A 3 D SMITH CHART FOR ACTIVE AN PASSIVE MICROWAVE CIRCUITS AND VISUAL COMPLEX ANALYSIS

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The paper explains for the first time the visual complex analysis properties of the very recently proposed 3D Smith for active and passive circuits. The authors develop their very actual concept with elements of complex analysis applied in the complex plane and in 3D (Riemann sphere). The geometrical properties of the different transformations of the complex plane are exploited and applied in microwave design.

Keywords: Smith chart, complex analysis

1. Introduction

In the era of Greek civilization, geometry was studied rigorously and put a firm theoretical basis for intellectual satisfaction, for intrinsic beauty of many geometrical results and the utility of the subject. Throughout the years the interest in geometry languished, it was thought old fashioned by the majority. One of the revival moments of geometry may be considered related with computer graphics and the first papers related to the fractals. The connection between nonscaling fractals, linked to one of the most difficult areas of classical mathematics \cite{1} and visual complex analysis made a subject like “mathematical analysis” to get a new perception.

The Smith chart is one of the most widely used charts in electrical and electronic engineering \cite{2}. Although invented in the late 30’s the diagram

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survived among the time and its use has grown steadily over the years as software
design tool and as a visualization instrument on the network analyzers. Its main
geometrical properties are related with the beauty of the Möbius transformations
and are also linked to topics from arts –Esher’s art [3] and arhitecture. In the recent
years several tries were done in order to increase the the capability of the original
Smith chart into 3d in order to contain a wider range of impedances as in [4,5].
These papers try to extend the previously 2d generalized approaches as [6] but
lack in a pure mathematical formalism and fail to include all the loads.
In our paper we focus our attention on the most recent proposed 3D Smith
chart [7,8]. This 3D Smith chart is created using the mathematical concept of the
Riemann sphere. The Riemann sphere is the key tool of visualizing inversive
transformations in the extended complex plane. It is totally difficult to visualize
the infinity far apart (as it may occur in active devices) on the generalized planar
Smith chart since an outer edge of the complex plane is vague definition for that.
Fortunately, Riemann interpreted the numbers in the extended complex plane as
points on a sphere. Using Ptolemy’s theory on celestial spheres, Riemann applied
a stereographic projection to perform a one-to-one correspondence between the
extended complex plane and the unity radius sphere (i.e., the unit sphere). The 3D
Smith chart is created using Riemann’s theory on the complex plane and applying
it to the Möbius transformation represented by the Smith chart mapping.
Considering a single point at infinity all the circles arcs that never met on the
Smith chart will get together in the South pole.

2. Möbius transformations, 2 D generalized Smith chart and the
maximum modulus theorem

The main governing equation for the Smith chart is (1) and represents the
reflection coefficient of the travelling waves

\[ \rho = \frac{z - 1}{z + 1} ; \quad z = r + jx = \frac{Z}{Z_0}. \]  

In the case of a real characteristic impedance \( Z_0 \) then all the normalized
impedances \( z \) that represent passive circuits will have a positive real part \( (r>0) \).
In this situation all the normalized passive loads will generate a reflection
coefficient that has a magnitude lower than 1. An example is presented in Fig.1
where the reflection coefficient for an arbitrarily shape in the RHP (right half
plane) is computed. The magnitude of the reflection coefficient in this case is
always bounded by one (the white circle in Fig 1.).
In (2) the general form of a Möbius transformation is given; a, b, c, d represent the coefficients of the Möbius transformation (ad-bc≠0). [M] stays for the matrix form corresponding to the Möbius transformation given by the reflection coefficient while M₁ shows the normalized Möbius transformation matrix in the Smith chart mapping case.

\[ \rho = \frac{Z - Z_c}{Z + Z_c}; \quad Z_c = R_c + jX_c; \quad \rho = \rho_x + j \rho_y \quad (2) \]

\[ \rho = \frac{Z - \frac{1}{R_c}}{Z + \frac{1}{R_c}}; \quad z_1 = \frac{Z}{R_c}; \quad x_c = \frac{X_c}{R_c} \quad (3) \]

\[ \rho = \frac{z_1 - \left(1 + j \cdot x_c\right)}{z_1 + 1 + j \cdot x_c}, \quad z_1 = r_1 + j \cdot x_1, \quad r_1 > 0 \quad (4) \]

In (3) one may see the general form of the reflection coefficient for complex characteristic impedances.
In (3) the reflection coefficient is rewritten so that $z_1$ stays for an impedance only in the RHP. However if one looks at (4) the reflection coefficient in the case of a complex characteristic impedance is just Möbius transformation of the RHP impedance plane with the coefficients: $a=1, b=-1-j\times x_c, c=1, d=1+j\times x_c$. If one considers $x_c=0.3$ in (6) this generates a mapping as in Fig 2. We may notice in Fig 2 that an arbitrarily shape in the RHP may generate a reflection coefficient that exceeds the unity. If we visualize this in a 3D manner we may get Fig 3. The interesting thing in Fig 3 is that the maximum magnitude of the reflection coefficient is obtained on the $x_1$ axes. This value is 1.09 for $x_c=0.3$.

However in order to give a general formula for the maximum magnitude of the reflection coefficient of a passive load one may treat (4) as a complex mapping generated by a Möbius transformation. Since $z_1$ is in this case just a complex number wherever in the RHP and $\rho_1$ is analytic in the RHP one may use the maximum modulus theorem in order to establish the boundary for $\rho_1$. 

![Fig. 2 Possible mapping of an impedance in the RHP into the reflection coefficient’s plane for a complex characteristic impedance](image)

![Fig. 3 Magnitude of the reflection coefficient for different values of $r_1$ and $x_1$ and a value of $x_c=0.3$](image)
The maximum modulus theorem [9] states that if a function is analytic in a domain then its maximum is achieved on the frontier. Applying this theorem for \( z_l \) belonging in the RHP we find out that the maximum is always on the \( x_1 \) axis of the \( z_l \) plane. The maximum value for the magnitude of the reflection coefficient is then obtained as:

\[
|\rho_1| = \frac{1 + x_c^2 + |x_c| \sqrt{1 + x_c^2}}{1 + x_c^2 - |x_c| \sqrt{1 + x_c^2}}
\]  

(5)

The value stated in (5) is the maximum value that the reflection coefficient may reach in a case of a complex impedance. One may see that considering \( x_c = 0 \) we get the well known \( \rho_1 = 1 \) while in the case of \( x_c = 0.3 \) from Fig 3 we get the value 1.09. The results obtained for \( \rho 1 \) in [10], [11], are a consequence of (5). The value is in perfect concordance with the results presented in [12] but the maximum modulus theorem applied to form (4) extend the point of view presented in [12] and prove that (5) is the global maximum for the magnitude of the reflection coefficient for passive loads when normalized to a complex impedance.

In Fig 4 is presented the mapping of an impedance from the RHP to the reflection coefficient’s plane. The extended 2 D Smith chart is also drawn.

In Figs 4-5 one may see the effect that an active device may generate. (the same for a complex normalization impedance). The reflection coefficient’s magnitude may exceed the unity and this leads to a reflection coefficient outside of the classical Smith chart (red-dashed).
Fig. 5. Mapping of an arbitrary shape in the reflection coefficients plane from the RHP and LHP using the extended Smith chart as a visual tool. The presence in the LHP makes the reflection coefficient jump outside the Smith chart.

3. A 3D Smith chart and its latitude-reflection coefficient correspondence

Since one needs different zoom scales in order to work with the phenomena presented in Fig 5, the letter [7] proposed the Riemann sphere as the new 3D Smith chart that may include any types of loads. However, the concept is just mathematically described.

We extend the ideas and present for the first time the connection between latitude and the reflection coefficient's plane and the steeographical map of the world and the Smith reflection plane (Fig 6 and Fig 7)

Fig. 6 A 3D Smith chart and the relationship between latitude and reflection coefficient
4. Conclusions

The paper used elements of visual complex analysis like the maximum modulus theorem or geometry of the Möbius transformations in order to establish a boundary for the reflection coefficient’s magnitude in the case of microwave circuits with complex characteristic impedance. It presented a new vision on the recently proposed 3D Smith chart and proposed a latitude–reflection coefficient scale for the 3D Smith chart. The results can be a useful starting point for a microwave engineer dealing with active circuits and complex characteristic impedances. Nevertheless the paper showed the connection between Möbius transformations and microwaves. Plotting the Möbius transformations that occur
in microwave theory on the 3D Smith chart one may get a handy tool to solve any matching problems.

REFERENCES

[8] www.3dsmithchart.com