GENERALIZED DUALITY FOR SEMI-INFINITE MINIMAX FRACTIONAL PROGRAMMING PROBLEM INVOLVING HIGHER-ORDER (Φ, ρ)-V-INVEXITY

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In [22], we introduced the concept of higher-order (Φ, ρ)-V-invexity and presented two types of higher-order dual models for a semi-infinite minimax fractional programming problem. Weak, strong and strict converse duality theorems were discussed under the assumptions of higher-order (Φ, ρ)-V-invexity to establish a relation between the primal and dual problems. In this paper, we give one more generalized dual model for semi-infinite minimax fractional programming problem involving higher-order (Φ, ρ)-V-invexity and prove duality results.

Keywords: Semi-infinite programming, minimax fractional programming, higher-order (Φ, ρ)-V-invexity, generalized duality.

MSC2010: 90C 46, 90C 29, 90C 32, 49J 52.

1. Introduction

A mathematical programming problem with a finite number of variables and infinitely many constraints is called a semi-infinite programming problem. Due to important applications of semi-infinite programming problem, a growing interest in semi-infinite optimization may be observed in the last two decades. Semi-infinite programming problems arise in several areas of modern research such as, robotic trajectory planning [23], production planning [26] digital filter design [18] and air pollution control [24], transportation theory [17], engineering design [20].

The study of higher order duality is significant due to the computational advantage over first order duality as it provides higher bounds for the value of the objective function of the primal problem when approximations are used, because there are more parameters involved.

Higher-order duality in nonlinear programming has been studied by some researchers, see [1, 2, 8, 10, 12, 13, 14, 15, 16, 19].

Recently, the concept of (Φ, ρ)-invexity has been introduced by Caristi et al. [9] to extend fundamental theoretical results of mathematical programming, while, Antczak [5, 6] introduced the notion of (Φ, ρ)-V-invexity by combining the concepts of (Φ, ρ)-invexity and the notions of V-invexity.

In [21], Sarita and Gupta proved duality theorems for higher-order Wolfe and Mond-Weir type duals of the vector optimization using the concept of higher-order (Φ, ρ)-V-invex function. In [22], we introduced the concept of higher-order (Φ, ρ)-V-invexity and presented two types of higher-order dual models for a semi-infinite minimax fractional programming

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problem. Weak, strong and strict converse duality theorems were discussed under the assumptions of higher-order (Φ,ρ)-V-invexity to establish a relation between the primal and dual problems. In this paper, we give one more generalized dual model for semi-infinite minimax fractional programming problem involving higher-order (Φ,ρ)-V-invexity and prove duality results.

In this paper, we consider the following semi-infinite minimax fractional programming problem involving higher-order (Φ,ρ)-V-invexity:

(P)
$$\min_{x\in X}\max_{1\le i\le p}\frac{f_i(x)}{g_i(x)}$$

$$\begin{aligned}\text{subject to } G_j(x,t) &\le 0, \text{ for all } t\in T_j, \quad j=1,2,...,q,\\ H_k(x,s) &= 0, \text{ for all } s\in S_k, \quad j=1,2,...,r,\\ x &\in X\end{aligned}$$

where $f_i : X \to R$, $g_i : X \to R, i \in I = \{1,2,...,p\}$, are real-valued functions defined on a nonempty open subset X of R^n such that, for each $i \in I$, $g_i(x) > 0$ for all $x \in X$, $T_j, j \in J = \{1,2,...,q\}$, and $S_k, k \in K = \{1,2,...,r\}$, are compact subsets of complete metric spaces, $x \to G_j(x,t)$ is a function on X for all $t \in T_j$, $x \to H_k(x,s)$ is a function on X for all $s \in S_k$, for each $j \in J$ and $k \in K, t \to G_j(x,t)$ and $s \to H_k(x,s)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$ satisfying the constraints of problem (P). Let $D = \{x \in X : G_j(x,t) \le 0, \text{ for all } t \in T_j, j = 1,2,...,q, H_k(x,s) = 0, \text{ for all } s \in S_k, k = 1,2,...,r\}$ be the set of all feasible solutions of (P).

The plan of the paper is as follows. In Section 2, we give some notation and definitions used throughout the paper. In Section 3, we discuss duality between the primal problem and a generalized dual model, and finally Conclusion are given in Section 4. Furthermore, the results obtained in this paper extend and generalize the results of Antczak and Zalmai [7] to a class of higher-order duality.

2. Notation and Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. Let X be an open subset of R^n. First, we recall the following definitions.

Definition 2.1 [7] The tangent cone to the feasible set D of problem (P) at $\bar{x} \in D$ is the set

$$T(D;\bar{x}) \equiv \{h \in R^n : h = \lim_{n\to\infty} t_n(x^n - \bar{x}) \text{ such that } x^n \in D,$$
$$\lim_{n\to\infty} x^n = \bar{x}, \text{ and } t_n > 0 \text{ for all } n = 1,2,....\}$$

Definition 2.2 [7] Let $\bar{x} \in D$. The linearizing cone at $\bar{x}$ for problem (P) is the set defined by

$$C(\bar{x}) \equiv \{h \in R^n : \langle \nabla G_j(\bar{x},t), h\rangle \le 0 \text{ for all } t \in \hat{T}_j(\bar{x}), j = 1,2,...,q,$$
$$\langle \nabla H_k(\bar{x},s), h\rangle = 0 \text{ for all } s \in S_k, k = 1,2,...,r\},$$

where $\hat{T}_j(\bar{x}) \equiv \{t \in T_j : G_j(\bar{x},t) = 0\}, j = 1,2,...,q,$and $<,>$ denotes the Euclidean inner product.

Definition 2.3 [7] The problem (P) satisfies the generalized Abadie constraint qualification at a given point $\bar{x} \in D$ if the following relation $C(\bar{x}) \subseteq T(D;\bar{x})$ holds.

Now, we give the definition of (strictly) higher-order (Φ,ρ)-V-invex function.

Let $f : X \to R^k$ and $\theta : X \times R^n \to R^k$ be differentiable functions. Also, consider the function $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi(x,\bar{x},.)$ is convex on R^{n+1}, $\Phi(x,\bar{x},(0,a)) \ge 0$ for all $x \in X$ and every $a \in R_+$, $\rho = (\rho_1,\rho_2,...,\rho_k) \in R^k$ and real-valued functions $\alpha_i : X \times X \to R_+ \setminus \{0\}, i = 1,2,...,k,$

Definition 2.4 [22] A function f is said to be higher-order (Φ, ρ)-V-invex at $\bar{x} \in X$ if

$$\begin{aligned} & f_i(x) - f_i(\bar{x}) - \theta_i(\bar{x}, p) + p^T \nabla_p \theta_i(\bar{x}, p) \\ \geq\ & \Phi(x, \bar{x}, \alpha_i(x, \bar{x})(\nabla_x f_i(\bar{x}) + \nabla_p \theta_i(\bar{x}, p), \rho_i)), i = 1, 2, ..., k, \end{aligned} \tag{1}$$

hold for all $(x, p) \in X \times R^n$.

If each function $f_i, i = 1, 2, ..., k$, satisfies the inequality (1) at each $x \in X$, then $f_i, i = 1, 2, ..., k$ is said to be higher-order (Φ, ρ_i)-α_i-invex at $\bar{x}$ on X.

Definition 2.5 [22] A function f is said to be strictly higher-order (Φ, ρ)-V-invex at $\bar{x} \in X$ if

$$\begin{aligned} & f_i(x) - f_i(\bar{x}) - \theta_i(\bar{x}, p) + p^T \nabla_p \theta_i(\bar{x}, p) \\ >\ & \Phi(x, \bar{x}, \alpha_i(x, \bar{x})(\nabla_x f_i(\bar{x}) + \nabla_p \theta_i(\bar{x}, p), \rho_i)), i = 1, 2, ..., k, \end{aligned} \tag{1'}$$

hold for all $(x, p) \in X \times R^n$.

If each function $f_i, i = 1, 2, ..., k$, satisfies the inequality (1') at each $x \in X$, then $f_i, i = 1, 2, ..., k$ is said to be strictly higher-order (Φ, ρ_i)-α_i-invex at $\bar{x}$ on X.

Remark 2.1 In order to define an analogous class of higher-order (Φ, ρ)-V-incave functions, the direction of the inequalities in (1) should be changed to the opposite one.

We shall need the following result in our discussion, which provides an alternative expression for the objective function of (P).

Lemma 2.1 [7] *For each $x \in X$,*

$$\varphi(x) \equiv \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)} = \max_{\lambda \in \Lambda} \frac{\sum_{i=1}^{p} \lambda_i f_i(x)}{\sum_{i=1}^{p} \lambda_i g_i(x)},$$

where $\Lambda = \{\lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \in R^p : \lambda_i \geq 0, \sum_{i=1}^{p} \lambda_i = 1\}$.

Now, we recall a necessary optimality result for (P) which was established in Zalmai and Zhang [29].

Theorem 2.1 *Let $\bar{x} \in D$ be an optimal solution for* (P) *with the corresponding optimal value equal to $\bar{v} = \max_{1 \leq i \leq p} \frac{f_i(\bar{x})}{g_i(\bar{x})}$, the functions $t \to G_j(x, t), t \in T_j$ and $s \to H_k(x, s), s \in S_k$ be continuously differentiable at $\bar{x}$, the generalized Abadie constraint qualification be satisfied at $\bar{x}$ and the set cone $\{\nabla G_j(\bar{x}, t) : t \in \hat{T}_j(\bar{x}), j = 1, 2, ..., q\}+$ span $\{\nabla H_k(\bar{x}, s) : s \in S_k, k = 1, 2, ..., r\}$ be closed. Then there exist $\bar{\lambda} \in \Lambda$ and integers $\bar{w}_0$ and $\bar{w}$, with $0 \leq \bar{w}_0 \leq \bar{w} \leq n + 1$, such that there exist $\bar{w}_0$ indices j_m, with $1 \leq j_m \leq q$, together with $\bar{w}_0$ points $t^m \in \hat{T}_{j_m}(\bar{x}) = \{t \in T_{j_m} : G_{j_m}(\bar{x}, t) = 0\}, m = 1, 2, ..., \bar{w}_0, \bar{w} - \bar{w}_0$ indices k_m, with $1 \leq k_m \leq r$, together with $\bar{w} - \bar{w}_0$ points $s^m \in S_{k_m}, m = \bar{w}_0 + 1, ..., \bar{w}$ and $\bar{w}$ real numbers $\bar{\mu}_m$ with $\bar{\mu}_m > 0, m = 1, 2, ..., \bar{w}_0$, such that the following conditions are satisfied:*

$$\sum_{i=1}^{p} \bar{\lambda}_i \{\triangledown f_i(\bar{x}) - \bar{v} \triangledown g_i(\bar{x})\} + \sum_{m=1}^{\bar{w}_0} \bar{\mu}_m \triangledown G_{j_m}(\bar{x}, t^m) + \sum_{l=\bar{w}_0+1}^{\bar{w}} \bar{\mu}_l \triangledown H_{k_l}(\bar{x}, s^l) = 0,$$

$$\bar{\lambda}_i[f_i(\bar{x}) - \bar{v} g_i(\bar{x})] = 0, i = 1, 2, ..., p.$$

We shall henceforth refer to $\bar{x} \in D$ as a normal feasible solution of problem (P) if the generalized Abadie constraint qualification is satisfied at $\bar{x}$, and if the set *cone* $\{\nabla G_j(\bar{x}, t) : t \in \hat{T}_j(\bar{x}), j = 1, 2, ..., q\}+$ *span* $\{\nabla H_k(\bar{x}, s) : s \in S_k, k = 1, 2, ..., r\}$ is closed, where *cone* (A) is a conic hull of A (i.e., the smallest convex cone containing A), *span* (A) is the linear hull of A (i.e., the smallest subspace containing A).

In the remainder of this paper, we shall assume that the functions $f_i, g_i, i = 1, 2, ..., p$, $\xi \to G_j(\xi, t)$ and $\xi \to H_k(\xi, s)$, are continuously differentiable on X for all $t \in T_j$, $j \in J$, and $s \in S_k$, $k \in K$. We denote

$$\mathbb{H} = \{(u, y, \lambda, \mu, v, w, w_0, J_{w_0}, K_{w\setminus w_0}, \bar{t}, \bar{s}) : u \in X, y \in R^n, \lambda \in \Lambda, v \in R_+, 0 \leq w_0 \leq w \leq n+1; \mu \in R^w, \mu_i > 0, 1 \leq i \leq w_0; J_{w_0} = \{j_1, j_2, ..., j_{w_0}\}, 1 \leq j_i \leq q, K_{w\setminus w_0} = \{k_{w_0+1}, k_{w_0+2}, ..., k_w\}, 1 \leq k_i \leq r, \bar{t} \equiv (t^1, t^2, ..., t^{w_0}), t^i \in T_{j_i}, \bar{s} \equiv (s^{w_0+1}, s^{w_0+2}, ..., s^w), s^i \in S_{k_i}\}.$$

3. Generalized duality model

In this section, we need some additional notation for the following generalized dual problem (D). Let w_0 and w be integers, with $1 \leq w_0 \leq w \leq n+1$, and let$\{J_0, J_1, ..., J_M\}$ and $\{K_0, K_1, ..., K_M\}$ be partitions of the sets $\{1, 2, ..., w_0\}$ and $\{w_0+1, ..., w\}$, respectively; thus, $J_i \subseteq \{1, 2, ..., w_0\}$ for each $i \in \{1, 2, ..., M\} \bigcup \{0\}$, $J_i \bigcap J_j = \emptyset$ for each $i, j \in \{1, 2, ..., M\} \bigcup \{0\}$ with $i \neq j$, and $\bigcup_{i=0}^{M} J_i = \{1, 2, ..., w_0\}$. Similar properties hold for $\{K_0, K_1, ..., K_M\}$. Moreover, if m_1 and m_2 are the numbers of the partitioning sets of $\{1, 2, ..., w_0\}$ and $\{w_0+1, ..., w\}$, respectively, then $M = \max\{m_1, m_2\}$ and $J_i = \emptyset$ or $K_i = \emptyset$, for $i > \min\{m_1, m_2\}$.

Now, we propose a generalized higher-order dual for (P) as follows:

(D) $$\sup_{(u,y,\lambda,\mu,v,w,w_0,J_{w_0},K_{w\setminus w_0},\bar{t},\bar{s})\in\mathbb{H}} v$$

$$\sum_{i=1}^{p} \lambda_i \{ \bigtriangledown f_i(u) - v \bigtriangledown g_i(u)\} + \sum_{m=1}^{w_0} \mu_m \bigtriangledown G_{j_m}(u, t^m) + \sum_{l=w_0+1}^{w} \mu_l \bigtriangledown H_{k_l}(u, s^l) +$$

$$\sum_{i=1}^{p} \lambda_i [\nabla_y (\mathcal{F}_i(u, y) - v\mathcal{G}_i(u, y))] + \sum_{m=1}^{w_0} \mu_m \nabla_y \mathcal{I}_{j_m}(u, t^m, y) + \sum_{l=w_0+1}^{w} \mu_l \nabla_y \mathcal{H}_{k_l}(u, s^l, y) = 0, \quad (2)$$

$$[f_i(u) - vg_i(u)] + \sum_{m\in J_0} \mu_m G_{j_m}(u, t^m) + \sum_{m\in K_0} \mu_m H_{k_m}(u, s^m) + [\mathcal{F}_i(u, y) - v\mathcal{G}_i(u, y)]$$
$$-y^T \nabla_y [\mathcal{F}_i(u, y) - v\mathcal{G}_i(u, y)] + \sum_{m\in J_0} \mu_m \mathcal{I}_{j_m}(u, t^m, y) - \sum_{m\in J_0} \mu_m y^T \nabla_y \mathcal{I}_{j_m}(u, t^m, y)$$
$$+ \sum_{m\in K_0} \mu_m \mathcal{H}_{k_m}(u, s^m, y) - \sum_{m\in K_0} \mu_m y^T \nabla_y \mathcal{H}_{k_m}(u, s^m, y) \geq 0, \ i = 1, 2, ..., p, \quad (3)$$

$$\sum_{m\in J_\iota} \mu_m G_{j_m}(u, t^m) + \sum_{m\in K_\iota} \mu_m H_{k_m}(u, s^m) + \sum_{m\in J_\iota} \mu_m \mathcal{I}_{j_m}(u, t^m, y) - \sum_{m\in J_\iota} \mu_m y^T \nabla_y \mathcal{I}_{j_m}$$
$$(u, t^m, y) + \sum_{m\in K_\iota} \mu_m \mathcal{H}_{k_m}(u, s^m, y) - \sum_{m\in K_\iota} \mu_m y^T \nabla_y \mathcal{H}_{k_m}(u, s^m, y) \geq 0, \ \iota = 1, 2, ..., M, \quad (4)$$

where $\mathcal{F}_i : X \times R^n \to R$, $\mathcal{G}_i : X \times R^n \to R, i = 1, 2, ..., p$, $\mathcal{I}_{j_m} : R^n \times T_j \times R^n \to R, m = 1, 2, ..., w_0$, and $\mathcal{H}_{k_l} : R^n \times S_k \times R^n \to R, l = w_0+1, ..., w$ are differentiable functions.

Let $\Theta = \{(u, y, \lambda, \mu, v, w, w_0, J_{w_0}, K_{w\setminus w_0}, \bar{t}, \bar{s}) \in \mathbb{H} :$ satisfying (2)-(4)$\}$ be the set all feasible solutions of dual problem (D). Moreover, let $\text{pr}_X\Theta = \{u \in X : (u, y, \lambda, \mu, v, w, w_0, J_{w_0}, K_{w\setminus w_0}, \bar{t}, \bar{s}) \in \Theta\}$ be the projection of Θ on X.

Remark 3.1 Let $\mathcal{F}_i(u, y) = \frac{1}{2} y^T \nabla^2 f_i(u) y$, $\mathcal{G}_i(u, y) = \frac{1}{2} y^T \nabla^2 g_i(u) y, i \in I$, $\mathcal{I}_{j_m}(u, t^m, y) = \frac{1}{2} y^T \nabla^2 G_{j_m}(u, t^m) y, m = 1, 2, ..., w_0$ and $\mathcal{H}_{k_m}(u, s^m, y) = \frac{1}{2} y^T \nabla^2 H_{k_m}(u, s^m) y, m = w_0 + 1, ..., w$. Then (D) reduces to the second order dual (SPD3) in [7]. If in addition, $y = 0$, then we get the dual (D) formulated in [30].

We take $\alpha_i : X \times X \to R_+ \setminus \{0\}, \hat{\rho}_i \in R, i = 1, 2, ..., p$ and $\beta_\iota : X \times X \to R_+ \setminus \{0\}, \tilde{\rho}_\iota \in R, \iota = 1, 2, ..., M$.

Theorem 3.1 (Weak Duality) *Let x and $(u, y, \lambda, \mu, v, w, w_0, J_{w_0}, K_{w\backslash w_0}, \bar{t}, \bar{s})$ be feasible points for* (P) *and* (D)*, respectively. Suppose that $[f_i(.) - vg_i(.)] + \sum_{m\in J_0} \mu_m G_{j_m}(., t^m) + \sum_{m\in K_0} \mu_m H_{k_m}(., s^m)$ for $i = 1, 2, ..., p$ is higher-order $(\Phi, \hat{\rho}_i)$-α_i-invex at u on $D \cup \mathrm{pr}_X\Theta$. Also, let $\sum_{m\in J_\iota} \mu_m G_{j_m}(., t^m) + \sum_{m\in K_\iota} \mu_m H_{k_m}(., s^m)$ for $\iota = 1, 2, ..., M$ be higher- order $(\Phi, \tilde{\rho}_\iota)$-$\beta_\iota$-invex at u on $D \cup \mathrm{pr}_X\Theta$ and the inequality*

$$\sum_{i=1}^{p} \lambda_i \rho_i + \sum_{\iota=1}^{M} \rho_\iota \geq 0$$

holds. Then $\varphi(x) \geq v$.

Proof. By the assumption of higher-order $(\Phi, \hat{\rho}_i)$-α_i-invexity of $[f_i(.) - vg_i(.)] + \sum_{m\in J_0} \mu_m G_{j_m} (., t^m) + \sum_{m\in K_0} \mu_m H_{k_m}(., s^m)$ for $i = 1, 2, ..., p$ at u on $D \cup pr_X\Theta$, we have

$$[f_i(x) - vg_i(x)] + \sum_{m\in J_0} \mu_m G_{j_m}(x, t^m) + \sum_{m\in K_0} \mu_m H_{k_m}(x, s^m) - \{[f_i(u) - vg_i(u)]$$
$$+ \sum_{m\in J_0} \mu_m G_{j_m}(u, t^m) + \sum_{m\in K_0} \mu_m H_{k_m}(u, s^m)\} - \{[\mathcal{F}_i(u, y) - v\mathcal{G}_i(u, y)]+$$
$$+ \sum_{m\in J_0} \mu_m \mathcal{I}_{j_m}(u, t^m, y) + \sum_{m\in K_0} \mu_m \mathcal{F}_{k_m}(u, s^m, y)\} + y^T \{\nabla_y [\mathcal{F}_i (u, y) - v\mathcal{G}_i (u, y)] +$$
$$+ \sum_{m\in J_0} \mu_m \nabla_y \mathcal{I}_{jm} (u, t^m, y) + \sum_{m\in K_0} \mu_m \nabla_y \mathcal{H}_{k_m}(u, s^m, y)\}$$
$$\geq \Phi(x, u, \alpha_i(x, u)([\bigtriangledown f_i(u) - v \bigtriangledown g_i(u)] + \sum_{m\in J_0} \mu_m \bigtriangledown G_{j_m}(u, t^m) + \sum_{m\in K_0} \mu_m \bigtriangledown H_{k_m}(u, s^m)$$
$$+[\nabla_y(\mathcal{F}_i(u, y) - v\mathcal{G}_i(u, y))] + \sum_{m\in J_0} \mu_m \nabla_y \mathcal{I}_{j_m}(u, t^m, y)$$
$$+ \sum_{m\in K_0} \mu_m \nabla_y \mathcal{H}_{k_m}(u, s^m, y), \hat{\rho}_i)), i = 1, 2, ..., p.$$

On utilizing the feasibility of x for (P) and dual constraint (3), the above inequality yields

$$[f_i (x) - vg_i] \geq \Phi (x, u, \alpha_i (x, u) [\nabla f_i (u) - v\nabla g_i (u)]) + \sum_{m\in J_0} \mu_m \nabla G_{jm} (u, t^m)$$
$$+ \sum_{m\in K_0} \mu_m \bigtriangledown H_{k_m}(u, s^m) + [\nabla_y (\mathcal{F}_i (u, y) - v\mathcal{G}_i (u, y))]+$$
$$+ \sum_{m\in J_0} \mu_m \nabla_y \mathcal{I}_{jm} (u, t^m, y) + \sum_{m\in K_0} \mu_m \nabla_y \mathcal{H}_{k_m}(u, s^m, y), \hat{\rho}_i)), i = 1, 2, ..., p.$$

Take $\tau = \sum_{i=1}^{p} \frac{\lambda_i}{\alpha_i(x, u)} + \sum_{\iota=1}^{M} \frac{1}{\beta_\iota(x, u)}$. It is easy to see that $\tau > 0$.

Now multiplying the above inequalities by $\frac{\lambda_i}{\tau\alpha_i(x, u)} \geq 0, i = 1, 2, ..., p$, then summing, we obtain

$$\sum_{i=1}^{p} \frac{\lambda_i}{\tau\alpha_i(x, u)} [f_i(x) - vg_i(x)] \geq \sum_{i=1}^{p} \frac{\lambda_i}{\tau\alpha_i(x, u)} \Phi(x, u, \alpha_i(x, u)([\bigtriangledown f_i(u) - v \bigtriangledown g_i(u)]$$
$$+ \sum_{m\in J_0} \mu_m \bigtriangledown G_{j_m}(u, t^m) + \sum_{m\in K_0} \mu_m \bigtriangledown H_{k_m}(u, s^m)$$

$$+[\nabla_y(\mathcal{F}_i(u,y)-v\mathcal{G}_i(u,y))]+\sum_{m\in J_0}\mu_m\nabla_y\mathcal{I}_{j_m}(u,t^m,y)$$
$$+\sum_{m\in K_0}\mu_m\nabla_y\mathcal{H}_{k_m}(u,s^m,y),\hat{\rho}_i)). \tag{5}$$

Now, by using higher-order $(\Phi,\tilde{\rho}_\iota)$-$\beta_\iota$-invexity of $\sum_{m\in J_\iota}\mu_m G_{j_m}(.,t^m)+\sum_{m\in K_\iota}\mu_m H_{k_m}(.,s^m)$ for $\iota=1,2,...,M$ at u on $D\cup \mathrm{pr}_X\Theta$, we obtain

$$\sum_{m\in J_\iota}\mu_m G_{j_m}(x,t^m)+\sum_{m\in K_\iota}\mu_m H_{k_m}(x,s^m)-\{\sum_{m\in J_\iota}\mu_m G_{j_m}(u,t^m)+\sum_{m\in K_\iota}\mu_m H_{k_m}(u,s^m)\}$$
$$-\{\sum_{m\in J_\iota}\mu_m\mathcal{I}_{j_m}(u,t^m,y)+\sum_{m\in K_\iota}\mu_m\mathcal{H}_{k_m}(u,s^m,y)\}+y^T\{\sum_{m\in J_\iota}\mu_m\nabla_y\mathcal{I}_{j_m}(u,t^m,y)+$$
$$+\sum_{m\in K_\iota}\mu_m\nabla_y\mathcal{H}_{k_m}(u,s^m,y)\}$$
$$\geq\Phi(x,u,\beta_\iota(x,u)(\sum_{m\in J_\iota}\mu_m\bigtriangledown G_{j_m}(u,t^m)+\sum_{m\in K_\iota}\mu_m\bigtriangledown H_{k_m}(u,s^m)+\sum_{m\in J_\iota}\mu_m\nabla_y\mathcal{I}_{j_m}(u,t^m,y)$$
$$+\sum_{m\in K_\iota}\mu_m\nabla_{yk_m}(u,s^m,y),\tilde{\rho}_\iota)),\ \ \iota=1,2,...,M.$$

On utilizing the feasibility of x for (P) and dual constraint (4), the above inequality yields

$$\Phi(x,u,\beta_\iota(x,u)(\sum_{m\in J_\iota}\mu_m\bigtriangledown G_{j_m}(u,t^m)+\sum_{m\in K_\iota}\mu_m\bigtriangledown H_{k_m}(u,s^m)+\sum_{m\in J_\iota}\mu_m\nabla_y\mathcal{I}_{j_m}(u,t^m,y)$$
$$+\sum_{m\in K_\iota}\mu_m\nabla_y\mathcal{H}_{k_m}(u,s^m,y),\tilde{\rho}_\iota))\leq 0,\iota=1,2,...,M. \tag{6}$$

Multiplying each inequality (6) by $\frac{1}{\tau\beta_\iota(x,u)}>0,\iota=1,2,...,M$, and then summing up these inequalities, we get

$$\sum_{\iota=1}^{M}\frac{1}{\tau\beta_\iota(x,u)}\Phi(x,u,\beta_\iota(x,u)(\sum_{m\in J_\iota}\mu_m\bigtriangledown G_{j_m}(u,t^m)+\sum_{m\in K_\iota}\mu_m\bigtriangledown H_{k_m}(u,s^m)$$
$$+\sum_{m\in J_\iota}\mu_m\nabla_y\mathcal{I}_{j_m}(u,t^m,y)+\sum_{m\in K_\iota}\mu_m\nabla_y\mathcal{H}_{k_m}(u,s^m,y),\tilde{\rho}_\iota))\leq 0. \tag{7}$$

By adding (5) and (7), we get

$$\sum_{i=1}^{p}\frac{\lambda_i}{\tau\alpha_i(x,u)}[f_i(x)-vg_i(x)]\geq\sum_{i=1}^{p}\frac{\lambda_i}{\tau\alpha_i(x,u)}\Phi(x,u,\alpha_i(x,u)([\bigtriangledown f_i(u)-v\bigtriangledown g_i(u)]$$
$$+\sum_{m\in J_0}\mu_m\bigtriangledown\mathcal{G}_{j_m}(u,t^m)+\sum_{m\in K_0}\mu_m\bigtriangledown\mathcal{H}_{k_m}(u,s^m)$$
$$+[\nabla_y(\mathcal{F}_i(u,y)-v\mathcal{G}_i(u,y))]+\sum_{m\in J_0}\mu_m\nabla_y\mathcal{I}_{j_m}(u,t^m,y)+\sum_{m\in K_0}\mu_m\nabla_y\mathcal{H}_{k_m}(u,s^m,y),\hat{\rho}_i))$$
$$+\sum_{\iota=1}^{M}\frac{1}{\tau\beta_\iota(x,u)}\Phi(x,u,\beta_\iota(x,u)(\sum_{m\in J_\iota}\mu_m\bigtriangledown\mathcal{G}_{j_m}(u,t^m)+\sum_{m\in K_\iota}\mu_m\bigtriangledown\mathcal{H}_{k_m}(u,s^m)$$
$$+\sum_{m\in J_\iota}\mu_m\nabla_y\mathcal{I}_{j_m}(u,t^m,y)+\sum_{m\in K_\iota}\mu_m\nabla_y\mathcal{H}_{k_m}(u,s^m,y),\tilde{\rho}_\iota)).$$

Using the convexity of $\Phi(x,u,(.,.))$ on R^{n+1}, the above inequality implies that

$$\sum_{i=1}^{p} \frac{\lambda_i}{\tau\alpha_i(x,u)}[f_i(x) - vg_i(x)] \geq \Phi\Bigg(x, u, \Bigg(\frac{1}{\tau}\Bigg[\sum_{i=1}^{p} \lambda_i[\bigtriangledown f_i(u) - v \bigtriangledown g_i(u)] +$$

$$+ \sum_{m\in J_0} \mu_m \bigtriangledown G_{j_m}(u, t^m) + \sum_{m\in K_0} \mu_m \bigtriangledown H_{k_m}(u, s^m) + \sum_{i=1}^{p} \lambda_i[\nabla_y(\mathcal{F}_i(u,y) - v\mathcal{G}_i(u,y))]$$

$$+ \sum_{m\in J_0} \mu_m \nabla_y \mathcal{I}_{j_m}(u, t^m, y) + \sum_{m\in K_0} \mu_m \nabla_y \mathcal{H}_{k_m}(u, s^m, y) + \sum_{\iota=1}^{M} \sum_{m\in J_\iota} \mu_m \bigtriangledown G_{j_m}(u, t^m)$$

$$+ \sum_{\iota=1}^{M} \sum_{m\in K_\iota} \mu_m \bigtriangledown H_{k_m}(u, s^m) + \sum_{\iota=1}^{M} \sum_{m\in J_\iota} \mu_m \nabla_y \mathcal{I}_{j_m}(u, t^m, y)$$

$$+ \sum_{\iota=1}^{M} \sum_{m\in K_\iota} \mu_m \nabla_y \mathcal{H}_{k_m}(u, s^m, y)\Bigg], \frac{1}{\tau}\Bigg[\sum_{i=1}^{p} \lambda_i \hat{\rho}_i + \sum_{\iota=1}^{M} \tilde{\rho}_\iota\Bigg]\Bigg)\Bigg).$$

Thus,

$$\sum_{i=1}^{p} \frac{\lambda_i}{\tau\alpha_i(x,u)}[f_i(x) - vg_i(x)] \geq \Phi\Bigg(x, u, \Bigg(\frac{1}{\tau}\Bigg[\sum_{i=1}^{p} \lambda_i[\bigtriangledown f_i(u) - v \bigtriangledown g_i(u)] +$$

$$+ \sum_{m=1}^{w_0} \mu_m \bigtriangledown G_{j_m}(u, t^m) + \sum_{m=w_0+1}^{w} \mu_m \bigtriangledown H_{k_m}(u, s^m) + \sum_{i=1}^{p} \lambda_i[\nabla_y$$

$$(\mathcal{F}_i(u,y) - v\mathcal{G}_i(u,y))] + \sum_{m=1}^{w_0} \mu_m \nabla_y \mathcal{I}_{j_m}(u, t^m, y)$$

$$+ \sum_{m=w_0+1}^{w} \mu_m \nabla_y \mathcal{H}_{k_m}(u, s^m, y)\Bigg], \frac{1}{\tau}\Bigg[\sum_{i=1}^{p} \lambda_i \hat{\rho}_i + \sum_{\iota=1}^{M} \tilde{\rho}_\iota\Bigg]\Bigg)\Bigg) \geq 0,$$

where the last inequality is according to the dual constraint (2), the hypothesis $\sum\limits_{i=1}^{p} \lambda_i \hat{\rho}_i + \sum\limits_{\iota=1}^{M} \tilde{\rho}_\iota \geq 0$, and the fact $\Phi(x, u, (0, a)) \geq 0$, $a \in R_+$. Hence

$$\sum_{i=1}^{p} \frac{\lambda_i}{\tau\alpha_i(x,u)}[f_i(x) - vg_i(x)] \geq 0.$$

Since $\tau > 0$, the above inequality gives

$$\sum_{i=1}^{p} \frac{\lambda_i}{\alpha_i(x,u)}[f_i(x) - vg_i(x)] \geq 0. \tag{8}$$

Using this inequality and Lemma 2.1., we see that

$$\varphi(x) = \max_{1 \leq i \leq p} \frac{f_i(x)}{g_i(x)} = \max_{\lambda\in\Lambda} \frac{\sum_{i=1}^{p} \lambda_i f_i(x)}{\sum_{i=1}^{p} \lambda_i g_i(x)} \text{ (by Lemma 2.1)}$$

$$\geq \frac{\sum_{i=1}^{p} \lambda_i f_i(x)}{\sum_{i=1}^{p} \lambda_i g_i(x)} \geq \nu \text{ (by (8))}$$

This completes the proof. □

The proof of the following Theorems 3.2 and 3.3 can be obtained by a similar way as Theorems 3.2 and 3.3 in [22]. Therefore, we simply state them here.

Theorem 3.2 (Strong duality) *Let $\bar{x}$ be a normal optimal solution for* (P). *Assume that*

$$\begin{cases} \mathcal{F}_i(\bar{x},0)=0;\ \nabla_y\ \mathcal{F}_i(\bar{x},0)=0, i=1,2,...,p, \\ \mathcal{G}_i(\bar{x},0)=0;\ \nabla_y\ \mathcal{G}_i(\bar{x},0)=0, i=1,2,...,p, \\ \mathcal{I}_{j_m}(\bar{x},t^m,0)=0; \nabla_y\ \mathcal{I}_{j_m}(\bar{x},t^m,0)=0, m=1,...,w_0, \\ \mathcal{H}_{k_l}(\bar{x},s^l,0)=0; \nabla_y\ \mathcal{H}_{k_l}(\bar{x},s^l,0)=0, l=w_0+1,...,w. \end{cases}$$

Then there exist $\bar{\lambda}\in\Lambda, \bar{v}\in R, 0\le\bar{w}_0\le\bar{w}\le n+1; \bar{\mu}\in R^{\bar{w}}, \bar{\mu}_m>0, 1\le m\le\bar{w}_0; J_{\bar{w}_0}=\{j_1,j_2,...,j_{\bar{w}_0}\}, 1\le j_m\le q, K_{\bar{w}\setminus\bar{w}_0}=\{k_{\bar{w}_0+1},k_{\bar{w}_0+2},...,k_{\bar{w}}\}, 1\le k_m\le r, \bar{\bar{t}}\equiv(t^1,t^2,...,t^{\bar{w}_0}), t^m\in T_{j_m}, \bar{\bar{s}}\equiv(s^{\bar{w}_0+1},s^{\bar{w}_0+2},...,s^{\bar{w}}), s^l\in S_{k_l}$ *such that* $(\bar{x},\bar{y}=0,\bar{\lambda},\bar{\mu},\bar{v},\bar{w},\bar{w}_0,J_{\bar{w}_0},K_{\bar{w}\setminus w_0},\bar{\bar{t}},\bar{\bar{s}})$ *is feasible for* (D) *and the corresponding objective values of* (P) *and* (D) *are equal. Further, if the conditions of the weak duality Theorem* 3.1 *holds for all feasible solutions of* (D), *then* $(\bar{x},\bar{y}=0,\bar{\lambda},\bar{\mu},\bar{v},\ \bar{w},\ \bar{w}_0,J_{\bar{w}_0},K_{\bar{w}\setminus w_0},\bar{\bar{t}},\bar{\bar{s}})$ *is optimal for* (D).

Theorem 3.3 (Strict converse duality) *Let $\bar{x}$ be a normal optimal solution of* (P) *and* $(\bar{u},\bar{y},\bar{\lambda},\bar{\mu},\bar{v},\ \bar{w},\ \bar{w}_0,J_{\bar{w}_0},K_{\bar{w}\setminus w_0},\bar{\bar{t}},\bar{\bar{s}})$ *be an optimal solution of* (D). *Suppose that*

$$[f_i(.)-\bar{v}g_i(.)]+\sum_{m\in J_0}\bar{\mu}_m G_{j_m}(.,t^m)+\sum_{m\in K_0}\bar{\mu}_m H_{k_m}(.,s^m)$$

for $i=1,2,...,p$ *is strictly higher-order* $(\Phi,\hat{\rho}_i)$-α_i*-invex at* $\bar{u}$ *on* $D\cup \mathrm{pr}_X\Theta$. *Also, let*

$$\sum_{m\in J_\iota}\bar{\mu}_m G_{j_m}(.,t^m)+\sum_{m\in K_\iota}\bar{\mu}_m H_{k_m}(.,s^m)$$

for $\iota=1,2,...,M$ *be higher-order* $(\Phi,\tilde{\rho}_\iota)$-$\beta_\iota$*-invex at* $\bar{u}$ *on* $D\cup \mathrm{pr}_X\Theta$ *and the inequality*

$$\sum_{i=1}^{p}\bar{\lambda}_i\hat{\rho}_i+\sum_{\iota=1}^{M}\tilde{\rho}_\iota\ge 0$$

holds. Then $\bar{x}=\bar{u}$, *that is,* $\bar{u}$ *is an optimal solution for* (P).

4. Conclusions

In this paper , we have formulated a higher-order generalized dual model for a semi-infinite minimax fractional programming problem and proved appropriate duality relations involving higher-order (Φ,ρ)-V-invex functions. The methods used here can be extended to the study of nonsmooth variational and nonsmooth control problems, which will orient the future research of the authors.

REFERENCES

[1] *I. Ahmad*, Higher-order duality in nondifferentiable minimax fractional programming involving generalized convexity, J. Inequal. Appl., 2012:306, 8 pages.

[2] *I. Ahmad, Z. Husain and S. Sharma*, Higher-order duality in nondifferentiable minimax programming with generalized type I functions, J. Optim. Theory Appl., 141(2009), 1-12.

[3] *T. Antczak.* Nonsmooth minimax programming under locally Lipschitz (Φ,ρ)-invexity, Appl. Math. Comput., 217(2011), 9607-9624.

[4] *T. Antczak,* Saddle point criteria and Wolfe duality in nonsmooth (Φ,ρ)-invex vector optimization problems with inequality and equality constraints, Int. J. Comput. Math., 92(2015), 1-26.

[5] *T. Antczak,* Sufficient optimality conditions for semi-infinite multiobjective fractional programming under (Φ,ρ)-V-invexity and generalized (Φ,ρ)-V-invexity, Filomat, 30(2016), 3649-3665.

[6] *T. Antczak,* Semi-infinite minimax fractional programming under (Φ,ρ)-V-invexity and generalized (Φ,ρ)-V-invexity. Optimality, Int. J. Oper. Res., 31(2018), 164-197.

[7] *T. Antczak and G. J. Zalmai*, Second order (Φ, ρ)-V-invexity and duality for semi-infinite minimax fractional programming, Appl. Math. Comput., 227(2014), 831-856.

[8] *M. Bhatia,* Higher-order duality in vector optimization over cones, Optim. Lett., 6(2012), 17-30.

[9] *G. Caristi, M. Ferrara and A. Ştefănescu*, Mathematical programming with (Φ, ρ)-invexity, in: V. I. Konnov, D. T. Luc, A. M. Rubinov (Eds.), Generalized Convexity and Related Topics, Lecture Notes in Economics and Mathematical Systems, vol. 583, Springer, Berlin, Heidelberg, New York, 2007, pp. 167-176.

[10] *X. Chen,* Higher-order symmetric duality in nondifferentiable multiobjective programming problems, J. Math. Anal. Appl., 290(2004), 423-435.

[11] *M. Ferrara and M.V. Ştefănescu,* Optimality conditions and duality in multiobjective programming with (Φ, ρ)-invexity, Yugoslav J. Oper. Res., 18(2008), 153-165.

[12] *A. Jayswal and K. Kummari,* Higher-order duality for multiobjective programming problem involving (Φ, ρ)-invex functions, J. Egypt. Math. Soc., 23(2015), 12-19.

[13] *A. Jayswal, I.M. Stancu-Minasian and D. Kumar,* Higher-order duality for multiobjective programming problems involving (F, α, ρ, d)-V-type I functions, J. Math. Model. Algorithms Oper. Res. 13(2014), 125-141.

[14] *C. Y. Jin and C.-Z. Cheng*, Higher-order duality for minimax fractional type programming involving symmetric matrices, Appl. Math., Irvine, 2(2011), 1387-1392.

[15] *M. Kapoor, S.K. Suneja and M.B. Grover,* Higher order optimality and duality in fractional vector optimization aver cones, Tamkang J. Math., 48(2017), 273-287.

[16] *D.S. Kim, H.S. Kang, Y.J. Lee and Y.Y. Seo,* Higher order duality in multiobjective programming with cone constraints, Optimization 59(2010), 29-43.

[17] *K.O. Kortanek and M.Yamasaki,* Semi-infinite transportation problems, J. Math. Anal. Appl., 88(1982), 555-565.

[18] *Y. Liu, C.H. Tseng and K.L. Teo,* A unified quadratic semi-infinite programming approach to time and frequency domain constrained digital filter design, Comm. Inf. Syst., 2(2002), 399-410.

[19] *S.K. Mishra, S.Y. Wang and K.K. Lai*, Higher-order duality for a class of nondifferentiable multiobjective programming problems, Int. J. Pure Appl. Math., 11(2004), 217-227.

[20] *E. Polak*, On the mathematical foundations of nondifferentiable optimization in engineering design, SIAM Rev., 29(1987), 21-89.

[21] *S. Sharma and S.K. Gupta,* Higher-order (Φ, ρ)-V-invexity and duality for vector optimization problems, Rend. Circ. Mat. Palermo, II. Ser.2, 65(2016), 351-364.

[22] *I.M. Stancu-Minasian, K.Kummari and A. Jayswal*, Duality for semi-infinite minimax fractional programming problem involving higher-order (Φ, ρ)-V-invexity, Numer. Func. Anal. Optim., 38(2017), 926-950.

[23] *A.I.F. Vaz, E.M.G.P. Fernandez and M.P.S.F. Gomes,* Robot trajectory planning with semi-infinite programming, Eur. J. Oper. Res., 153(2004), 607-617.

[24] *A.I.F. Vaz and E.C. Ferreira,* Air pollution control with semi-infinite programming, Appl. Math. Model., 33(2009), 1957-1969.

[25] *R.U. Verma,* Semi-infinite fractional programming, Infosys Science Foundation Series in Mathematical Sciences, Springer-Verlag, Singapore XI+291pages, 2017.

[26] *D. Wang and S.-C. Fang,* A semi-infinite programming model for earliness/tardiness production planning with a genetic algorithm, Comput. Math. Appl., 31(1996), 95-106.

[27] *G. J. Zalmai and Q. Zhang*, Global parametric sufficient optimality conditions for semi-infinite discrete minmax fractional programming problems involving generalized (η, ρ)-invex functions, Acta Math. Appl. Sinica, 23 (2007), 217-234.

[28] *G. J. Zalmai and Q. Zhang,* Parametric duality models for semi-infinite multiobjective fractional programming problems containing generalized (α, η, ρ)-V-invex functions, Acta Math. Appl. Sinica, 29 (2013), 225-240.

[29] *G. J. Zalmai and Q. Zhang*, Global nonparametric sufficient optimality conditions for semi-infinite discrete minimax fractional programming problems involving generalized (η, ρ)-invex functions, Numer. Func. Anal. Optim., 28 (2007), 173-209.

[30] *G. J. Zalmai and Q. Zhang*, Parametric duality models for semi-infinite discrete minimax fractional programming problems containing generalized (α, η, ρ)-V-invex functions, Global J. Math. Math. Sci., 1 (2005), 39-58.

[31] *J. Zhang*, Generalized convexity and higher order duality for mathematical programming problem. Ph.D. Thesis, La Trobe University, Australia, 1998.