LAST MULTIPLIERS ON WEIGHTED MANIFOLDS AND THE WEIGHTED LIOUVILLE EQUATION

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We study the notion of last multipliers as time-independent solutions of the Liouville equation of transport in weighted (Riemannian) manifolds. On this way, several results from previous papers are generalized in a larger framework.

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1. Introduction

In January 1838, Joseph Liouville (1809-1882) published a note ([9]) on the time-dependence of the Jacobian of the ”transformation” exerted by the solution of an ODE on its initial condition. In modern language if \( A = A(x) \) is the vector field corresponding to the given ODE and \( m = m(t, x) \) is a smooth function (depending also of the time \( t \) ) then the main equation of the cited paper is:

\[
\frac{dm}{dt} + m \cdot \text{div} A = 0 \quad \text{(LE)}
\]
called, by then, the Liouville equation. The notion of last multiplier was introduced by Carl Gustav Jacob Jacobi (1804-1851) in "Vorlesungen über Dynamik", edited by R. F. A. Clebsch in Berlin in 1866. So, sometimes is used under the name of Jacobi (last) multiplier. Since then, this tool for understanding ODE was intensively studied by mathematicians in the usual Euclidean space \( \mathbb{R}^n \), conform the bibliography of [1]. In [2] we have obtained that, placed in a general oriented manifold, the last multipliers are the autonomous solutions of (LE). Moreover, in the series of papers [1]-[4] we consider these notions in some important frameworks as Riemannian, Poisson and Lie algebroids geometries. Let us remark that a Sturm-Liouville operator was studied in Riemannian manifolds by Prof. dr. C. Udrişte and I. Ţevy in [15].

The aim of the present note is to discuss some results of this useful theory extended to a new framework namely weighted manifolds. Our study is based on the excellent book [7] where this concept is considered from the point of view of geometrical analysis, more precisely the heat kernel is computed. Let us remark

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that a relationship between the heat equation and the general method of multipliers is well-known; see the examples from [13, p. 364].

The content of the paper is as follows. The first section is a review of definition of last multipliers and previous important results. The next section starts the new framework given by a *weighted oriented manifold* and presents the associated Liouville equation and last multipliers. The last section is devoted to the weighted Riemannian manifolds and assuming a Helmholtz type decomposition, some examples are given.

2. General facts about last multipliers

Let $M$ be a real, smooth, $n$-dimensional manifold, $C^\infty (M)$ the algebra of smooth real functions on $M$, $\mathcal{X} (M)$ the Lie algebra of vector fields and $\Lambda^k (M)$ the $C^1 (M)$-module of $k$-differential forms, $0 \leq k \leq n$. Suppose that $M$ is orientable with the fixed volume form $V \in \Lambda^n (M)$ and for a fixed $A \in \mathcal{X} (M)$ let us consider the $(n - 1)$-form $\Omega = i_A V \in \Lambda^{n-1} (M)$.

**Definition 2.1** ([5, p. 107], [11, p. 428]) The function $m \in C^1 (M)$ is called a *last multiplier* of $A$ if

$$d (m \Omega) := (dm) \wedge \Omega + m d \Omega = 0 \quad (2.1)$$

Let $LM (A)$ and $FInt(A)$ be respectively the set of last multipliers and first integrals for $A$.

In dimension 2 the notions of last multiplier and integrating factor are identical and Sophus Lie gave a method to associate a last multiplier to every symmetry vector field of $A$ (Theorem 1.1 in [8, p. 752]). The Lie method is extended to any dimension in [11].

Characterizations of $LM (A)$ can be obtained in terms of Witten's differential [16] and Marsden's differential [10] but we present here only the last since the former appears in [2, p. 458]. If $f \in C^\infty (M)$ the Marsden deformation of the differential is $d^f : \Lambda^* (M) \to \Lambda^{*+1} (M)$ defined by:

$$d^f (\omega) = \frac{1}{f} d (f \omega) \quad (2.2)$$

and whence $m$ is a last multiplier if and only if $\Omega$ is $d^m$-closed.

The following characterization of last multipliers will be useful:

**Lemma 2.2** ([11, p. 428]) $m \in C^\infty (M)$ belongs to $LM (A)$ if and only if:

$$A (m) + m \cdot div A = 0 \quad (2.3)$$

where $div A$ is the divergence of $A$ with respect to volume form $V$.

**Remarks 2.3** (i) The equation (2.3) is the time-independent version of the Liouville equation studied in [2] on manifolds. An important feature of equation (2.3) is that it does not always admit solutions conform [6, p. 269].

(ii) A first result given by (2.3) is the case of solenoidal i.e. divergence-free vector fields: $LM (A) = FInt (A)$. The importance of this result is shown by the fact
that three remarkable classes of solenoidal vector fields are provided by: Killing vector fields in Riemannian geometry, Hamiltonian vector fields in symplectic geometry and Reeb vector fields in contact geometry (in particular $K$-almost contact geometry). Also, there are many equations of mathematical physics which are modeled by a solenoidal vector field.

(iii) For the general case, namely $A$ is not solenoidal, the ratio of two last multipliers is a first integral and conversely, the product between a first integral and a last multiplier is a last multiplier. Since $FInt(A)$ is a subalgebra in $C^\infty(M)$ it results that $LM(A)$ is a $FInt(A)$-module.

(iv) Recalling the formulae:
\[
\text{div}(fX) = X(f) + f\text{div}X
\]
(2.4)
it follows that $m \in LM(A)$ if and only if the vector field $mA$ is solenoidal i.e. $\text{div}(mA) = 0$. Then $LM(A)$ is a linear subspace in $C^\infty(M)$.

(v) To the vector field $A$ we can associate an adjoint $A^*$, acting on functions in the following manner, [14]:
\[
A^*(m) = -A(m) - m\text{div}A.
\]
Then, another simple characterization is: $LM(A) = FInt(A^*)$.

An important structure generated by a last multiplier is given by:

**Proposition 2.4** ([2, p. 459]) Let $m \in C^\infty(M)$ be fixed. The set of vector fields admitting $m$ as last multiplier is a Lie subalgebra in $\mathcal{X}(M)$.

3. Last multipliers on weighted oriented manifolds

We extend the framework of previous section in the following manner:

**Definition 3.1** i) A weighted oriented manifold is a triple $(M, V, \Upsilon)$ with $(M, V)$ as above and $\Upsilon \in C^\infty_+(M)$ i.e. $\Upsilon$ is a smooth and strictly positive function on $M$.

(ii) Following the expression (1.2) we define the weighted divergence of $X \in \mathcal{X}(M)$ as:
\[
\text{div}_\Upsilon X = \frac{1}{\Upsilon}\text{div}(\Upsilon X).
\]
(3.1)

(iii) $m \in C^\infty(M)$ is a $\Upsilon$-weighted last multiplier for $A$ if it is a solution of the weighted Liouville equation:
\[
A(m) + m\text{div}_\Upsilon A = 0.
\]
(3.2)
Let $\Upsilon LM(A)$ be the set of these functions and with a subscript ”$+$” we will denote the subsets of strictly positive functions.

**Remarks 3.2** i) The weighted Liouville equation can be read as follows: $m$ is an ”eigenvector” of $A$ considered as derivation over the real algebra $C^\infty(M)$ with $-\text{div}_\Upsilon A$ as ”eigenvalue”.

(ii) If $m \in C^\infty_+(M)$ then (1.4) yields the following expression of (3.2):
\[
A(ln(m\Upsilon)) + \text{div}A = 0
\]
(3.3)
which means that $\Upsilon LM_+(A) = \frac{1}{\Upsilon}LM_+(A)$. □
For the general case, two situations when $\Upsilon LM(A)$ is completely determined are provided by the following result:

**Proposition 3.3** 

i) If $\Upsilon \in F I^+(A)$ then: $\Upsilon LM(A) = LM(A)$.

ii) If $A$ is divergence-free then: $\Upsilon LM(A) = \frac{1}{F} F Int(A)$.

**Proof** The equation (3.2) has the form:

$$A(m) + \frac{m}{\Upsilon} A(\Upsilon) + m div A = 0 \quad (3.4)$$

and then both implications above are immediately. □

The next result is a natural extension of the Proposition 2.4:

**Proposition 3.4** Let $m \in C^\infty_+(M)$ be fixed. Then the set of vector fields $X$ with $m \in \Upsilon LM_+(X)$ is a Lie subalgebra in $X(M)$.

**Proof** Obviously, the result can be obtained from Proposition 2.4 and the Remark 3.2 but we prefer to present a direct proof based on the identity:

$$div [X, Y] = X(div Y) - Y(div X) \quad (3.5)$$

Let $X, Y$ with the above property. Then:

$$[X, Y](ln(m\Upsilon)) + div([X, Y]) = X(Y(ln(m\Upsilon))) - Y(X(ln(m\Upsilon))) + X(div Y) - Y(div X) = 0$$

which gives the conclusion. □

4. Last multipliers on weighted Riemannian manifolds

A more interesting framework is provided by [7, p. 67]:

**Definition 4.1** A weighted manifold is a triple $(M, g = <, >, \Upsilon)$ with $(M, g)$ a Riemannian manifold.

On any weighted manifold there exists an induced volume form $V = V_g$. Let $\omega \in \Lambda^1(M)$ be the $g$-dual of $A$ and $\delta$ the co-derivative operator $\delta : \Lambda^*(M) \rightarrow \Lambda^{* - 1}(M)$. Then:

$$div_{V_g} A = -\delta \omega, \quad A(f) = g^{-1}(df, \omega) \quad (4.1)$$

and the condition (3.3) means:

$$g^{-1}(d(ln(m\Upsilon)), \omega) = \delta \omega \quad (4.2)$$

It follows that $m \in \Upsilon LM_+(A)$ if and only if $\omega$ belongs to the kernel of the differential operator: $g^{-1}(d(ln(m\Upsilon)), \cdot) - \delta : \Lambda^1(M) \rightarrow \Lambda^0 = C^\infty(M)$.

For the general case of $m$ an important fact is given by the product rule for divergence ([7, p. 69]):

$$div_{\mu}(fX) = g(\nabla f, X) + f div_{\mu}X \quad (4.3)$$

where $\nabla f$ is the $g$-gradient of $f$ and then the weighted Liouville equation (3.2) reads:

$$div_{\mu}(mA) = 0 \quad (4.4)$$

which means that $\Upsilon LM(A)$ is a is a "measure of how far away" is $A$ from being $\mu$-divergence-free.
An important tool in the Riemannian case is the weighted Laplacian ([7, p. 68]):

$$\Delta_{\mu} = \text{div}_{\mu} \circ \nabla.$$  \hspace{1cm} (4.5)

Now, assume that the vector field $A$ admits a Helmholtz type decomposition:

$$A = X + \nabla u$$  \hspace{1cm} (4.6)

where $X$ is a solenoidal vector field and $u \in C^\infty(M)$; for example if $M$ is compact such decompositions always exist. From $\nabla u(m) = \langle \nabla u, \nabla m \rangle$ it follows that (4.2) becomes:

$$X(m) + \langle \nabla u, \nabla m \rangle + m[X(\ln \Gamma) + \Delta_{\mu} u] = 0.$$  \hspace{1cm} (4.7)

**Example 4.1** $u$ is a $\Upsilon$-last multiplier of $A = X + \nabla u$ if and only if:

$$X(u) = -u[X(\ln \Upsilon) + \Delta_{\mu} u] - \|\nabla u\|^2_g.$$  \hspace{1cm} (4.8)

Suppose that $M$ is a cylinder $M = I \times N$ with $I \subseteq \mathbb{R}$ and $N$ a $(n - 1)$-manifold; then for $X = -\frac{1}{2} \frac{\partial}{\partial t} \in \mathcal{X}(I)$ which is divergence-free with respect to $V = dt \wedge V_N$ with $V_N$ a volume form on $N$, the previous relation yields:

$$u_t = 2 \left[ u(-\frac{1}{2} (\ln \Upsilon)_t + \Delta u) + \|\nabla u\|^2_g \right].$$  \hspace{1cm} (4.9)

By the product rule for the weighted Laplacian ([12, p. 55]):

$$\langle \nabla f, \nabla g \rangle = \frac{1}{2} (\Delta_{\mu} (fg) - f \cdot \Delta_{\mu} g - g \cdot \Delta_{\mu} f)$$  \hspace{1cm} (4.10)

the previous equation becomes:

$$u_t = -u(\ln \Upsilon)_t + \Delta_{\mu} (u^2)$$  \hspace{1cm} (4.11)

In particular, if $\Upsilon \in C^\infty_+(N)$ we get:

$$u_t = \Delta_{\mu} (u^2)$$  \hspace{1cm} (4.12)

which is a weighted version of the nonlinear parabolic equation of porous medium type.

**Example 4.2** Returning to (4.6) suppose that $X = 0$. The condition (4.7) reads:

$$m \cdot \Delta_{\mu} u + \langle \nabla u, \nabla m \rangle = 0$$  \hspace{1cm} (4.13)

which is equivalent, via (4.10) to:

$$\Delta_{\mu} (um) + m \cdot \Delta_{\mu} u = u \cdot \Delta_{\mu} m.$$  \hspace{1cm} (4.14)

which yields:

**Proposition 4.3** Let $u, m \in C^\infty(M)$ such that $u \in \Upsilon LM(\nabla m)$ and $m \in \Upsilon LM(\nabla u)$. Then $u \cdot m$ is a $\Upsilon$-harmonic function on $M$. $u \in \Upsilon LM(\nabla u)$ if and only if $u^2$ is a $\Upsilon$-harmonic function on $M$.

**Proof** Adding to (4.14) a similar relation with $u$ replaced by $m$ gives the conclusion. \quad $\square$

**Example 4.4.** The gradient of distance function with respect to a 2D rotationally symmetric metric
Let $M$ be a 2D manifold with local coordinates $(t, \theta)$ endowed with a rotationally symmetric metric $g = dt^2 + \varphi^2(t)d\theta^2$ conform [12, p. 11]. Let the smooth function $u(t, \theta) = t$ which appear as a distance function with respect to the given metric. Then $\nabla u = \frac{\partial}{\partial t}$ and $\Delta_u u = \frac{1}{\varphi} \frac{\partial \varphi}{\partial t}$; the equation (3.13) is:

$$m \cdot \left( \frac{\varphi}{\varphi} \frac{\partial}{\partial t} \right) + \frac{\partial m}{\partial t} = 0$$

(4.15)

with the solutions: $m = \frac{c f(\theta)}{\varphi}$ for $c \in \mathbb{R}$. Therefore $\frac{1}{\varphi} LM(\frac{\partial}{\partial t}) = \mathbb{R} \cdot C^\infty([0, 2\pi])$.

REFERENCES