NUMERICAL SOLUTION OF STOCHASTIC FRACTIONAL PDES BASED ON TRIGONOMETRIC WAVELETS

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This article presents a trigonometric wavelet–based implicit numerical method in order to approximate the solutions of a class of stochastic fractional partial differential equations (SFPDEs). The existence and uniqueness of mild solution is studied. Furthermore, the convergence analysis of the proposed method is investigated. Finally, some numerical examples are performed to show the validity of theoretical results.

Keywords: trigonometric wavelets, Caputo fractional derivative, stochastic fractional partial differential equations, implicit numerical method, convergence.

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1. Introduction

During the past decades, fractional equations have received significat attention. They have been applied to model various phenomena in physics biology, engineering [14], viscoelasticity [11], finance [18], hydrology [20] and control systems [12]. Many of these phenomena have some uncertainty, therefore to have more accurate solutions, we need them be considered as a stochastic equation. Debbi and Dozzi [6] proved the existence, uniqueness and regularity of the solution for a stochastic fractional partial differential equation driven by a time white noise in one dimension. Sakthivel et al. [19] proved the existence of mild solutions for a class of fractional stochastic differential equations with impulses in Hilbert spaces. They studied the existence and uniqueness of mild solutions for semi-linear stochastic fractional differential equations with nonlocal conditions, finite delay and without impulse. Mijena and Nane [13] considered fractional heat equation on unbounded domains, with a non-linear random external force, involving space-time white noise. Sufficient conditions for the existence and uniqueness of mild solutions are derived. Anh et al. [2] studied the weak-sense solution of the fractional in space and time SPDE with Dirichlet boundary conditions. The driven process is constructed from time fractional integration of the space-time Gaussian white noise. In [3] Chen et al. studied the existence and uniqueness of the mild solution of the linear stochastic partial differential equation of fractional orders both in time and space. Their proof are based on some properties of the fundamental solutions represented in terms of the Fox H-functions. Printems in [15] applied finite difference method for time discretization based on the spectral properties of the linear operator of SPDE.

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The aim of this paper is to generalize some results of [15] to nonlinear stochastic fractional partial differential equation (1) driven by a time white noise. Our method is based on trigonometric wavelet approximations [16], some of these functions can be reshaped to satisfy on boundary conditions exactly. The numerical experiments support our conjecture. We study the existence and uniqueness of the solution of the equation formally given by

$$du = \left(Au + \frac{\partial^\alpha}{\partial x^\alpha} (F(u)) \right) dt + dW, 0 < \alpha \leq 1, \tag{1}$$

$$u(x,0) = u_0 \in H = L^2(\mathbb{T}),\ \mathbb{T} = (0, 2\pi), \tag{2}$$

where $A = \frac{\partial^2}{\partial x^2}$, and u is a H-valued random process, $\frac{\partial^\alpha}{\partial x^\alpha}$ denote Caputo fractional partial derivative of α order defined as in [14] by

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} \frac{\partial u(s,t)}{\partial s} ds, \tag{3}$$

where $0 < \alpha \leq 1$ and $\Gamma(\cdot)$ is the gamma function:

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx. \tag{4}$$

Moreover, let the boundary condition $u(0,t) = u(2\pi,t) = 0$ and $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz function. The numerical experiments support our claim, also this procedure can be adapted to extend SFPDE (1) to

$$du = \left(Au + \sum_k \frac{\partial^{\alpha_k}}{\partial x^{\alpha_k}} (F_k(u)) \right) dt + dW, \quad \forall k, \quad 0 \leq \alpha_k \leq 1, \tag{5}$$

where A denotes an unbounded, self-adjoint operator that is combination of even order differential operators

$$A = \sum_{j=0}^{N} \frac{\partial^{2j}}{\partial x^{2j}}.$$

The rest of the paper is organized as follows: In Section 2 we review the concept of multi-resolution analysis in wavelet bases. This is one of the key concepts that will be used in the paper. Section 3 investigate the existence of mild solution of the SFPDE. The convergence of the proposed method is explained in Section 4. In Section 5, numerical examples are presented, in order to illustrate the results. Finally, the conclusion are given in Section 6.

Nomenclatures

$\frac{\partial^\alpha}{\partial x^\alpha}$	the Caputo fractional derivative of α order
$\Gamma(\cdot)$	the gamma function
$V^\perp$	the orthogonal complement of V
ψ	a wavelet
ϕ	a scaling function
$\delta_{k,n}$	the Kronecker delta function
$(-A)^{\frac{\alpha}{2}}$	the fractional power of the differential operator A for $0 \le \alpha \le 1$
$H^\alpha(0, T)$	the fractional Sobolev space on the interval $(0, T)$
$W^{p,k}$	the Sobolev space
$L^2(\Omega)$	a standard space, composed of (measurable) functions f on Ω such that $\int_\Omega \|f\|^2 dx < \infty$
L^2	$L^2(0, 2\pi)$
D_L^α	the Riemann-Liouville fractional derivative

2. Preliminaries

As we know, the multi-resolution analysis is one of the best framework to describe wavelets. From [16], we have the following results.

Definition 2.1. *(Multi-resolution analysis): A multi-resolution analysis of $L^2(\mathbb{T})$ with inner product $\langle .,. \rangle$, is defined as a sequence of closed subspaces $V_j \subset L^2(\mathbb{T})$, $j \in \mathbb{Z}$ with the following properties*

(i) $V_j \subset V_{j+1}$, $-1 \le j \in \mathbb{Z}$,

(ii) $\bigcup_{j=-1}^{\infty} V_j$ *is dense in* $L^2(\mathbb{T})$ *and* $\bigcap_{j=-1}^{\infty} V_j = \{0\}$.

Let $W_j = V_{j+1} \cap V_j^\perp$, then we have $V_{j+1} = V_j \oplus W_j$ and $L^2 = V_0 \oplus \left(\oplus_{j=0}^{\infty} W_j\right)$. In this work V_j and W_j are defined by

$$V_j = \operatorname{span}\left\{1, \cos x, \dots, \cos(2^{j+1}-1)x, \sin x, \dots, \sin 2^{j+1}x\right\}, \tag{6}$$

and

$$W_j = \operatorname{span}\left\{\cos 2^{j+1}x, \dots, \cos(2^{j+2}-1)x, \sin(2^{j+1}+1)x, \dots, \sin 2^{j+2}x\right\}. \tag{7}$$

Alternative bases for these spaces are

$$V_j = \operatorname{span}\left\{\phi^0_{j,n}, \phi^1_{j,n}: \quad n = 0, \dots, 2^{j+1}-1\right\},$$

$$W_j = \operatorname{span}\left\{\psi^0_{j,n}, \psi^1_{j,n}: \quad n = 0, \dots, 2^{j+1}-1\right\}.$$

For simplicity, we define the following notation

$$D_l(x) = \frac{1}{2} + \sum_{k=1}^{l} \cos kx, \quad \widetilde{D}_l(x) = \sum_{k=1}^{l} \sin kx.$$

We let $\phi^0_{j,0}$ and $\phi^1_{j,0}$ denote scaling functions and $\psi^0_{j,0}$ and $\psi^1_{j,0}$ denote wavelets

$$\phi^0_{j,0}(x) = \frac{1}{2^{2j+1}} \sum_{l=0}^{2^{j+1}-1} D_l(x),$$
$$\phi^1_{j,0}(x) = \frac{1}{2^{2j+1}} \left(\widetilde{D}_{2^{j+1}}(x) + \frac{1}{2}\sin(2^{j+1}x) \right),$$
$$\psi^0_{j,0}(x) = \frac{1}{2^{j+1}} \cos 2^{j+1}x + \frac{1}{3 \cdot 2^{2j+1}} \sum_{l=2^{j+1}+1}^{2^{j+2}-1} \left(3 \cdot 2^{j+1} - l\right) \cos lx,$$
$$\psi^1_{j,0}(x) = \frac{1}{3 \cdot 2^{2j+1}} \sum_{l=2^{j+1}+1}^{2^{j+2}-1} \sin lx + \frac{1}{2^{2j+3}} \sin 2^{j+2}x.$$

Let $x_{j,n} = \frac{n\pi}{2^j}$, for $n = 0, \ldots 2^{j+1}-1$ and $\phi^0_{j,n}(x) = \phi^0_{j,0}(x - x_{j,n})$, the same definition is used for $\phi^1_{j,n}(x)\,, \psi^0_{j,n}(x)$ and $\psi^1_{j,n}(x)\,.$

Theorem 2.1. [16] *(Interpolatory properties of the scaling function). The following interpolatory properties hold for each k, $n = 0, 1, \ldots, 2^{j+1}-1$:*

$$\phi^0_{j,n}\left(x_{j,k}\right) = \delta_{k,n}, \quad \left(\phi^0_{j,n}\left(x_{j,k}\right)\right)' = 0,$$
$$\phi^1_{j,n}\left(x_{j,k}\right) = 0, \quad \left(\phi^1_{j,n}\left(x_{j,k}\right)\right)' = \delta_{k,n},$$

where

$$\delta_{k,n} = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

We want to reshape scaling functions such that satisfy the boundary conditions exactly, for this aim redefining $\phi^0_{j,0}$ as following is enough,

$$\phi^0_{j,0} = \cos(2^j x) - \cos(2^{j-1}x)$$

Let $f_{|\ V_j}$ denote the projection $f \in L^2(\mathbb{R})$ onto V_j. We can obtain the scaling expansion of $f_{|\ V_j}$ as

$$f_{|\ V_j}(x) = \sum_{i=0}^{N'} a^0_{j\,i}\phi^0_{j\,i}(x) + a^1_{j\,i}\phi^1_{j\,i}(x) = \left[\ a_j\ \right][\Phi], \tag{8}$$

where

$$N' = 2^{j+1}-1, [a_j]^T = \left[a^0_{j0},\, a^0_{j1},\, \ldots,\, a^0_{j\,N'},\, a^1_{j0},\, a^1_{j1},\, \ldots,\, a^1_{j\,N'}\right],$$

and

$$[\Phi] = \left[\phi^0_{j0},\, \phi^0_{j1},\, \ldots,\, \phi^0_{j\,N'},\, \phi^1_{j0},\, \phi^1_{j1},\, \ldots,\, \phi^1_{j\,N'}\right].$$

In this article we need to approximate differential operator $\frac{\partial^r}{\partial x^r}$ on V_j. Therefore, we use a collocation method to approximate derivative operators, which requires exact derivative values at specific points on the domain. For this aim we must firstly construct the matrix P_j that converts F the vector of coefficients $f_{|\ V_j}$ into $\overline{F}$ the vector of values of f in $y_{j,n} = \frac{(2n+1)\pi}{2^{j+2}}$, for $n = 0, \ldots 2^{j+2}-1$.

$$P_j F = \overline{F}, \tag{9}$$

where

$$F=\left[a_{j\,0}^{0},\,a_{j\,1}^{0},\,\ldots\,,\,a_{j\,N'}^{0},\,a_{j\,0}^{1},\,a_{j\,1}^{1},\,\ldots\,,\,a_{j\,N'}^{1}\right]^{T},$$

and

$$[P_j]_{m,n}=\begin{cases}\phi^{0}_{j,n-1}(y_{jm}), & n=1,\,\ldots,\,2^{j+1},\\ \phi^{1}_{j,n-2^{j+1}-1}(y_{jm}), & n=2^{j+1}+1,\,\ldots,\,2^{j+2}.\end{cases}$$

Next, we construct the matrix P_j^r which convert F the vector of coefficients of $f_{|\ V_j}$ into $\overline{F}^r$ the actual values of $\frac{\partial^r f}{\partial x^r}$ in $y_{j,n}$,

$$P_j^r F=\overline{F}^r, \tag{10}$$

where

$$\overline{F}^r=\left[\frac{\partial^r}{\partial x^r}f_{|\ V_j}\left(y_{j0}\right),\,\frac{\partial^r}{\partial x^r}f_{|\ V_j}\left(y_{j1}\right),\,\ldots\,,\,\frac{\partial^r}{\partial x^r}f_{|\ V_j}\left(y_{j,2^{j+2}-1}\right)\right]^T,$$

and

$$\left[P_j^r\right]_{m,n}=\begin{cases}\frac{\partial^r}{\partial x^r}\phi^{0}_{j,n-1}(y_{jm}), & n=1,\,\ldots,\,2^{j+1},\\ \frac{\partial^r}{\partial x^r}\phi^{1}_{j,n-2^{j+1}-1}(y_{jm}), & n=2^{j+1}+1,\,\ldots,\,2^{j+2}.\end{cases}$$

Therefore M^r the derivatives matrix will be as the following

$$M^r=(P_j)^{-1}\times P_j^r. \tag{11}$$

For defining the fractional power $(-A)^{\frac{\alpha}{2}}$ of the differential operator A for $0\le\alpha\le 1$,let $0<\lambda_1<\lambda_2<\ldots$ be the eigenvalue and ζ_k , the corresponding eigenfunctions of $-A$. Then $(-A)^{\frac{\alpha}{2}}$ the power of the differential operator A is defined as following

$$\begin{cases}(-A)^{\frac{\alpha}{2}}u=\sum_{k=1}^{\infty}\lambda_k^{\frac{\alpha}{2}}(u,\zeta_k)_{L^2}\zeta_k,\ u\in D\left((-A)^{\frac{\alpha}{2}}\right),\\ D((-A)^{\frac{\alpha}{2}})=\left\{u\in L^2:\sum_{k=1}^{\infty}\lambda_k^{\frac{\alpha}{2}}\left|(u,\zeta_k)_{L^2}\right|^2<\infty\right\},\\ |u|_{D((-A)^{\frac{\alpha}{2}})}=\left(\sum_{k=1}^{\infty}\lambda_k^{\frac{\alpha}{2}}\left|(u,\zeta_k)_{L^2}\right|^2\right)^{\frac{1}{2}}.\end{cases} \tag{12}$$

According to [7], the domain $D\left((-A)^{\frac{\alpha}{2}}\right)$ can be described as follows:

$$D\left((-A)^{\frac{\alpha}{2}}\right)=\begin{cases}H^{\alpha}(\mathbb{T}), & 0\le\alpha<\frac{1}{2},\\ 0H^{\alpha}(\mathbb{T}), & \frac{1}{2}<\alpha\le 1,\\ \left\{u\in H^{\frac{1}{2}}(\mathbb{T}):\int_0^{2\pi}x^{-1}|u(x,t)|^2dx<\infty,\forall t\in[0,T]\right\}, & \alpha=\frac{1}{2}.\end{cases} \tag{13}$$

Moreover, in V_j we can define the representation matrix of $(-A)^{\alpha}$ with respect to the trigonometric scaling functions. by $TD^{\alpha}T^{-1}$ where D is representation matrix of $-A$ with respect to $\{\zeta_k\}$ (7) and T denote the change of basis matrix from $\{\zeta_k\}$ to trigonometric scaling functions $\{\Phi_k\}$.

3. **Existence and uniqueness**

In this section we show existence and uniqueness of mild solution of (1) which is defined to be the solution of the following integral equation,

$$u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}\frac{\partial^{\alpha}}{\partial x^{\alpha}}F\left(u(s)\right)ds+\int_0^t e^{(t-s)A}dW(s). \tag{14}$$

We will study equation (1) a. s. $\omega\in\Omega$ and prove local existence in time, for this aim we need to prove the following lemma.

Lemma 3.1. *For any $T > 0$ and $u \in C\left([0,T]; L^1(0,a)\right)$*

$$\int_0^T |e^{tA}\frac{\partial^\alpha}{\partial x^\alpha}u(x,t)|_H dt \leq C\left(T^{\frac{3-2\alpha}{4}} + T\right)\sup_{0\leq t\leq T}|u(x,t)|_{L^1(0,a)}.$$

Proof. From [8], we know $|\frac{\partial^\alpha}{\partial x^\alpha}u|_H$ and $|u|_{H^\alpha}$ are equivalent, so

$$|\frac{\partial^\alpha}{\partial x^\alpha}e^{tA}u|_H \leq C_1|(-A)^{\frac{\alpha}{2}}e^{tA}u|_H. \tag{15}$$

Using the Sobolev embedding theorem, $W^{\frac{1}{2},1} \subseteq W^{0,2}$, so we have

$$|(-A)^{\frac{\alpha}{2}}e^{tA}u|_{W^{0,2}(0,a)} \leq C_2|(-A)^{\frac{\alpha}{2}}e^{tA}u|_{W^{\frac{1}{2},1}(0,a)}. \tag{16}$$

On the other hand for any $s_1 \leq s_2$ in $\mathbb{R}$, and $s \geq 1$, e^{tA} maps $W^{s_1,s}(0,a)$ into $W^{s_1,s}(0,a)$, for all $t > 0$. Moreover the following estimate holds [17]

$$|e^{tA}z|_{W^{s_2,s}(0,a)} \leq C_3\left(t^{\frac{s_1-s_2}{2}} + 1\right)|z|_{W^{s_1,s}(0,a)} \quad \forall z \in W^{s_1,s}(0,a). \tag{17}$$

Let $s_1 = -\alpha$, $s_2 = \frac{1}{2}$ and $s = 1$ so from (15) and (16) we have

$$\begin{aligned}|e^{tA}\frac{\partial^\alpha}{\partial x^\alpha}u|_{W^{0,2}(0,a)} \leq & C_1C_2C_3\left(t^{\frac{-\alpha-\frac{1}{2}}{2}} + 1\right)|A^{\frac{\alpha}{2}}e^{tA}u|_{W^{-\alpha,1}(0,a)},\\ |e^{tA}\frac{\partial^\alpha}{\partial x^\alpha}u|_H \leq & C_1C_2C_3\left(t^{\frac{-\alpha-\frac{1}{2}}{2}} + 1\right)|e^{tA}u|_{L^1(0,a)}.\end{aligned} \tag{18}$$

Integrating over t completes the proof.

□

Theorem 3.1. *Let $|u_0|_H < m$, for a $T^* > 0$ there exists a unique mild solution $u \in \Sigma(m, T^*)$ of the equation (1) where*

$$\Sigma(m,T^*) = \left\{v \in C\left([0,T^*]; L^2(0,T^*)\right) : |v(t)|_{L^2(0,T^*)} \leq m, \forall t \in [0,T^*]\right\}.$$

Proof. The proof is mentioned from [4]. At first we define the operator $\mathcal{L}$,

$$\mathcal{L}u(t) := e^{tA}u_0 + \int_0^t e^{(t-s)A}\frac{\partial^\alpha}{\partial x^\alpha}F\left(u(s)\right)ds + \int_0^t e^{(t-s)A}dW(s). \tag{19}$$

Let u and v be two elements of $\Sigma(m,T^*)$ such that $u(0) = v(0) = u_0$, using lemma 3.1 we have

$$\begin{aligned}|\mathcal{L}(u) - \mathcal{L}(v)|_H \leq & \int_0^t \left|e^{(t-s)A}\frac{\partial^\alpha}{\partial x^\alpha}\left(F(u) - F(v)\right)\right|_H dt\\ \leq & C'C\left({T^*}^{\frac{3-2\alpha}{4}} + T^*\right)\sup_{0\leq t\leq T}|u - v|_{L^1(0,a)}.\end{aligned} \tag{20}$$

For small T^* we have $C'C\left({T^*}^{\frac{3-2\alpha}{4}} + T^*\right) < 1$, so $\mathcal{L}$ is a contraction on $\Sigma(m,T^*)$, repeating the argument on $[0,T^*], [T^*, 2T^*], ...,$ we find a unique mild solution for all $t > 0$.

□

4. The approximation of stochastic fractional pde

We choose a semi-implicit scheme to approximate the solution of (1) in the subspace V_j,

$$\mathcal{U}_j^{n+1} = \mathcal{U}_j^n - h\left(\mathbf{A}(\theta\mathcal{U}_j^{n+1} + (1-\theta)\mathcal{U}_j^n) + M^\alpha P_j^{-1}F(P_j\mathcal{U}_j^n)\right) + P_j^{-1}\Delta W_n \qquad (21)$$

where $\Delta W_n := W((n+1)\Delta t) - W(n\Delta t)$ is normally distributed with mean 0 and variance Δt and $\mathbf{A} = M^2$, $\mathcal{U}_j^n = [\mathcal{U}_{j\,0}^{0,\,n}, \dots \mathcal{U}_{j\,N'}^{0,\,n}, \mathcal{U}_{j\,0}^{1,\,n}, \dots \mathcal{U}_{j\,N'}^{1,\,n}]$ and $u_j^n(x) = \sum_{i=0}^{N'} \mathcal{U}_{j\,i}^{0,\,n}\phi_{j\,i}^0(x) + \sum_{i=0}^{N'} \mathcal{U}_{j\,i}^{1,\,n}\phi_{j\,i}^1(x)$, $u_j^n(x)$ is approximate solution of (1) in the V_j. Now by multiplying (21) by P_j we obtain

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - h\left(P_j\mathbf{A}P_j^{-1}(\theta\mathbf{u}_j^{n+1} + (1-\theta)\mathbf{u}_j^n) + P_jM^\alpha P_j^{-1}F(\mathbf{u}_j^n)\right) + \Delta W_n. \qquad (22)$$

where $P_j\mathcal{U}_j^n = \mathbf{u}_j^n$ and $\mathbf{u}_j^n = \left[u_j^n(y_{j0}), u_j^n(y_{j1}), \dots, u_j^n\left(y_{j,2^{j+2}-1}\right)\right]^T$. To simplify the writing, $\mathbf{u}_j^n$, $P_j\mathbf{A}P_j^{-1}$ and $P_jM^\alpha P_j^{-1}$ denoted by u^n, $\mathcal{A}$ and $\mathcal{M}^\alpha$ respectively, then (22) can be written as

$$u^{n+1} = S_hu^n - h\left(I + h\theta\mathcal{A}\right)^{-1}\mathcal{M}^\alpha F(u^n) + \left(I + h\theta\mathcal{A}\right)^{-1}\Delta W_n, \qquad (23)$$

where

$$S_h = \left(I + h\theta\mathcal{A}\right)^{-1}\left(I - h(1-\theta)\mathcal{A}\right).$$

We know the mild solution of (1) is as following

$$u(x, nh) = e^{nhA}u_0(x) - \int_0^{nh} e^{(nh-s)A}\frac{\partial^\alpha}{\partial x^\alpha}F(u(x,s))ds + \int_0^{nh} e^{(nh-s)A}dW(s). \qquad (24)$$

We set for any n

$$e_n = u(nh) - u^n, \qquad (25)$$

where

$$u(nh) = [u(0, nh), u(2^j, nh), u(2\times 2^j, nh), ..., u(a, nh)]^T,$$

In the next section we show

$$\lim_{N\to\infty}\mathbb{P}\left\{\max_{0\le n\le N}|e_n| \ge \epsilon\right\} = 0. \qquad (26)$$

4.1. Convergence

Consider the following SPDE,

$$u_t = Au + f(u) + dW, \qquad (27)$$
$$u(0) = u_0,$$

where $A = \frac{\partial^2}{\partial x^2}$ and the function f is mapping from $H = L^2(0,a)$ into $D\left((-A)^{-r}\right)$, for some $r \in [0,1]$, also we assume that there exists $L_f \in \mathbb{R}^+$ such that

$$|(-A)^{-r}\left(f(u) - f(v)\right)|_H \le L_f|u - v|_H, \qquad (28)$$

$$|(-A)^{-r}f(u)|_H \le L_f\left(1 + |u|_H\right). \qquad (29)$$

From lemma 2.1 in [15] and theorem 1.1 in [10] we have following inequalities which are used for showing convergence of the method,

$$\left|(-A)^a e^{tA}\right|_{L(H)} \le C_a t_k^{-a}, \quad t>0, \tag{30}$$

$$\left|S_h^k - e^{t_k A}\right|_{L(H)} \le \frac{C(\theta)}{k}, \quad k \ge 1, \tag{31}$$

$$\left|(-A)^{-b}(e^{tA}-I)\right|_{L(H)} \le C_b t^b. \tag{32}$$

Lemma 4.1. *If $u \in C([0,T];H)$ is the mild solution of (1) then*

$$|u(t)|_H \le \sqrt{\mu}\left(e^{\frac{1}{2}Ct}\right). \tag{33}$$

Moreover we have

$$\lim_{M\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T]} |u(t)| \ge M\right\} = 0. \tag{34}$$

Proof. Let $\{W^n\}$ as following

$$\mathcal{W}^n(t) := \int_0^t e^{(t-s)A} dW^n(s) \longrightarrow W(t) \tag{35}$$

Consider the following equation

$$dv^n = \left(Av^n + \frac{\partial^\alpha}{\partial x^\alpha}\left(F(v^n + \mathcal{W}^n)\right)\right)dt, \tag{36}$$

Using theorem 3.1 v^n the mild solution of (36) on $[0,T_n]$ exists such that $T_n \longrightarrow T^*$, $v^n \longrightarrow u$ and

$$v^n = \int_0^t e^{(t-s)A}\frac{\partial^\alpha}{\partial x^\alpha}\left(F(v^n+\mathcal{W}^n)\right)ds. \tag{37}$$

By multiplying v^n by (36) and then integrating over $[0,1]$ we have

$$\frac{1}{2}\frac{\partial}{\partial t}|v^n|_H^2 = \int_0^1 v^n v_{xx}^n dx + \int_0^1 v^n \frac{\partial^\alpha}{\partial x^\alpha}\left(F(v^n+\mathcal{W}^n)\right)dx \tag{38}$$

$$\le -\int_0^1 (v_x^n)^2 dx + |v^n|_H^2 dx + |F(v^n+\mathcal{W}^n)|_\alpha. \tag{39}$$

Since F is a locally Lipschitz function with μ_n denoting $\sup_{t\in[0,T]}|\mathcal{W}^n|_\infty$, we can get from Poincaré inequality

$$\frac{1}{2}\frac{\partial}{\partial t}|v^n|_H^2 \le -|v^n|_{H^1}^2 + |v^n|_{H^1}^2 + C|v^n|_{L^2}^2 + C\mu_n, \tag{40}$$

by Gronwall's inequality

$$|v^n|_H^2 \le \mu_n\left(e^{Ct}\right), \tag{41}$$

through n tends to infinity we get

$$|u(t)|_H \le \sqrt{\mu}\left(e^{\frac{1}{2}Ct}\right). \tag{42}$$

Then

$$\log|u(t)|_H \le \frac{1}{2}\log(\mu) + \frac{1}{2}Ct, \tag{43}$$

$$\sup_{t\in[0,T]} \log|u(t)|_H \le \frac{1}{2}\log(\mu) + \frac{1}{2}CT. \tag{44}$$

By Jensen inequality we find

$$\mathbb{E}\left(\sup_{t\in[0,T]}\log|u(t)|_H\right) \leq \frac{1}{2}\log(\mathbb{E}(\mu)) + \frac{1}{2}CT. \tag{45}$$

By the Chebyshev inequality we get

$$\mathbb{P}\left(\sup_{t\in[0,T]}\log|u(t)|_H \geq \log(M)\right) \leq \frac{\frac{1}{2}\log(\mathbb{E}(\mu)) + \frac{1}{2}CT}{\log(M)}, \tag{46}$$

hence

$$\lim_{M\to\infty}\mathbb{P}\left(\sup_{t\in[0,T]}|u(t)|_H \geq M\right) = 0. \tag{47}$$

□

The proof of convergence of the method for (1) relies mainly on the following theorems and lemma.

Theorem 4.1. [15], [9] *Let u be the solution of (27) on* $[0,T]$ *and assumptions (28) and (29) hold , then* $\{u^n\}_{n\geq 0}$ *is convergent in probability to u , where* $\{u^n\}_{n\geq 0}$ *comes from*

$$u^{n+1} = u^n - h\left(\mathcal{A}(\theta u^{n+1} + (1-\theta)u^n) + f(u^n)\right) + \Delta W_n, . \tag{48}$$

Moreover, for any $\widetilde{\gamma} < \frac{1}{4}$, *this scheme is of order in probability* $\widetilde{\gamma}$ *in* $L^2(0,1)$.*Indeed, we have*

$$\lim_{C\to\infty}\limsup_{h\to 0}\mathbb{P}\left\{\max_{1\leq n\tau\leq T}|e_n| \geq Ch^{\widetilde{\gamma}}\right\} = 0, \tag{49}$$

for any $\widetilde{\gamma} < \frac{1}{4}$.

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $supp(\varphi) \subset]-2,+2[$, $\varphi(x) = 1$, if $|x| \leq 1$. For any $R > 0$, we put

$$\varphi_R(x) = \varphi\left(\frac{x}{R}\right), \tag{50}$$

and

$$f_R(u) = \varphi_R(|u|)f(u). \tag{51}$$

Lemma 4.2. *For any* $s > \frac{1+2\alpha}{4}$ *and* $R > 0$, *there exist some* $C(s,R) > 0$ *such that*

$$|f_R(u) - f_R(v)|_{D((-A)^{-s})} \leq C(s,R)|u-v|, \tag{52}$$

for any u and v in H, where $f = \frac{\partial^\alpha}{\partial x^\alpha}F$.

Proof. We denote Riemann-Liouville fractional derivative by D_L^α ,

$$D_L^\alpha g(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}g(t)dt, \quad 0 \leq \alpha < 1. \tag{53}$$

Let $\phi \in H^{p+\alpha}$ for some $p > \frac{1}{2}$. Using fractional variant of the integration by parts formula [1] for all $|u|_H \leq M$, $M > 0$ we have

$$\int \phi\cdot\frac{\partial^\alpha}{\partial x^\alpha}F(u) = \int F(u)\cdot D_L^\alpha\phi \leq C(M)|u|_H|D_L^\alpha\phi|_\infty \leq C(M)Mc|A^{\frac{\alpha}{2}}\phi|_\infty, \tag{54}$$

since $H^p \subseteq L_\infty$ we can control the right term by $|\phi|_{H^{p+\alpha}}$. Hence $\frac{\partial^\alpha}{\partial x^\alpha}F(u)$ is in the dual of $H^{p+\alpha}$, i.e. $\frac{\partial^\alpha}{\partial x^\alpha}F(u) \in D\left((-A)^{-\frac{p+\alpha}{2}}\right)$. This concludes the inequality (52). □

Lemma 4.3. [15], [9] *Let $R > 0$ and*

$$\tau_R = \inf\{t \in [0,T] | \, |u_R(t)| \geq R\} \wedge T.$$

Then we have

- $\{\tau_R\}_{R>0}$ *is non-decreasing with respect to R,*
- $\lim_{R\to+\infty} \tau_R = T$*, a.s.,*
- $\forall t \leq \tau_R, \quad u_R(t) = u(t)$ *a.s., where u satisfies the mild formulation, a.s.,*

$$u(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A} \frac{\partial^\alpha}{\partial x^\alpha} (F(u))\, ds + \int_0^t e^{(t-s)A} dW(s). \tag{55}$$

The following theorem is proved by employing a strategy similar to that underlying the proof of lemma 4.8 in [15].

Theorem 4.2. *Let u be the mild solution of (1) and let u^n and e_n be given by (22) and (25) respectively, then the scheme (22) is convergent in probability to the mild solution.*

$$\lim_{N\to\infty} \mathbb{P}\left\{\max_{0\leq n\leq N} |e_n| \geq \epsilon\right\} = 0, \tag{56}$$

$$\lim_{M\to\infty} \lim_{N\to\infty} \sup \mathbb{P}\left\{\max_{0\leq n\leq N} |u^n| \geq M\right\} = 0. \tag{57}$$

Proof. At first we define the following random variables

$$\theta_R = \inf\{t \leq T, |u(t)| \geq R-1\}\, a.s., \tag{58}$$

$$n_\epsilon = \min\{n \leq N, |u^n - u(nh)| \geq \epsilon\}\, a.s.. \tag{59}$$

Let

$$\mathbb{A} = \left\{\max_{0\leq n\leq N} |e_n| \geq \epsilon\right\}, \quad \mathbb{B} = \{\theta_R < T\} \text{ and } \quad \mathbb{C} = \{n_\epsilon h < T\}, \tag{60}$$

so we have

$$\begin{aligned} \mathbb{A} =& (\mathbb{B} \cup (\mathbb{B}^c \cap (\mathbb{C} \cup \mathbb{C}^c))) \cap \mathbb{A} \\ =& (\mathbb{B} \cap \mathbb{A}) \cup (\mathbb{B}^c \cap \mathbb{C} \cap \mathbb{A}) \cup (\mathbb{B}^c \cap \mathbb{C}^c \cap \mathbb{A}), \end{aligned} \tag{61}$$

Using the definitions of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ the last set equals empty, hence

$$\mathbb{A} \subset \mathbb{B} \cup (\mathbb{B}^c \cap \mathbb{C}), \tag{62}$$

therefore

$$\begin{gathered} \left\{\max_{0\leq n\leq N} |e_n| \geq \epsilon\right\} \subset \{\theta_R < T\} \cup \{n_\epsilon h < T \leq \theta_R\}, \\ \mathbb{P}\left\{\max_{0\leq n\leq N} |e_n| \geq \epsilon\right\} \leq \mathbb{P}\{\theta_R < T\} + \mathbb{P}\{n_\epsilon h < T \leq \theta_R\}, \end{gathered} \tag{63}$$

from $T \leq \theta_R$ we have for any $n \leq N, \quad |u(nh)| \leq R-1 < R$, in other hand we know that $n_\epsilon h < T$, this means $n_\epsilon < N$ so

$$\begin{aligned} |u^{n_\epsilon-1} - u((n_\epsilon - 1))| < \epsilon < 1 \Longrightarrow& |u^{n_\epsilon-1}| < 1 + |u((n_\epsilon-1))|, \\ \Longrightarrow& |u^{n_\epsilon-1}| < 1 + R - 1 = R. \end{aligned} \tag{64}$$

We set for any $0 \leq n \leq N$, $e_{n,R} = u_R^n - u_R(nh)$, where

$$\begin{cases} u_R^{n+1} = u_R^n - \left(\mathcal{A}(\theta u_R^{n+1} + (1-\theta)u_R^n) + f(u_R^n)\right) + \Delta W_n, \\ u_R^0 = u_0, \end{cases} \tag{65}$$

then using lemma 4.3 we find $u(nh) = u_R(nh)$,

$$u^{n_\epsilon} = u_R^{n_\epsilon} \Longrightarrow e_{n_\epsilon} = e_{n_\epsilon,R}, \tag{66}$$

so from (63) we have

$$\mathbb{P}\left\{\max_{0\le n\le N} |e_n| \ge \epsilon\right\} \le \mathbb{P}\left\{\theta_R < T\right\} + \mathbb{P}\left\{\max_{0\le n\le N} |e_{n,R}| \ge \epsilon\right\}, \tag{67}$$

then using Chebyshev inequality we get

$$\mathbb{P}\left\{\max_{0\le n\le N} |e_n| \ge \epsilon\right\} \le \mathbb{P}\left\{\theta_R < T\right\} + \frac{1}{\epsilon}\mathbb{E}\left\{\max_{0\le n\le N} |e_{n,R}|^p\right\}^{\frac{1}{p}}, \tag{68}$$

hence with regard to (52) we can use the theorem 4.1,

$$\begin{aligned}\mathbb{P}\left\{\max_{0\le n\le N} |e_n| \ge \epsilon\right\} &\le \mathbb{P}\left\{\theta_R < T\right\} + \frac{C(p)}{\epsilon} h^{\frac{\gamma}{p}} \\ &\le \mathbb{P}\left\{\theta_R < T\right\} + 0 \quad (as\, N \longrightarrow \infty) \\ &\le 0 \quad (as\, R \longrightarrow \infty).\end{aligned}$$

To proof another inequality (57) we consider the following

$$\mathbb{P}\left\{\max_{0\le n\le N} |u_n| \ge M\right\} \le \mathbb{P}\left\{\max_{0\le n\le N} |e_n| \ge \epsilon\right\} + \mathbb{P}\left\{\sup_{t\in[0,T]} |u(t)| \ge M - \epsilon\right\}.$$

At last using (56) and lemma 4.1 we get the result. □

5. **Numerical results**

In this section we apply our numerical method for $\theta = 0.6$ to several SPDEs and report its convergence and performance. We do not have the exact solution for these example, and so we examine the error by $\mathbb{E}|u_j^n - u_{j=7}^n|$. For each $\{\Delta x = \pi 2^{-j-1}, \Delta t = \frac{T}{n}\}$, 10000 runs are performed with different samples of noise (via MATLAB software) and their averages computed.

Example 5.1. *Consider the following stochastic fractional PDE with regard to general equation (5),*

$$d_t u\left(x,t\right) = 4\pi^2 u_{xx}\left(x,t\right) + 2\pi \frac{\partial^\alpha}{\partial x^\alpha} u\left(x,t\right) + 2\pi u_x\left(x,t\right) + dW(t)$$

for $(x,t) \in [0,2\pi]\times[0,1]$ *and* $\alpha = 0.9$, *with the initial function*

$$u(x,0) = \frac{5}{2\pi^2} x(2\pi - x),$$

and the following boundary conditions

$$u(0,t) = u(2\pi,t) = 0.$$

In figure (1), we see the results for $\mathbb{E}(u(x,0.3))$, *in different resolutions* j. *In this case, we choose* $\Delta t = 0.01$, $\Delta x = \pi 2^{-j-1}$. *In figure (2), approximate solution for* $\mathbb{E}(u(x,t))$, $j = 6$, *is shown.*

Example 5.2. *Consider the following stochastic fractional PDE ,*

$$d_t u\left(x,t\right) = 4\pi^2 u_{xx}\left(x,t\right) + \pi \frac{\partial^\alpha}{\partial x^\alpha} u^2\left(x,t\right) + dW(t),$$

for $(x,t) \in [0,2\pi]\times[0,1]$ *and* $\alpha = 0.9$, *with the initial function*

$$u(x,0) = sin(\frac{x}{2}),$$

TABLE 1. Comparison of the error in different levels for example 5.1 at $t = 0.3$ with $\Delta t = 0.01$.

	V_3/V_7	V_4/V_7	V_5/V_7	V_6/V_7
L_∞ *Error*	*0.02666*	*0.00901*	*0.00576*	*0.00517*
L^2 *Error*	*0.04787*	*0.01685*	*0.01059*	*0.01043*

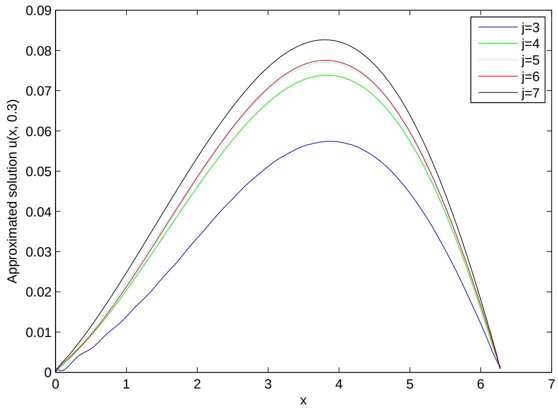

FIGURE 1. Approximate solution of example 5.1 in different resolutions at $t = 0.3$.

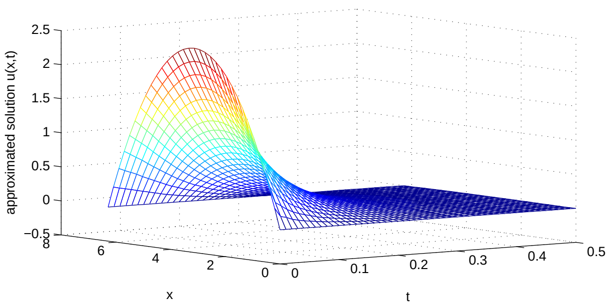

FIGURE 2. Mean solutions of example 5.1 using proposed scheme.

and the following boundary conditions

$$u(0, t) = u(2\pi, t) = 0.$$

In figure (3), we see the results for $\mathbb{E}(u(x, 0.3))$*, in different resolutions* j*. In this case, we choose* $\Delta t = 0.01$*,* $\Delta x = \pi 2^{-j-1}$*. In figure (4), the approximate solution* $\mathbb{E}(u(x, t))$*,* $j = 6$ *is shown.*

TABLE 2. Comparison of the error in different levels for example 5.2 at $t = 0.3$ with $\Delta t = 0.01$.

	V_3/V_7	V_4/V_7	V_5/V_7	V_6/V_7
L_∞ *Error*	*0.00427*	*0.00404*	*0.00266*	*0.00046*
L^2 *Error*	*0.00665*	*0.00716*	*0.00432*	*0.00098*

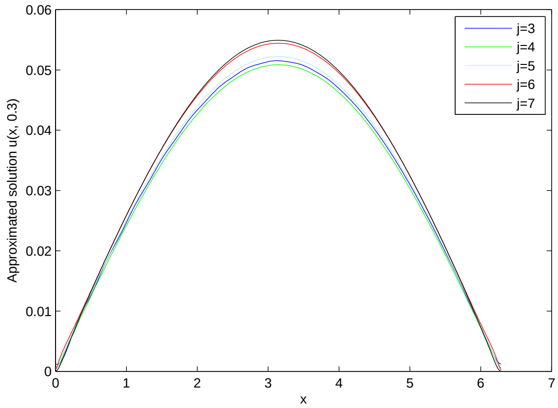

FIGURE 3. Approximate solution of example 5.2 in different resolutions at $t = 0.3$.

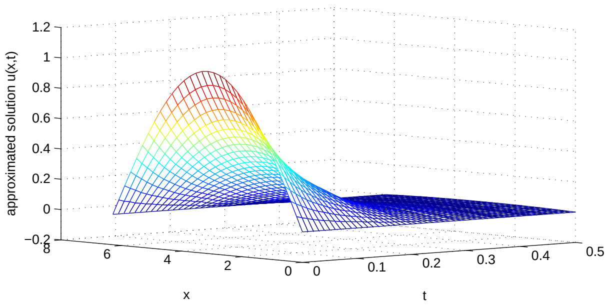

FIGURE 4. Mean solutions of example 5.2 using proposed scheme.

6. Conclusion

In this paper we have proposed an implicit method based on trigonometric wavelets to solve SFPDEs. We extended the result of [15] for the SPDEs in which the drift is more general and has fractional derivative. Numerical examples revealed the efficiency and accuracy of the proposed method. Also, we investigated the existence and uniqueness of the solutions and convergence of the method.

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