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ON THE DETERMINATION OF A COMPLEX FINITE JACOBI MATRIX FROM SPECTRAL DATA

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In this paper, we study the necessary and sufficient conditions for solvability of an inverse spectral problem for finite order complex Jacobi matrices (tri-diagonal symmetric matrices with complex entries). The problem is to reconstruct the complex Jacobi matrix from the spectral data consisting of eigenvalues and normalizing numbers of this matrix. An explicit procedure of reconstruction of the matrix from the spectral data is given.

Keywords: Jacobi matrix, resolvent function, spectral data, inverse spectral problem.

MSC2010: 15A29.

1. Introduction

Jacobi matrices appear in a variety of applications. Spectral theory of Jacobi matrices plays a fundamental role in the investigation of the classical moment problem and in the theory of orthogonal polynomials [1, 2, 3]. Inverse spectral problems for Jacobi matrices form a powerful tool for solving nonlinear discrete dynamical systems (see [11, 15, 17] and references given therein).

An $N \times N$ complex Jacobi matrix is a matrix of the form

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \tag{1}$$

where for each n, a_n and b_n are arbitrary complex numbers such that a_n is different from zero:

$$a_n, b_n \in \mathbb{C}, \ a_n \neq 0. \tag{2}$$

A distinguishing feature of the Jacobi matrix (1) from other matrices is that the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$ is equivalent to the second order linear difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \tag{3}$$

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$$n \in \{0, 1, \ldots, N-1\}, \quad a_{-1} = a_{N-1} = 1,$$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions $y_{-1} = y_N = 0$. This allows, using techniques from the theory of three-term linear difference equations [2], to develop a thorough analysis of the eigenvalue problem $Jy = \lambda y$.

Eq. (3) with the boundary conditions $y_{-1} = y_N = 0$, arises, for example, in the discretization of the (continuous) Sturm-Liouville eigenvalue problem

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + q(x)y(x) = \lambda y(x), \quad x \in [a, b], \tag{4}$$

$$y(a) = y(b) = 0,$$

where $[a, b]$ is a finite interval. To the equation in (4) considered on the semi-infinite interval $[0, \infty)$ or on the whole real axis $(-\infty, \infty)$ there correspond infinite Jacobi matrices.

Quantities connected with the eigenvalues and eigenvectors of the matrix are called the spectral chracteristics of the matrix. The general inverse spectral problem is to reconstruct the matrix given some of its spectral characteristics (spectral data). Many versions of the inverse spectral problem for finite and infinite Jacobi matrices have been investigated in the literature and many procedures and algorithms for their solution have been proposed (see [2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14]). Some of them form analogs of problems of inverse Sturm-Liouville theory [4, 16], in which a coefficient-function or "potential" in a second order differentrial equation is to be recovered, given either the spectral function, or alternatively given two sets of eigenvalues corresponding to two given boundary conditions at one end, the boundary condition at the other end being fixed.

Note that, in general, one spectrum consisting of the eigenvalues of the Jacobi matrix does not determine this matrix. It turns out that the eigenvalues together with the normalizing numbers or the so-called "two spectra" are enough to determine the Jacobi matrix in essential uniquely.

The spectral data, consisting of eigenvalues and normalizing numbers of the matrix J given by (1) and (2), is introduced as follows [6]. Let $R(\lambda) = (J - \lambda I)^{-1}$ be the resolvent of the matrix J (by I we denote the identity matrix of needed dimension) and e_0 be the N-dimensional column vector with the components $1, 0, \ldots, 0$. The rational function

$$w(\lambda) = -\langle R(\lambda)e_0, e_0\rangle = \left\langle (\lambda I - J)^{-1}e_0, e_0\right\rangle, \tag{5}$$

introduced earlier in [14], we call the *resolvent function* of the matrix J, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{C}^N$. This function is known also as the Weyl-Titchmarsh function of J. Denote by $\lambda_1, \ldots, \lambda_p$ all the distinct eigenvalues of the matrix J and by $m_1, \ldots, m_p$ their multiplicities, respectively, as the zeros of the characteristic polynomial $\det(J - \lambda I)$, so $1 \le p \le N$, $m_1 + \ldots + m_p = N$, and

$$\det(\lambda I - J) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p}. \tag{6}$$

We can decompose the rational function $w(\lambda)$ into partial fractions to get:

$$w(\lambda) = \sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(\lambda - \lambda_k)^j}, \tag{7}$$

where β_{kj} are some complex numbers uniquely determined by the matrix J. For each $k \in \{1, \ldots, p\}$ the (finite) sequence $\{\beta_{k1}, \ldots, \beta_{km_k}\}$ is called the *normalizing chain* (of the matrix J) associated with the eigenvalue λ_k.

The collection of the eigenvalues and normalizing numbers,

$$\{\lambda_k,\ \beta_{kj}\ (j = 1, \ldots, m_k;\ k = 1, \ldots, p)\}, \tag{8}$$

of the matrix J of the form (1), (2) is called the *spectral data* of this matrix.

The *inverse spectral problem* is to reconstruct the matrix using the eigenvalues and normalizing numbers (spectral data) of the matrix.

In this paper, we present solutions of the direct and inverse spectral problems for complex finite Jacobi matrices in terms of eigenvalues and normalizing numbers. These problems were elaborated by the author before in [6, 7] using the obtained there solution of the direct and inverse problems with respect to the so-called *generalized spectral function* that is a linear functional on the linear space of all polynomials with complex coefficients. In this paper we present a straightforward solution of the direct and inverse problems for spectral data consisting of eigenvalues and normalizing numbers of the complex matrix, not using the solution of the inverse problem for the generalized spectral function.

2. Spectral data

Given a Jacobi matrix J of the form (1) with the entries in (2), denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of Eq. (3) satisfying the initial conditions

$$P_{-1}(\lambda) = 0,\ P_0(\lambda) = 1; \quad Q_{-1}(\lambda) = -1,\ Q_0(\lambda) = 0. \tag{9}$$

For each $n \geq 0$, $P_n(\lambda)$ is a polynomial of degree n and is called a polynomial of the first kind and $Q_n(\lambda)$ is a polynomial of degree $n-1$ and is known as a polynomial of the second kind. These polynomials can be found recurrently from Eq. (3) using initial conditions (9).

Lemma 2.1. *(*[8, Lemma 1]*) The equality*

$$\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda) \tag{10}$$

holds so that the eigenvalues (counted according to their multiplicities) of the matrix J coincide with the zeros (counted according to their multiplicities) of the polynomial $P_N(\lambda)$.

Lemma 2.2. *(*[7, Lemma 3.1]*) The entries $R_{nm}(\lambda)$ of the matrix $R(\lambda) = (J - \lambda I)^{-1}$ (resolvent of J) are of the form*

$$R_{nm}(\lambda) = \begin{cases} P_n(\lambda)[Q_m(\lambda) + M(\lambda)P_m(\lambda)], & 0 \leq n \leq m \leq N-1, \\ P_m(\lambda)[Q_n(\lambda) + M(\lambda)P_n(\lambda)], & 0 \leq m \leq n \leq N-1, \end{cases} \tag{11}$$

where

$$M(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{12}$$

According to (5), (11), (12) and using initial conditions (9), we get

$$w(\lambda) = -R_{00}(\lambda) = -M(\lambda) = \frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{13}$$

By (10) and (6) we have $P_N(\lambda) = c(\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_p)^{m_p}$, where c is a nonzero constant. Therefore from (13) we can get for the resolvent function $w(\lambda)$ the decomposition (7) with

$$\beta_{kj} = \frac{1}{(m_k-j)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\left[(\lambda-\lambda_k)^{m_k}\frac{Q_N(\lambda)}{P_N(\lambda)}\right]$$
$$(j=1,\dots,m_k;\ k=1,\dots,p)$$

called the *normalizing numbers* of the matrix J.

Determination of the spectral data (8) of a given Jacobi matrix is called the *direct spectral problem* for this matrix.

The resolvent function $w(\lambda)$ of the matrix J can be constructed by using Eq. (13). Another convenient formula for computing the resolvent function is ([6, Section 5])

$$w(\lambda) = -\frac{\det(J_1-\lambda I)}{\det(J-\lambda I)} = \frac{\det(\lambda I-J_1)}{\det(\lambda I-J)}, \tag{14}$$

where J_1 is the truncated matrix obtained from J by deleting its first row and first column.

It follows from (14) that $\lambda w(\lambda)$ tends to 1 as $\lambda\to\infty$. Therefore multiplying (7) by λ and passing then to the limit as $\lambda\to\infty$, we find that

$$\sum_{k=1}^{p}\beta_{k1} = 1. \tag{15}$$

3. The orthogonality relations

Lemma 3.1. *(*[7, Lemma 3.2]*) Let $R_{nm}(\lambda)$ $(n,m=0,1,\dots,N-1)$ be entries of the matrix $R(\lambda)=(J-\lambda I)^{-1}$. For any vector $f=\{f_n\}_{n=0}^{N-1}\in\mathbb{C}^N$ and each $n\in\{0,1,\dots,N-1\}$, the representation*

$$\sum_{n=0}^{N-1} R_{nm}(\lambda)f_m = -\frac{f_n}{\lambda} + r_n(\lambda) \tag{16}$$

holds and there exist sufficiently large positive constants Λ and C such that

$$|r_n(\lambda)| \le \frac{C}{|\lambda|^2} \tag{17}$$

for all $n\in\{0,1,\dots,N-1\}$ and all $\lambda\in\mathbb{C}$ with $|\lambda|\ge\Lambda$.

Let us define a linear functional Ω on the linear space of all polynomials in λ with complex coefficients as follows: if $G(\lambda)$ is a polynomial then the value $\langle\Omega, G(\lambda)\rangle$ of the functional Ω on the element (polynomial) G is

$$\langle\Omega, G(\lambda)\rangle = \sum_{k=1}^{p}\sum_{j=1}^{m_k}\beta_{kj}\frac{G^{(j-1)}(\lambda_k)}{(j-1)!}, \tag{18}$$

where λ_k, β_{kj} $(j=1,\dots,m_k;\ k=1,\dots,p)$ are the spectral data of the matrix J, $G^{(n)}(\lambda)$ denotes the n-th order derivative of $G(\lambda)$ with respect to λ.

Theorem 3.1. *The "orthogonality" relations*

$$\langle \Omega, P_m(\lambda)P_n(\lambda)\rangle = \delta_{mn}, \quad m,n \in \{0,1,\dots,N-1\}, \tag{19}$$

hold, where δ_{mn} is the Kronecker delta.

Proof. Let f be an arbitrary element (column vector) of $\mathbb{C}^N$, with the components $f_0, f_1, \dots, f_{N-1}$. Writing (16) for this vector f and then integrating both sides, we obtain for each $n \in \{0,1,\dots,N-1\}$,

$$f_n = -\frac{1}{2\pi i}\oint_{\Gamma_r}\left\{\sum_{m=0}^{N-1} R_{nm}(\lambda) f_m\right\} d\lambda + \frac{1}{2\pi i}\oint_{\Gamma_r} r_n(\lambda)d\lambda, \tag{20}$$

where r is a sufficiently large positive number, Γ_r is the circle in the λ-plane, of radius r centered at the origin which encloses all the eigenvalues $\lambda_1, \dots, \lambda_p$ of J. By (11), (12), (13), and the fact that the integral of a regular function over Γ_r is zero, we have

$$-\frac{1}{2\pi i}\oint_{\Gamma_r}\left\{\sum_{m=0}^{N-1} R_{nm}(\lambda) f_m\right\} d\lambda = \frac{1}{2\pi i}\oint_{\Gamma_r} w(\lambda)F(\lambda)P_n(\lambda)d\lambda, \tag{21}$$

where

$$F(\lambda) = \sum_{m=0}^{N-1} f_m P_m(\lambda). \tag{22}$$

Next, applying the residue theorem and using the Leibnitz formula

$$[u(\lambda)v(\lambda)]^{(n)} = \sum_{j=0}^{N}\binom{n}{j} u^{(n-j)}(\lambda)v^{(j)}(\lambda),$$

we find, taking into account (7),

$$\frac{1}{2\pi i}\oint_{\Gamma_r} w(\lambda)F(\lambda)P_n(\lambda)d\lambda = \sum_{k=1}^{p}\mathrm{Res}_{\lambda=\lambda_k}[w(\lambda)F(\lambda)P_n(\lambda)]$$

$$= \sum_{k=1}^{p}\frac{1}{(m_k-1)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-1}}{d\lambda^{m_k-1}}\left[(\lambda-\lambda_k)^{m_k} w(\lambda)F(\lambda)P_n(\lambda)\right]$$

$$= \sum_{k=1}^{p}\frac{1}{(m_k-1)!}\lim_{\lambda\to\lambda_k}\sum_{j=0}^{m_k-1}\binom{m_k-1}{j}\left[\frac{d^{m_k-1-j}}{d\lambda^{m_k-1-j}}(\lambda-\lambda_k)^{m_k} w(\lambda)\right]$$

$$\times\frac{d^j}{d\lambda^j}[F(\lambda)P_n(\lambda)] = \sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(j-1)!}\left\{\frac{d^{j-1}}{d\lambda^{j-1}}[F(\lambda)P_n(\lambda)]\right\}_{\lambda=\lambda_k}. \tag{23}$$

Substituting (23) in (21) and then (21) in (20), passing then to the limit as $r \to \infty$, and taking into account that the second integral in the right side of (20) tends to zero by (17), we get

$$f_n = \sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(j-1)!}\left\{\frac{d^{j-1}}{d\lambda^{j-1}}[F(\lambda)P_n(\lambda)]\right\}_{\lambda=\lambda_k}, \quad n \in \{0,1,\dots,N-1\}, \tag{24}$$

where $F(\lambda)$ is defined by (22). Now by using (18), formula (24) can be written in the form $f_n = \langle \Omega, F(\lambda)P_n(\lambda)\rangle$, $n \in \{0, 1, \ldots, N-1\}$. Substituting here (22), we can write

$$f_n = \sum_{m=0}^{N-1} f_m \langle \Omega, P_m(\lambda)P_n(\lambda)\rangle, \quad n \in \{0, 1, \ldots, N-1\}. \tag{25}$$

Since the numbers $f_0, f_1, \ldots, f_{N-1}$ in (25) are arbitrary, it follows that the "orthogonality" relations (19) hold. □

4. **The fundamental equation of inverse problem**

As the $P_n(\lambda)$ is a polynomial of degree n, we can write the representation

$$P_n(\lambda) = \alpha_n \left(\lambda^n + \sum_{k=0}^{n-1} \chi_{nk}\lambda^k \right), \quad n \in \{0, 1, \ldots, N\}, \tag{26}$$

where α_n, χ_{nk} are some complex numbers. Substituting (26) in (3), we find that the coefficients a_n, b_n of system (3) and the quantities α_n, χ_{nk} of decomposition (26), are interconnected by the equations

$$a_n = \frac{\alpha_n}{\alpha_{n+1}} \quad (0 \le n \le N-2), \quad \alpha_0 = 1, \ \alpha_N = \alpha_{N-1}, \tag{27}$$

$$b_n = \chi_{n,n-1} - \chi_{n+1,n} \quad (0 \le n \le N-1), \quad \chi_{0,-1} = 0. \tag{28}$$

Let us set

$$s_l = \left\langle \Omega, \lambda^l \right\rangle = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \binom{l}{j-1} \beta_{kj} \lambda_k^{l-j+1}, \quad l = 0, 1, 2, \ldots\ , \tag{29}$$

where $\binom{l}{j-1}$ is a binomial coefficient and we put $\binom{l}{j-1} = 0$ if $j - 1 > l$.

Using the decomposition $\lambda^j = \sum_{i=0}^{j} c_{ji}P_i(\lambda)$, $j \in \{0, 1, \ldots, N\}$ it is easy to see that the relations (19) are equivalent to the relations

$$\langle \Omega, \lambda^m P_n(\lambda)\rangle = \frac{\delta_{mn}}{\alpha_n}, \quad m, n \in \{0, 1, \ldots, N-1\}. \tag{30}$$

Replacing $P_n(\lambda)$ in (30) by its expansion given in (26), we obtain

$$s_{n+m} + \sum_{k=0}^{n-1} \chi_{nk} s_{k+m} = 0, \quad m = 0, 1, \ldots, n-1, \ n \in \{1, 2, \ldots, N-1\}, \tag{31}$$

$$s_{2n} + \sum_{k=0}^{n-1} \chi_{nk} s_{k+n} = \frac{1}{\alpha_n^2}, \quad n \in \{0, 1, \ldots, N-1\}. \tag{32}$$

Note also that since $P_N(\lambda_k) = P_N'(\lambda_k) = \ldots = P_N^{(m_k-1)}(\lambda_k) = 0$, $k = 1, \ldots, N$, we have, according to (18), $\langle \Omega, \lambda^m P_N(\lambda)\rangle = 0$, $m = 0, 1, 2, \ldots$. This gives

$$s_{N+m} + \sum_{k=0}^{N-1} \chi_{Nk} s_{k+m} = 0, \ m = 0, 1, 2, \ldots .$$

Therefore Eq. (31) holds also for $n = N$ and we can write

$$s_{n+m} + \sum_{k=0}^{n-1} \chi_{nk} s_{k+m} = 0, \quad m = 0, 1, \ldots, n-1, \; n \in \{1, 2, \ldots, N\}. \tag{33}$$

Eq. (33) is the *fundamental equation* (a discrete version of the Gelfand-Levitan equation [4]) of the inverse problem, in the sense that it enables the problem to be formally solved. For, if we are given the spectral data (8), we can find the quantities s_l from (29) and then we consider the inhomogeneous system of linear algebraic equations (33) with unknowns $\chi_{n0}, \chi_{n1}, \ldots, \chi_{n,n-1}$, for every fixed $n \in \{1, 2, \ldots, N\}$. If this system is uniquely solvable, and $s_{2n} + \sum_{k=0}^{n-1} \chi_{nk} s_{k+n} \neq 0$ for $n \in \{1, 2, \ldots, N-1\}$, then the entries a_n, b_n of the required matrix J can be found from (27) and (28), respectively, α_n being found from (32). Below in Theorem 5.1 of Section 5 we give the conditions under which the indicated procedure of solving the inverse problem is rigorously justificated.

Given a collection (8), define the numbers s_l ($l = 0, 1, 2, \ldots$) by (29) and using these numbers introduce the determinants

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \quad n = 0, 1, 2, \ldots . \tag{34}$$

Lemma 4.1. *(*[10, Lemma 2]*) Given any collection (8), for the determinants D_n defined by (34) and (29) we have $D_n = 0$ for $n \geq N$, where $N = m_1 + \ldots + m_p$.*

Lemma 4.2. *(*[10, Lemma 3]*) If collection (8) is the spectral data of a matrix J of the form (1) with entries belonging to the class (2), then for the determinants D_n defined by (34) and (29) we have $D_n \neq 0$ for $n \in \{0, 1, \ldots, N-1\}$.*

5. Solution of the inverse problem

The following theorem gives a complete solution of the inverse spectral problem.

Theorem 5.1. *Let an arbitrary collection (8) of numbers be given, where $1 \leq p \leq N$, $m_1, \ldots, m_p$ are positive integers with $m_1 + \ldots + m_p = N$, $\lambda_1, \ldots, \lambda_p$ are distinct complex numbers. In order for this collection to be the spectral data for a Jacobi matrix J of the form (1) with entries belonging to the class (2), it is necessary and sufficient that the following two conditions are satisfied:*

- **(i):** $\sum_{k=1}^{p} \beta_{k1} = 1$;
- **(ii):** $D_n \neq 0$, *for* $n \in \{1, 2, \ldots, N-1\}$, *where D_n is the determinant defined by (34), (29).*

Under the conditions (i) and (ii) the entries a_n and b_n of the matrix J for which the collection (8) is spectral data, are recovered by the formulae

$$a_n = \frac{\pm\sqrt{D_{n-1} D_{n+1}}}{D_n}, \quad n \in \{0, 1, \ldots, N-2\}, \; D_{-1} = 1, \tag{35}$$

$$b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}}, \quad n \in \{0, 1, \ldots, N-1\}, \; \Delta_{-1} = 0, \; \Delta_0 = s_1, \tag{36}$$

where D_n is defined by (34), (29), and Δ_n is the determinant obtained from the determinant D_n by replacing in D_n the last column by the column with the components $s_{n+1}, s_{n+2}, \ldots, s_{2n+1}$.

Proof. The necessity of the conditions of Theorem 5.1 follows from (15) and Lemma 4.2. The proof of the sufficiency we shall give in several stages.

(a) Given a collection (8) satisfying the conditions (i) and (ii) of Theorem 5.1, consider Eq. (33) for fixed $n \in \{1, 2, \ldots, N\}$ with the unknowns χ_{nk}, $k = 0, 1, \ldots, n-1$, in which s_l are found with the aid of the collection (8) from expression (29). The determinant of system (33) coincides with the D_{n-1} and is different from zero by the condition (ii) of the theorem. Therefore system (33) is uniquely solvable. The solution can be found by making use of Cramer's rule. For this purpose, denote by $D_{n-1}^{(k)}$ $(k = 0, 1, \ldots, n-1)$ the determinat that is obtained from the determinant D_{n-1} by replacing in D_{n-1} the $(k+1)$-th column by the column with the components $s_n, s_{n+1}, \ldots, s_{2n-1}$. Then we have

$$\chi_{nk} = -\frac{D_{n-1}^{(k)}}{D_{n-1}}, \quad k = 0, 1, \ldots, n-1. \tag{37}$$

Using this expression for χ_{nk} we have

$$s_{2n} + \sum_{k=0}^{n-1} \chi_{nk} s_{k+n} = s_{2n} - \sum_{k=0}^{n-1} \frac{D_{n-1}^{(k)}}{D_{n-1}} s_{k+n}$$

$$= \frac{1}{D_{n-1}} \left(D_{n-1} s_{2n} - \sum_{k=0}^{n-1} D_{n-1}^{(k)} s_{k+n} \right) = \frac{D_n}{D_{n-1}} \neq 0. \tag{38}$$

Given the solution $(\chi_{nk})_{k=0}^{n-1}$ of the fundamental equaton (33), we find α_n from (32) which gives, by (38),

$$\frac{1}{\alpha_n^2} = \frac{D_n}{D_{n-1}}, \quad n \in \{0, 1, \ldots, N-1\}, \quad D_{-1} = 1, \tag{39}$$

with $\alpha_0 = 1$ and set $\alpha_N = \alpha_{N-1}$. Then we construct polynomials $P_n(\lambda)$ by

$$P_n(\lambda) = \alpha_n \left(\lambda^n + \sum_{k=0}^{n-1} \chi_{nk} \lambda^k \right), \quad n \in \{0, 1, \ldots, N\}. \tag{40}$$

Given collection (8) let us set

$$\langle \Omega, G(\lambda) \rangle = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \beta_{kj} \frac{G^{(j-1)}(\lambda_k)}{(j-1)!}, \tag{41}$$

where $G(\lambda)$ is any polynomial. We can show that for the constructed polynomials $P_n(\lambda)$ the orthogonality relations in (19) hold with this functional Ω. In fact, it is enough to show that (30) holds. But it is straightforward to get (30) from (33) and (32).

(b) We can show as in [6, pp.9–10] that the polynomials $P_n(\lambda)$, $n = 0, 1, \ldots, N$, constructed in accordance with (40) with the aid of the numbers χ_{nk} and α_n obtained by (37) and (39), satisfy the equations

$$\begin{aligned} b_0P_0(\lambda)+a_0P_1(\lambda) &= \lambda P_0(\lambda), \\ a_{n-1}P_{n-1}(\lambda)+b_nP_n(\lambda)+a_nP_{n+1}(\lambda) &= \lambda P_n(\lambda), \end{aligned} \tag{42}$$

$$n\in\{1,2,\ldots,N-1\},\ a_{N-1}=1,$$

where the coefficients a_n, b_n are given by the expressions

$$a_n=\frac{\alpha_n}{\alpha_{n+1}}\quad (0\le n\le N-2),\quad \alpha_0=1,\ \alpha_N=\alpha_{N-1}, \tag{43}$$

$$b_n=\chi_{n,n-1}-\chi_{n+1,n}\quad (0\le n\le N-1),\quad \chi_{0,-1}=0. \tag{44}$$

Now letting Δ_n be the determinant obtained from the determinant D_n by replacing in D_n the last column by the column with the components s_{n+1}, s_{n+2}, $\ldots$, s_{2n+1}, we get from (43), (44), by virtue of (37), (39), the formulae (35), (36).

(c) We show that $P_N(\lambda)=c(\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_p)^{m_p}$, where c is a nonzero constant. This will mean, in particular, that $\lambda_1,\ldots,\lambda_p$ are eigenvalues of the constructed matrix J with the entries a_n, b_n, of multiplicities $m_1,\ldots,m_p$, respectively. Set

$$H(\lambda)=(\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_p)^{m_p}. \tag{45}$$

Let us show that there exists a constant c such that

$$a_{N-2}P_{N-2}(\lambda)+b_{N-1}P_{N-1}(\lambda)+cH(\lambda)=\lambda P_{N-1}(\lambda) \tag{46}$$

for all λ. If we prove this, then from (42) with $n=N-1$ we get that $P_N(\lambda)=cH(\lambda)$. Since $\deg P_n(\lambda)=n$ $(0\le n\le N-1)$, $\deg H(\lambda)=N$, the polynomials $P_0(\lambda),\ldots,P_{N-1}(\lambda),H(\lambda)$ form a basis of the linear space of all polynomials of degree $\le N$ with complex coefficients. Therefore we have the decomposition

$$\lambda P_{N-1}(\lambda)=cH(\lambda)+\sum_{n=0}^{N-1}c_nP_n(\lambda), \tag{47}$$

where $c,c_0,c_1,\ldots,c_{N-1}$ are some complex constants. By (45) and (41) it follows that $\langle\Omega,H(\lambda)P_n(\lambda)\rangle=0,\ \ n\in\{0,1,\ldots,N\}$. Hence taking into account the relations (19) and $a_n=\langle\Omega,\lambda P_n(\lambda)P_{n+1}(\lambda)\rangle$, $(0\le n\le N-2)$; $b_n=\left\langle\Omega,\lambda P_n^2(\lambda)\right\rangle$, $(0\le n\le N-1)$, which follow from (42), we find from (47) that $c_n=0$ $(0\le n\le N-3)$, $c_{N-2}=a_{N-2}$, $c_{N-1}=b_{N-1}$. So (46) is shown.

It remains to show that for each $k\in\{1,\ldots,p\}$ the sequence $\{\beta_{k1},\ldots,\beta_{km_k}\}$ is the normalizing chain of the constructed matrix J, associated with the eigenvalue λ_k. Since we have already shown that λ_k is an eigenvalue of the matrix J of multiplicity m_k, the normalizing chain of J associated with the eigenvalue λ_k has the form $\{\widetilde{\beta}_{k1},\ldots,\widetilde{\beta}_{km_k}\}$. We have to show that $\widetilde{\beta}_{kj}=\beta_{kj}$ $(j=1,\ldots,m_k,\ k=1,\ldots,p)$. Let us set

$$\left\langle\widetilde{\Omega},G(\lambda)\right\rangle=\sum_{k=1}^{p}\sum_{j=1}^{m_k}\widetilde{\beta}_{kj}\frac{G^{(j-1)}(\lambda_k)}{(j-1)!},$$

where $G(\lambda)$ is any polynomial. By the direct spectral problem we have

$$\left\langle\widetilde{\Omega},P_m(\lambda)P_n(\lambda)\right\rangle=\delta_{mn},\quad m,n\in\{0,1,\ldots,N-1\}. \tag{48}$$

From (48) and (19) we get $\left\langle \widetilde{\Omega}, P_m(\lambda)P_n(\lambda)\right\rangle = \langle \Omega, P_m(\lambda)P_n(\lambda)\rangle$, $m, n \in \{0, 1, \ldots, N-1\}$. Since the $2N-1$ polynomials $P_n(\lambda) = P_0(\lambda)P_n(\lambda)$ $(n = 0, 1, \ldots, N-1)$, $P_m(\lambda)P_{N-1}(\lambda)$ $(m = 1, 2, \ldots, N-1)$ of degree $\le 2N-2$ are linearly independent (they have distinct degrees), any polynomial of degree $\le 2N-2$ can be represented as a linear combination of these polynomials. Consequently, $\left\langle \widetilde{\Omega}, G(\lambda)\right\rangle = \langle \Omega, G(\lambda)\rangle$ for all polynomials $G(\lambda)$ of degree $\le 2N-2$. Hence

$$\sum_{k=1}^{p}\sum_{j=1}^{m_k}(\widetilde{\beta}_{kj} - \beta_{kj})\frac{G^{(j-1)}(\lambda_k)}{(j-1)!} = 0$$

for all polynomials $G(\lambda)$ of degree $\le 2N-2$. Note that $2N-2 \ge N$ for $N \ge 2$. Since the $G^{(j-1)}(\lambda_k)$ can be arbitrary numbers (by the Hermite general interpolation theorem), it follows that $\widetilde{\beta}_{kj} - \beta_{kj} = 0$ $(j = 1, \ldots, m_k,\ k = 1, \ldots, p)$. Theorem 5.1 is completely proved. □

It follows from the above solution of the inverse problem that the matrix (1) is not uniquely restored from the spectral data. This is linked with the fact that the a_n are determined from (35) uniquely up to a sign. Thus, we can say that the inverse problem with respect to the spectral data is solved uniquely up to signs of the off-diagonal elements of the recovered Jacobi matrix.

6. **Examples and discussion of Theorem 5.1**

Note that in the case of arbitrary real distinct numbers $\lambda_1, \ldots, \lambda_N$ and positive numbers $\beta_1, \ldots, \beta_N$ the condition (ii) of Theorem 5.1 is satisfied automatically and in this case we have $D_n > 0$, for $n \in \{1, 2, \ldots, N-1\}$, (see [8, Lemma 7]). However, in the case of complex valued spectral data the condition (ii) of Theorem 5.1 may not be satisfied automatically.

1. Let $N = 2$ and take as collection (8) the data$\{\lambda_1,\ \lambda_2,\ \beta_1,\ \beta_2\}$, where λ_1, λ_2, β_1, and β_2 are arbitrary complex numbers such that $\lambda_1 \ne \lambda_2$, $\beta_1 \ne 0$, $\beta_2 \ne 0$, $\beta_1 + \beta_2 = 1$. Since in this case $D_{-1} = 1$, $D_0 = s_0 = 1$, $D_1 = \beta_1\beta_2(\lambda_1 - \lambda_2)^2 \ne 0$, we see that all the conditions of Theorem 5.1 are satisfied.

2. Let again $N = 2$ and take as collection (8) the data $\{\lambda_1,\ \beta_{11} = 1,\ \beta_{12}\}$, where λ_1 and β_{12} are arbitrary complex numbers (hence we get $N = 2$, $p = 1$, $m_1 = 2$) . We have $D_{-1} = 1$, $D_0 = s_0 = 1$, $D_1 = -\beta_{12}^2$. We see that the condition $D_1 \ne 0$ of Theorem 5.1 is equivalent to the condition that $\beta_{12} \ne 0$.

3. Let now $N = 3$ and as the collection (8) we take $\{\lambda_1,\ \lambda_2,\ \lambda_3,\ \beta_1,\ \beta_2,\ \beta_3\}$, where λ_1, λ_2, λ_3, β_1, β_2, β_3 are arbitrary complex numbers such that $\lambda_1 \ne \lambda_2$, $\lambda_1 \ne \lambda_3$, $\lambda_2 \ne \lambda_3$, $\beta_1 \ne 0$, $\beta_2 \ne 0$, $\beta_3 \ne 0$, $\beta_1 + \beta_2 + \beta_3 = 1$. We have

$$D_1 = \beta_1\beta_2(\lambda_1 - \lambda_2)^2 + \beta_1\beta_3(\lambda_1 - \lambda_3)^2 + \beta_2\beta_3(\lambda_2 - \lambda_3)^2,$$

$$D_2 = \beta_1\beta_2\beta_3(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2,$$

We see that the condition $D_1 \ne 0$ is not satisfied automatically and therefore one must require $D_1 \ne 0$ as a condition. For example, if $\beta_1 = \beta_2 = \beta_3 = 1/3$, $\lambda_1 = (1 \pm i\sqrt{3})/2$, $\lambda_2 = 1$, $\lambda_3 = 0$ then $D_1 = 0$.

7. Conclusions

In this work, a thorough spectral analysis of the finite order complex Jacobi matrices has been carried out. In particular, the concept of spectral data for such matrices has been introduced. The spectral data consist of the complex-valued eigenvalues and associated normalizing numbers derived by decomposing the resolvent function of the Jacobi matrix into partial fractions using the eigenvalues. The crucial point is derivation of the orthogonality relations (Theorem 3.1 in Section 3). Then the inverse problem has been studied and discussed how to reconstruct the complex Jacobi matrices from the spectral data. The uniqueness and existence results for solution of the inverse problem have been established and an explicit procedure of reconstruction of the matrix from the spectral data has been given.

A distinguishing feature of the Jacobi matrices from other matrices is that they are related to certain three-term recursion relations (second order linear difference equations). This allows, using techniques from the theory of linear difference equations, to develop more detailed analysis of the eigenvalue problem for Jacobi matrices. Spectral and inverse spectral problems for Jacobi matrices play a fundamental role in the investigation of completely integrable nonlinear lattices, in particular, in the investigation of the Toda lattices.

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