= ||a \cdot n_\alpha - \phi(a)n_\alpha||_{S\otimes_p S} + ||\phi(a)n_\alpha - n_\alpha \cdot a||_{S\otimes_p S} \to 0, \end{aligned} \tag{7}$$

for each $a \in S(G)$. Also we have

$$\phi \circ \pi_{S(G)}(n_\alpha) = \phi(R(m_\alpha) * i_0^2 * L(m_\alpha)) = \phi(R(m_\alpha))\phi(L(m_\alpha)) \to 1. \tag{8}$$

Let N be a w^*-cluster point of (n_α) in $(S(G) \otimes_p S(G))^{**}$. Combining (7) and (8) with the facts

$$a \cdot n_\alpha \xrightarrow{w^*} a \cdot N, \quad n_\alpha \cdot a \xrightarrow{w^*} N \cdot a, \quad \pi^{**}_{S(G)}(n_\alpha) \xrightarrow{w^*} \pi^{**}_{S(G)}(N) \quad (a \in (S(G))$$

we have

$$a \cdot N = N \cdot a, \quad \tilde{\phi} \circ \pi^{**}_{S(G)}(N) = 1 \quad (a \in (S(G))).$$

Define a map $\rho : S(G) \to (S(G) \otimes_p S(G))^{**}$ by $\rho(a) = a \cdot N$ for every $a \in S(G)$. It is easy to see that ρ is a bounded $S(G)$-bimodule morphism and $\tilde{\phi} \circ \pi^{**}_{S(G)} \circ \rho(a) = \tilde{\phi} \circ \pi^{**}_{S(G)}(a \cdot N) = \phi(a)$, so $S(G)$ is ϕ-biflat.
(ii)$\Rightarrow$(i) is clear by Corollary 3.1.
(iii)$\Leftrightarrow$(i) is clear by [2, Corollary 3.4]. □

Corollary 3.2. *If $S(G)^{**}$ is $\tilde{\phi}$-biflat then G is amenable.*

Proof. Since $S(G)$ has a left approximate identity, by Theorem 2.2 $\tilde{\phi}$-biflatness of $S(G)^{**}$ implies that $S(G)$ is left ϕ-amenable. Hence by [2, Corollary 3.4] G is an amenable group. □

Remark 3.1. The converse of previous Corollary is also true, whenever G is compact group. To see this, let $\hat{G}$ be the dual group of G which consists of all non-zero continuous homomorphism $\rho : G \to \mathbb{T}$. Since G is compact, $\hat{G} \subseteq L^\infty(G) \subseteq L^1(G)$. It is well-known that every character $\phi \in \Delta(L^1(G))$ has the form $\phi_\rho(f) = \int_G \overline{\rho(x)} f(x)dx$, where dx is the normalized Haar measure and $\rho \in \hat{G}$, for more details see [9, Theorem 23.7]. Clearly we have

$$\rho * f = f * \rho = \phi_\rho(f)\rho, \quad \phi_\rho(f)(\rho) = 1 \qquad (f \in L^1(G)).$$

Note that by [2, Lemma 2.2], $\Delta(S(G))$ is same as $\Delta(L^1(G))$. Now pick $f_0 \in S(G)$ which $\phi_\rho(f_0) = 1$. Since $\rho * f_0 = f_0 * \rho = \phi_\rho(f_0)\rho = \rho$, we have $\rho \in S(G)$. On the other hand since $\rho \in S(G)$, two maps $F \mapsto F\rho$ and $F \mapsto \rho F$ are w^*-continuous on $S(G)^{**}$, we have $F\rho = \rho F = \tilde{\phi}_\rho(F)\rho$ for all $F \in S(G)^{**}$. Hence the map $K : S(G)^{**} \to (S(G)^{**} \otimes S(G)^{**})^{**}$ defined by $K(F) = F \cdot \rho \otimes \rho$ is a bounded $S(G)^{**}$-bimodule morphism which satisfies

$$\tilde{\tilde{\phi}}_\rho \circ \pi^{**}_{S(G)^{**}} \circ K(F) = \tilde{\phi}_\rho(F) \quad (F \in S(G)^{**}).$$

It follows that $S(G)^{**}$ is $\tilde{\phi}_\rho$-biflat.

4. Bounded character biflat Banach algebras

Definition 4.1. Let A be a Banach algebra. A is called character biflat if for each $\phi \in \Delta(A)$ there exists a bounded A-bimodule morphism $\rho_\phi : A \to (A \otimes_p A)^{**}$ such that

$$\tilde{\phi} \circ \pi^{**}_A \circ \rho_\phi(a) = \phi(a) \quad (a \in A).$$

A is called bounded character biflat if A is character biflat and there exists $C > 0$ such that $||\rho_\phi|| < C$, for all $\phi \in \Delta(A)$.

It is easy to see that every biflat Banach algebra is bounded character biflat but the converse is not always true. At the following example we give a bounded character biflat Banach algebra which is not biflat.

Example 4.1. Consider the semigroup $\mathbb{N}_\wedge$, with the semigroup operation $m\wedge n = \min\{m, n\}$, where m and n are in $\mathbb{N}$. $\Delta(\ell^1(\mathbb{N}_\wedge))$ consists of the all functions $\phi_n : \ell^1(\mathbb{N}_\wedge) \to \mathbb{C}$ defined by $\phi_n(\sum_{i=1}^{\infty} \alpha_i\delta_i) = \sum_{i=n}^{\infty} \alpha_i$ for every $n \in \mathbb{N}$, where δ_i is point mass at $\{i\}$. See [3] for more details about the semigroup algebra $\ell^1(\mathbb{N}_\wedge)$. Author with A. Pourabbas in [20, Example 5.3] showed that $\ell^1(\mathbb{N}_\wedge)$ with respect to the $\ell^1(\mathbb{N}_\wedge)$-bimodule map $\rho_1 : \ell^1(\mathbb{N}_\wedge) \to (\ell^1(\mathbb{N}_\wedge))^{**}$ given by $\rho_1(a) = a \cdot \delta_1 \otimes \delta_1 \quad (a \in \ell^1(\mathbb{N}_\wedge))$ is ϕ_1-biflat. Also for each $n > 1$, set $\rho_n : \ell^1(\mathbb{N}_\wedge) \to (\ell^1(\mathbb{N}_\wedge) \otimes_p \ell^1(\mathbb{N}_\wedge))^{**}$ given by

$$\rho_n(a) = a \cdot \delta_n - \delta_{n-1} \otimes \delta_n - \delta_{n-1} \quad (a \in \ell^1(\mathbb{N}_\wedge)).$$

It is easy to see that

$$\phi_n \circ \pi_{\ell^1(\mathbb{N}_\wedge)} \circ \rho_n(a) = \phi(a) \quad (a \in \ell^1(\mathbb{N}_\wedge))$$

and $||\rho_n|| \leq 4$ for every $n \in \mathbb{N}$. It follows that $\ell^1(\mathbb{N}_\wedge)$ is bounded character biflat. But $\ell^1(\mathbb{N}_\wedge)$ is not biflat Banach algebra. To see this suppose conversely that $\ell^1(\mathbb{N}_\wedge)$ is biflat. Since $(\delta_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $\ell^1(\mathbb{N}_\wedge)$ see [3, Proposition 3.3.1], biflatness of $\ell^1(\mathbb{N}_\wedge)$ implies that $\ell^1(\mathbb{N}_\wedge)$ is amenable. Then [5, Theorem 2] follows that $E_{\mathbb{N}_\wedge}$ the set of idempotents of $\mathbb{N}_\wedge$ must be finite but as we know $\{\delta_n | n \in \mathbb{N}\}$ is an infinite subset of $E(\mathbb{N}_\wedge)$ which is impossible.

Let A be a Banach algebra and $\Delta(A)$ be a non-empty set. A is called C-left ϕ-amenable, if there exists $C > 0$ such that for each $\phi \in \Delta(A)$ and $m_\phi \in A^{**}$ which satisfies

$$am_\phi = \phi(a)m_\phi, \quad \tilde{\phi}(m) = 1$$

we have $||m_\phi|| < C$. A subset Y of a metric space (X, d) is called uniformly discrete if there exists a $\epsilon > 0$ such that for each x, y in X, $d(x, y) > \epsilon$.

Lemma 4.1. *Let A be a Banach algebra with a bounded left approximate identity. If A is bounded character biflat, then $\Delta(A)$ is a uniformly discrete subset of A^*.*

Proof. Suppose that A is bounded character biflat. Let $\phi \in \Delta(A)$ and ρ_ϕ be a bounded A-bimodule morphism such that

$$\tilde{\phi} \circ \pi_A^{**} \circ \rho_\phi(a) = \phi(a) \quad (a \in A),$$

which $||\rho_\phi||$ is bounded by some $C > 0$. Suppose that $(e_\alpha)_\alpha$ is a bounded left approximate identity for A with bound $K > 0$. It is easy to see that a net $(\rho_\phi(e_\alpha))$ is a bounded net in A^{**} with bound CK. Using Alauglu's Theorem, after passing to a subnet, we can assume that $\rho_\phi(e_\alpha) \xrightarrow{w^*} E$ for some E in A^{**} which $||E|| < CK$. Now similar to the arguments as in the proof of Theorem 2.1 set $m_\phi = [\lambda^{**} \circ (id_A \otimes \overline{\phi})^{**} \circ (id_A \otimes q)^{**}(E)]$. Using the similar method as in Theorem 1 and also Theorem 2 we can see that

$$am_\phi = \phi(a)m_\phi \to 0 \quad \tilde{\phi}(m_\phi) \to 1 \qquad (a \in A).$$

Hence m_ϕ is a left ϕ-mean and also the net $(m_\phi)_{\phi\in\Delta(A)}$ is a bounded net with bound MCK, where $M > 0$. So A is left MCK-ϕ-amenable for all $\phi \in \Delta(A)$. Applying [4, Corollary 2.2] one can see that $\Delta(A)$ is a uniformly discrete subset of A^*. □

5. Application to biflatness of upper triangular Matrix algebras

In this section we study the biflatness of some matrix algebras via the notion of ϕ-biflatness and right ϕ-amenability.

Proposition 5.1. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that I is a two-sided closed ideal of A which $\phi|_I \neq 0$. A is approximately left(right) ϕ-amenable if and only if I is approximately left(right) $\phi|_I$-amenable, respectively.*

Proof. For if part, suppose that A is approximately left ϕ-amenable. Then there exists a net $(m_\alpha)_\alpha$ in A such that $am_\alpha - \phi(a)m_\alpha \to 0$ and $\phi(m_\alpha) = 1$ for all $a \in A$. Pick $i_0 \in I$ which $\phi(i_0) = 1$. Set $n_\alpha = m_\alpha i_0$, then (n_α) is a net in I which

$$||in_\alpha - \phi(i)n_\alpha|| = ||im_\alpha i_0 - \phi(i)m_\alpha i_0|| \leq ||im_\alpha - \phi(i)m_\alpha||||i_0|| \to 0 \quad (i \in I)$$

and

$$\phi(n_\alpha) = \phi(m_\alpha i_0) = \phi(m_\alpha) = 1.$$

Hence I is approximately left $\phi|_I$-amenable.

For converse, suppose that I is approximately left $\phi|_I$-amenable. Then there exists a net (m_α) in I such that $im_\alpha - \phi(i)m_\alpha \to 0$ and $\phi(m_\alpha) = 1$ for all $i \in I$. Pick $i_0 \in I$ which $\phi(i_0) = 1$. Consider

$$\begin{aligned} ||am_\alpha - \phi(a)m_\alpha|| &= ||am_\alpha - ai_0m_\alpha + ai_0m_\alpha - \phi(a)m_\alpha|| \\ &\leq ||am_\alpha - ai_0m_\alpha|| + ||ai_0m_\alpha - \phi(a)m_\alpha|| \\ &\leq ||m_\alpha - i_0m_\alpha||||a|| + ||ai_0m_\alpha - \phi(ai_0)m_\alpha|| \\ &\to 0 \quad (a \in A) \end{aligned}$$

and $\phi(m_\alpha) = 1$. Then A is approximately left ϕ-amenable.

The proof of right case is same as the left one. □

Let A be a Banach algebra and I be a totally ordered set. By $UP_I(A)$ we denote the set of $I \times I$ upper triangular matrices which its entries come from A and

$$||(a_{i,j})_{i,j \in I}|| = \sum_{i,j \in I} ||a_{i,j}|| < \infty.$$

With matrix operations and $|| \cdot ||$ as a norm, $UP_I(A)$ becomes a Banach algebra. These algebras are similar (in properties) to the ℓ^1- Munn algebras. Existence of bounded approximate identity for ℓ^1- Munn algebras has been studied in [6] by Esslamzadeh. Using this approach Ramsden in [17] characterized biprojectivity and biflatness of some semigroup algebras which are related to a class of ℓ^1- Munn algebras.

Lemma 5.1. *Let A be a Banach algebra with a left (right) identity and I be a totally ordered set. Then $UP_I(A)$ has a left (right) approximate identity, respectively.*

Proof. It is clear that $UP_I(A)$ has left identity, whenever I is a finite set. Then suppose that I is an infinite set. Put $F(I)$ for the set of all finite subsets of I and 1_A for a left identity of A. Let $b = (b_{i,j})_{i,j \in I}$ be an arbitrary element of $UP_I(A)$. Then there exists an element $F \in F(I)$ such that $\sum_{i,j \in I-F} ||b_{i,j}|| < \epsilon$. Define $e_F = (a_{i,j})_{i,j \in I}$ with $a_{i,j} = 1_A$ whenever $i = j \in F$, otherwise $a_{i,j} = 0$.

$$||e_Fb - b|| = || \sum_{i,j \in I-F} b_{i,j}|| \leq \sum_{i,j \in I-F} ||b_{i,j}|| < \epsilon.$$

It means $UP_I(A)$ has a left approximate identity. Right case is similar to the left one. □

Theorem 5.1. *Let A be a Banach algebra with a right identity and $\Delta(A) \neq \emptyset$ and also let $(I, \leq)$ be a totally ordered set which has a smallest element. $UP_I(A)$ is biflat if and only if A is biflat and I is singleton.*

Proof. Only if part is clear.

Suppose $UP_I(A)$ is biflat. Then $UP_I(A)$ is ψ-biflat for every $\psi \in \Delta(UP_I(A))$. Let $i_0 \in I$ be a smallest element of I with respect to $\leq$ and $\phi \in \Delta(A)$. Define $\psi_{i_0}((a_{i,j})_{i,j\in I}) = \phi(a_{i_0,i_0})$, for every $(a_{i,j})_{i,j\in I} \in UP_I(A)$. It is easy to see that ψ_{i_0} is a character on $UP_I(A)$. Then $UP_I(A)$ is ψ_{i_0}-biflat. Using previous Lemma and Theorem 2.1, one can see that $UP_I(A)$ is right ψ_{i_0}-amenable. Let

$$J = \{(a_{i,j})_{i,j\in I} \in UP_I(A) | a_{i,j} = 0 \quad \text{for} \quad i \neq i_0\}.$$

It is easy to see that J is a closed ideal of $UP_I(A)$ and $\psi_{i_0}|_J \neq 0$. Thus by the right version of [13, Lemma 3.1] we have J is right ψ_{i_0}-amenable. Using the right version of [13, Theorem 1.4] there exists a bounded net (j_α) in J such that

$$j_\alpha j - \psi_\phi(j) j_\alpha \to 0, \quad \psi_\phi(j_\alpha) = 1 \quad (j \in J). \tag{9}$$

Suppose that I has at least two elements. We claim that $|I| = 1$. Suppose conversely that $|I| > 1$. Let a_0 be an element in A such that $\phi(a_0) = 1$. Set $j = \begin{pmatrix} 0 & a_0 & \cdots & a_0 & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ : & : & : & : & : \\ 0 & 0 & \cdots & 0 & \cdots \\ : & : & : & : & : \end{pmatrix}$

We know that for every α the element j_α has a form $\begin{pmatrix} j^\alpha_{i_0} & \cdots & j^\alpha_i & \cdots \\ 0 & \cdots & 0 & \cdots \\ : & : & : & : \\ 0 & \cdots & 0 & \cdots \\ : & : & : & : \end{pmatrix}$, where $j^\alpha_i \in A$ for every $i \in I$. Now put j and j_α in (9) we have $j^\alpha_0 a_0 \to 0$. Since ϕ is continuous, we have $\phi(j^\alpha_0) \to 0$. On the other hand $\psi_\phi(j_\alpha) = \phi(j^\alpha_0) = 1$ which is a contradiction. So I must be singleton and the proof is complete. □

Lemma 5.2. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that x_0 is an element in A which satisfies $ax_0 = x_0 a$ and $\phi(x_0) = 1$, for every $a \in A$. Then $UP_{\mathbb{N}\cup\{0\}}(A)$ is approximately ψ-inner amenable, for some $\psi \in \Delta(UP_{\mathbb{N}\cup\{0\}}(A))$.*

Proof. Suppose that $I = \mathbb{N} \cup \{0\}$. Define $\psi(\sum_{i,j\in I} a_{i,j}) = \phi(a_{0,0})$, it is clear that ψ is a character on $UP_I(A)$. Put $F(I)$ for the set of all finite subsets of I. Let a be an arbitrary element of $UP_I(A)$ and $F \in F(I)$ be such that $\sum_{i,j\in I-F} ||a_{i,j}|| < \frac{\epsilon}{||x_0||}$. Set

$$n_F = \max\{n | i_n \in F\}.$$

Define $a_{n_F} = \sum_{i,j\in\{1,2,\ldots,n_F\}} a_{i,j}$ with $a_{i,j} = x_0$ whenever $i = j \in \{1, 2, ..., n_F\}$, otherwise $a_{i,j} = 0$. Consider

$$||aa_{n_F} - a_{n_F}a|| \leq ||x_0|| || \sum_{i,j\in I-F} a_{i,j}|| < \epsilon, \quad \psi(a_{n_F}) = \phi(x_0) = 1.$$

Then $UP_I(A)$ is approximately ψ-inner amenable. □

Theorem 5.2. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that x_0 is an element in A which satisfies $ax_0 = x_0 a$ and $\phi(x_0) = 1$, for every $a \in A$. Then $UP_{\mathbb{N}\cup\{0\}}(A)$ is not biflat.*

Proof. Let $I = \mathbb{N} \cup \{0\}$. Suppose conversely that $UP_I(A)$ is biflat. Then $UP_I(A)$ is ψ-biflat, where ψ is the character which we defined as in the proof of Lemma 5.2. Using the Lemma 5.2, $UP_I(A)$ is approximately ψ-inner amenable. Thus by Proposition 2.1, we have $UP_I(A)$ is right approximate ψ-amenable. Set

$$J = \{(a_{i,j})_{i,j\in I} \in UP_I(A) | a_{i,j} = 0 \quad \text{for} \quad i \neq 0\}.$$

It is easy to see that J is a closed ideal of $UP_I(A)$ and $\psi|_J \neq 0$. By Proposition 5.1, we have J is approximately right ψ-amenable. Following the same way as in the proof of Theorem 5.1, we have a contradiction. □

Corollary 5.1. *Let G be a SIN group. Then $UP_{\mathbb{N}\cup\{0\}}(S(G))$ is not biflat.*

Proof. It is well-known that, if G is an SIN group, then $S(G)$ has a central approximate identity, say $(e_\alpha)_{\alpha\in I}$, see [14]. It follows that $ae_\alpha = e_\alpha a$ and $\phi(e_\alpha) \to 1$. Replacing e_α with $\frac{e_\alpha}{\phi(e_\alpha)}$ we can assume that $\phi(e_\alpha) = 1$. Applying Theorem 5.2, one can see that $UP_{\mathbb{N}\cup\{0\}}(S(G))$ is not biflat. □

Corollary 5.2. *Let A be a commutative Banach algebra which $\Delta(A)$ is non-empty. Then $UP_{\mathbb{N}\cup\{0\}}(A)$ is not biflat.*

Remark 5.1. Suppose that X is a compact space. Then $C(X)$ is an amenable Banach algebra [19, Example 2.3.4]. Since amenability implies the biflatness, $C(X)$ is biflat but using previous Corollary we can see that $UP_{\mathbb{N}}(C(X))$ is not biflat.
Set $M_I(A)$, for the set of all $I \times I$-matrices, say $(a_{i,j})_{i,j\in I}$, which $(a_{i,j})$ comes from A and $\sum_{i,j} ||a_{i,j}|| < \infty$. Note that $UP_I(A)$ is a subalgebra of $M_I(A)$. In the case of $I = \mathbb{N}$ and $A = \mathbb{C}$, $M_I(A)$ is biprojective so is biflat see [17, Proposition 2.7] but by previous Corollary $UP_I(A)$ is not biflat.

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