ON CERTAIN CLASS OF UNIVALENT FUNCTIONS WITH CONIC DOMAINS INVOLVING SOKÓŁ - NUNOKAWA CLASS

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The aim of this investigation is to introduce a new class of analytic univalent functions $k-\mathcal{MN}$ that are connected to domains bounded by conic sections and obtain certain differential subordination results involving $k-\mathcal{MN}$. Certain comparisons of the differential subordination is being analyzed with the classical results existing in the literature. Apart from obtaining other results related to the class $k-\mathcal{MN}$, we also obtain a containment relation between the class $k-\mathcal{MN}$ and the class of starlike functions under certain condition. A slight improvement of a recent work of On some class of convex functions, C. R. Math. Acad. Sci. Paris, **353** *(2015), 427–431, by Sokół and Nunokawa* [19] *is also obtained.*

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of all functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{S}$ is said to be convex if and only if $\Re\left(zf''(z)/f'(z)\right) > -1$ for $z \in \mathbb{U}$ and a function $f \in \mathcal{S}$ is said to be starlike if and only if $\Re\left(zf'(z)/f(z)\right) > 0$ for $z \in \mathbb{U}$. The class of all convex and starlike functions are denoted by $\mathcal{K}$ and $\mathcal{S}^*$ respectively. Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ such that $\Re(p(z)) > 0$ in $\mathbb{U}$. In 1999, Kanas and Wiśniowska [7] (and [8]) introduced the class of k-uniformly convex functions, denoted by k-$\mathcal{UCV}$ and the class of k-starlike functions, denoted by k-$\mathcal{ST}$ respectively. The analytic conditions of these classes are the following (see [6],[7],[8] for details).

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For $0 \le k < \infty$,

$$k - \mathcal{UCV} = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \mathbb{U} \right\} \tag{2}$$

and

$$k - \mathcal{ST} = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in \mathbb{U} \right\}. \tag{3}$$

Note that for $k = 1$, we get the well known classes of $\mathcal{UCV}$ and $\mathcal{S}_p$ studied by Goodman [3],[4] and Rønning [15], while for $k = 0$, we get defined above families of convex $\mathcal{K}$ and starlike $\mathcal{S}^*$ functions, respectively.

It is easy to see that the conditions (2) and (3) may be rewritten into the form

$$\Re(p(z)) > k|p(z) - 1| \quad (z \in \mathbb{U}), \tag{4}$$

where $p(z) = 1 + zf''(z)/f'(z)$ or $p(z) = zf'(z)/f(z)$ is a function from the class $\mathcal{P}$.

Also, it is easy to see that $p(\mathbb{U})$ is the conic domain

$$\Omega_k = \{\omega \in \mathbb{C} : \Re(\omega) > k|\omega - 1|\} \tag{5}$$

or

$$\Omega_k = \left\{ \omega = u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}, \tag{6}$$

where $0 \le k < \infty$. Note that Ω_k is such that $1 \in \Omega_k$ and $\partial\Omega_k$ is a curve defined by

$$\partial\Omega_k = \left\{ \omega = u + iv : u^2 = k^2(u-1)^2 + k^2v^2 \right\}. \tag{7}$$

Elementary computations show that Ω_k represents a conic section symmetric about the real axis. It follows that the domain Ω_k is bounded by an ellipse for $k > 1$, by a parabola for $k = 1$ and by a hyperbola if $0 < k < 1$. Finally, for $k = 0$, Ω_k is the right half plane.

Recently Sokół and Nunokawa [19] defined an interesting new class of functions $\mathcal{MN}$ defined by

$$\mathcal{MN} = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in \mathbb{U} \right\}. \tag{8}$$

It is clear that $\mathcal{MN} \subset \mathcal{K}$. Motivated by the work [19] (related works are also done in [2],[5],[13],[14],[16],[17],[18]) and the conic domain defined by Kanas and Wiśniowska [7], one may construct a new class $k - \mathcal{MN}$ defined by

$$k - \mathcal{MN} = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in \mathbb{U} \right\}. \tag{9}$$

A function f is subordinate to the function g, written as $f \prec g$, provided that there is an analytic function $w(z)$ defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g[w(z)]$ for $z \in \mathbb{U}$. In particular, if the function g is univalent in $\mathbb{U}$ then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Denoting by p_k the conformal mapping that maps $\mathbb{U}$ onto Ω_k, we obtain the family of conformal mappings depending on $k\,(k \in [0, \infty))$. Therefore, for each fixed k, the family $k - \mathcal{UCV}$ is the family of all $f \in \mathcal{S}$ for which $1 + \dfrac{zf''(z)}{f'(z)} \prec p_k(z)$ and $k - \mathcal{ST}$ consists of all functions $f \in \mathcal{S}$ such that $\dfrac{zf'(z)}{f(z)} \prec p_k(z), z \in \mathbb{U}$.

The idea of subordination was used for defining many of classes of functions studied in geometric function theory. For obtaining the main result, we shall use the methods of differential subordinations. The theory of differential subordinations were introduced by Miller and Mocanu in [11] and [12].

The purpose of the present paper is to obtain interesting new results for the class $k-\mathcal{MN}$ by using the method of differential subordination to improve a recent result obtained in [19]. Finally, we compare our conclusions with the classical results in the univalent function theory.

2. Main Results

Let us start with the following theorem to prove that the class $k-\mathcal{MN}$ is non empty. In fact, the theorem will show that there are plenty of functions in the class $k-\mathcal{MN}$.

Theorem 2.1. *A function*

$$f(z)=\frac{z}{1-Az} \tag{10}$$

is in the class $k-\mathcal{MN}$ if and only if

$$|A|\leq\frac{1}{1+k},\ \ k\geq 0. \tag{11}$$

Proof. If f is given by (10), then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)=\Re\left(\frac{1+Az}{1-Az}\right) \tag{12}$$

and

$$k\left|\frac{zf'(z)}{f(z)}-1\right|=k\left|\frac{Az}{1-Az}\right|. \tag{13}$$

We know that $f\in k-\mathcal{MN}$ if and only if

$$k\left|\frac{Az}{1-Az}\right|<\Re\left(\frac{1+Az}{1-Az}\right). \tag{14}$$

It is suffices to study for $|z|=1$. Setting $|A|=r$ and $Az=re^{i\theta}$ in (14), then

$$k\left|\frac{re^{i\theta}}{1-re^{i\theta}}\right|<\Re\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right). \tag{15}$$

Following a computation,

$$\Re\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)=\frac{1-r^2}{|1-re^{i\theta}|^2}. \tag{16}$$

From (15) and (16), we get

$$\frac{kr}{|1-re^{i\theta}|}\leq\frac{1-r^2}{|1-re^{i\theta}|^2}. \tag{17}$$

On simplification, we easily get

$$kr\leq\frac{1-r^2}{[1-2r\cos\theta+r^2]^{\frac{1}{2}}}. \tag{18}$$

The right-hand side of (18) is seen to have a minimum for $\theta=\pi$ and this minimal value is $1-r$. Hence, a necessary and sufficient condition for (18) is $kr\leq 1-r$ or $|A|=r\leq\dfrac{1}{1+k}$. This completes the proof of Theorem 2.1. □

Corollary 2.1. *The function $f(z)=\dfrac{z}{1-z}$ belongs to the class $k-\mathcal{MN}$ if and only if $k=0$.*

The following proposition follows directly from the definition.

Proposition 1. *If $k_1\geq k_2$, then $k_1-\mathcal{MN}\subset k_2-\mathcal{MN}$.*

Next, we state a basic lemma and Theorems which are required prove our main results.

Lemma 2.1. [11] *Let h be an analytic function on $\overline{\mathbb{U}}$ except for at most one pole on $\partial\mathbb{U}$, and univalent on $\overline{\mathbb{U}}$, p be an analytic function in $\mathbb{U}$ with $p(0) = h(0)$ and $p(z) \neq p(0), z \in \mathbb{U}$. If p is not subordinate to h, then there exist points $z_0 \in \mathbb{U}, \zeta_0 \in \partial\mathbb{U}$ and $m \geq 1$ for which*

$$p(|z| < |z_0|) \subset h(\mathbb{U}),\ p(z_0) = h(\zeta_0),\ z_0 p'(z_0) = m\zeta_0 h'(\zeta_0).$$

Theorem 2.2. [8] *If $f \in \mathcal{S}^*(\alpha)$ for some $\alpha \in [1/2, 1]$, then*

$$\Re\left(\frac{f(z)}{z}\right) > \frac{1}{3 - 2\alpha}. \tag{19}$$

Theorem 2.3. [8] *If $\Re\left(\sqrt{f'(z)}\right) > \alpha$ for some $\alpha \in [1/2, 1]$, then*

$$\Re\left(\frac{f(z)}{z}\right) > \frac{2\alpha^2 + 1}{3}. \tag{20}$$

Theorem 2.4. *Let $k \in [0, \infty)$. Also, let p be an analytic function in the unit disk such that $p(0) = 1$. If*

$$\Re\left(p(z) + \frac{zp'(z)}{p(z)}\right) - k\,|p(z) - 1| > 0, \tag{21}$$

then

$$p(z) \prec \frac{1 + (1 - 2\alpha)z}{1 - z} = h(z), \tag{22}$$

where $\alpha \geq \alpha(k)$, and $\alpha(k)$ is given by

$$\alpha(k) = \frac{1}{4}\left[\sqrt{\left(\frac{1 - 2k}{1 + k}\right)^2 + \frac{8}{(1 + k)}} - \left(\frac{1 - 2k}{1 + k}\right)\right]. \tag{23}$$

Proof. We may assume that $\alpha \geq \frac{1}{2}$, since we have the condition, $\Re\left(p(z) + \frac{zp'(z)}{p(z)}\right) > 0$ implies that at least $\Re(p(z)) > \frac{1}{2}$. Suppose now, on the contrary, that $p \not\prec h$. Then by Lemma 2.1 of Miller and Mocanu [11], there exist a $z_0 \in \mathbb{U}$, $\zeta_0 \in \partial\mathbb{U}$, $\zeta_0 \neq 1$ and $m \geq 1$, such that

$$p(z_0) = \alpha + ix,\ \ z_0 p'(z_0) = my,\ \ \text{where}\ \ y \leq -\frac{(1 - \alpha)^2 + x^2}{2(1 - \alpha)},\ \ x, y \in \mathbb{R}.$$

Making use the above relations, we have

$$\begin{aligned}
&\Re\left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right) - k\,|p(z_0) - 1| \\
&= \Re\left(\alpha + ix + \frac{my}{\alpha + ix}\right) - k\,|\alpha - 1 + ix| \\
&= \alpha + \frac{\alpha m y}{\alpha^2 + x^2} - k\sqrt{(1 - \alpha)^2 + x^2} \\
&\leq \alpha - \frac{\alpha}{2(1 - \alpha)}\frac{(1 - \alpha)^2 + x^2}{\alpha^2 + x^2} - k\sqrt{(1 - \alpha)^2 + x^2} \\
&= r(x)
\end{aligned}$$

The function $r(x)$ is even as regards x. Now, we show that $r(x)$ attains its maximum at $x = 0$ when $\frac{1}{2} \leq \alpha < 1$. Clearly,

$$r'(x) = -x \left[\frac{\alpha(2\alpha - 1)}{(1-\alpha)(\alpha^2 + x^2)^2} + \frac{k}{\sqrt{(1-\alpha)^2 + x^2}} \right].$$

Equating $r'(x) = 0$, one can easily see that $r'(x) = 0$ if and only if $x = 0$. For $\alpha \geq \frac{1}{2}$, the expression $(2\alpha - 1)$ is nonnegative. Therefore, we have

$$r''(0) = \frac{-1}{\alpha(1-\alpha)} \left[(2\alpha - 1) + k\alpha \right] < 0.$$

Therefore, we have $r(x)$ has a maximum at $x = 0$. Hence,

$$r(x) \leq r(0) = \alpha - \frac{(1-\alpha)}{2\alpha} - k(1-\alpha) = 0, \tag{24}$$

for $\alpha = \alpha(k)$, as given by (23), which contradicts the assumption. This essentially completes the proof of Theorem 2.4. □

Applying Theorem 2.4 we may formulate the following:

Theorem 2.5. *If* $0 \leq k < \infty$, *then* $k - \mathcal{MN} \subset \mathcal{S}^*(\alpha)$, *where* $\mathcal{S}^*(\alpha)$ *is the class of starlike functions of order* α, $\alpha \geq \alpha(k)$ *and* $\alpha(k)$ *is given by* (23).

Proof. Let $f \in k - \mathcal{MN}$. Then, by (9)

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \mathbb{U}. \tag{25}$$

Setting $p(z) = \dfrac{zf'(z)}{f(z)}$, $p(0) = 1$, the above condition can be written as

$$\Re\left(p(z) + \frac{zp'(z)}{p(z)}\right) > k\,|p(z) - 1| \tag{26}$$

or

$$\Re\left(p(z) + \frac{zp'(z)}{p(z)}\right) - k\,|p(z) - 1| > 0. \tag{27}$$

Now applying Theorem 2.4, we obtain the assertion as stated in Theorem 2.5. □

In view of Theorem 2.2 and Theorem 2.3, we compare the classical results concerning the right half plane with these obtained for domains bounded by conic sections. This comparison shows that such a substitution provides "a step to the right". For instance classical results give

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \Rightarrow \Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2} \Rightarrow \Re\left(\frac{f(z)}{z}\right) > \frac{1}{2},$$

whereas, Theorem 2.2 and Theorem 2.4, gives us that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \Rightarrow \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \Rightarrow \Re\left(\frac{f(z)}{z}\right) > \eta,$$

That is,

$$f \in k - \mathcal{MN} \Rightarrow \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \Rightarrow \Re\left(\frac{f(z)}{z}\right) > \eta,$$

where $\alpha = \alpha(k) = \frac{1}{4}\left[\sqrt{\left(\frac{1-2k}{1+k}\right)^2 + \frac{8}{(1+k)}} - \left(\frac{1-2k}{1+k}\right)\right]$ and

$\eta = \frac{1}{3-2\alpha}$.

Setting $k = 1$, we obtain from the above the following:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \quad \Rightarrow \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{8}\left[\sqrt{17}+1\right] \approx 0.64$$
$$\Rightarrow \quad \Re\left(\frac{f(z)}{z}\right) > \frac{4}{11-\sqrt{17}} \approx 0.58.$$

That is,

$$f \in \mathcal{MN} \Rightarrow \Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{8}\left[\sqrt{17}+1\right] \Rightarrow \Re\left(\frac{f(z)}{z}\right) > \frac{4}{11-\sqrt{17}} \approx 0.58. \tag{28}$$

The above observation in (28) may stated as below.

Corollary 2.2. *If* $f \in \mathcal{MN}$ *then* $\mathcal{MN} \subset \mathcal{S}^*(\alpha_0)$, *where* $\alpha = 0.64038\cdots$.

It is to be remarked here that Kanas [8] has obtained a similar improvement of order of starlikeness for uniformly convex functions from $\frac{3}{4} = 0.75$ to 0.705 by applying the method of differential subordination techniques for domains involving conic section. Our corollary 2.2 is for the class $\mathcal{MN}$. Although it is a slight improvement, the technique adopted by us is different. Moreover, we have also given one more implication in (28) relating to the class $\mathcal{MN}$, **which is a slight improvement of the recent result obtained by Sokół and Nunokawa** [19] as the order of starlikeness is slightly increased.

Remark 2.1. Observe that in the case, $k = 0$, we recover the classical sharp result that each convex function is starlike at least of order $1/2$.

Furthermore, we know that,

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \Rightarrow \Re\left(\sqrt{f'(z)}\right) > \frac{1}{2} \Rightarrow \Re\left(\frac{f(z)}{z}\right) > \frac{1}{2},$$

whereas, Theorem 2.4 and Theorem 2.3, gives us

$$\Re\left(\sqrt{f'(z)} + \frac{zf''(z)}{2f'(z)}\right) > k\left|\sqrt{f'(z)} - 1\right| \Rightarrow \Re\left(\sqrt{f'(z)}\right) > \alpha \Rightarrow \Re\left(\frac{f(z)}{z}\right) > \delta,$$

where $\alpha = \alpha(k) = \frac{1}{4}\left[\sqrt{\left(\frac{1-2k}{1+k}\right)^2 + \frac{8}{(1+k)}} - \left(\frac{1-2k}{1+k}\right)\right]$ and $\delta = \frac{2\alpha^2+1}{3}$.

Setting $k = 1$, we obtain from the above the following:

$$\Re\left(\sqrt{f'(z)} + \frac{zf''(z)}{2f'(z)}\right) > \left|\sqrt{f'(z)} - 1\right| \quad \Rightarrow \quad \Re\left(\sqrt{f'(z)}\right) > \frac{1}{8}\left[\sqrt{17}+1\right] \approx 0.64$$
$$\Rightarrow \quad \Re\left(\frac{f(z)}{z}\right) > \frac{4}{11-\sqrt{17}} \approx 0.60.$$

For $k=0$, we get one more classical result as given below

$$\Re\left(\sqrt{f'(z)}+\frac{zf''(z)}{2f'(z)}\right)>0 \quad\Rightarrow\quad \Re\left(\sqrt{f'(z)}\right)>\frac{1}{2}\Rightarrow\Re\left(\frac{f(z)}{z}\right)>\frac{1}{2}.$$

Setting $p(z)=f(z)/zandp(z)=f'(z)$, Theorem 2.4 reduces to the following corollaries:

Corollary 2.3. *Let* $0\le k<\infty$*. Let* $f\in\mathcal{A}$ *is analytic in* $\mathbb{U}$*. Then*

$$\Re\left(\frac{zf'(z)}{f(z)}+\frac{f(z)}{z}-1\right)>k\left|\frac{f(z)}{z}-1\right|\Rightarrow\Re\left(\frac{f(z)}{z}\right)>\alpha(k). \tag{29}$$

Corollary 2.4. *Let* $0\le k<\infty$*. Let* $f\in\mathcal{A}$ *is analytic in* $\mathbb{U}$*. Then*

$$\Re\left(\frac{zf''(z)}{f'(z)}+f'(z)\right)>k\left|f'(z)-1\right|\Rightarrow\Re\left(f'(z)\right)>\alpha(k). \tag{30}$$

For $k=1$ and $p(z)=\dfrac{f(z)}{z}$ in Theorem 2.4, we obtain the following corollary

Corollary 2.5.

$$\Re\left(\frac{zf'(z)}{f(z)}+\frac{f(z)}{z}-1\right)>\left|\frac{f(z)}{z}-1\right|\Rightarrow\Re\left(\frac{f(z)}{z}\right)>\frac{1}{2}. \tag{31}$$

For $k=1, p(z)=f'(z)$ in Theorem 2.4, we obtain the following corollary

Corollary 2.6.

$$\Re\left(\frac{zf''(z)}{f'(z)}+f'(z)\right)>\left|f'(z)-1\right|\Rightarrow\Re\left(f'(z)\right)>\frac{1}{2}. \tag{32}$$

From (9) we obtain

$$\Re\left(p(z)+\frac{zp'(z)}{p(z)}\right)>k\left|p(z)-1\right|\ge k\ \Re\left(1-p(z)\right)\quad(z\in\mathbb{U}) \tag{33}$$

and hence

$$\Re\left(p(z)+\frac{zp'(z)}{(1+k)p(z)}\right)>\frac{k}{k+1}. \tag{34}$$

The above inequality is equivalent to the familiar Briot-Bouquet differential subordination of the form,

$$p(z)+\frac{zp'(z)}{(1+k)p(z)}\prec\frac{1+\left(\frac{1-k}{1+k}\right)z}{1-z}.$$

In view of Theorem 3.3d, ([11], p.109), we obtain

$$p(z)\prec q(z)=\frac{1}{2\int_0^1\left(\frac{1-k}{1-tz}\right)^{\frac{4}{1+k}}tdt} \tag{35}$$

$$=\left({}_2F_1\left(\frac{4}{1+k},1,3;\frac{z}{(z-1)}\right)\right)^{-1}=g_k(z)(say). \tag{36}$$

A simple computation yields

$$g_k(z)=(1-z)^{\left(\frac{4}{1+k}\right)}\left(1+\frac{8}{3(1+k)}z+\frac{(5+k)}{(1+k)^2}z^2+\frac{8(5+k)(3+k)}{15(1+k)^3}z^3+\cdots\right). \tag{37}$$

In order to prove the results involving coefficient inequalities, we need the following lemmas.

Lemma 2.2. [1] *If* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}$, *then for each* $k \geq 1$, $|p_k| \leq 2$, *and*

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}. \tag{38}$$

Lemma 2.3. [10] *If* $q(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is an analytic function with positive real part in* $\mathbb{U}$, *then*

$$|c_2 - v c_1^2| \leq 2 \max\{1; |2v - 1|\}. \tag{39}$$

In particular, if v *is a real number, then*

$$|c_2 - v c_1^2| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0, \\ 2 & \text{if } 0 \leq v \leq 1, \\ 4v - 2 & \text{if } v \geq 1. \end{cases}$$

When $v < 0$ *or* $v > 1$, *the equality in* (39) *holds true if and only if* $q(z) = \dfrac{1+z}{1-z}$ *or one of its rotations. If* $0 < v < 1$, *then the equality holds true if and only if* $q(z) = \dfrac{1+z^2}{1-z^2}$ *or one of its rotations. If* $v = 0$, *then the equality holds true if and only if*

$$q(z) = \left(\frac{1}{2} + \frac{\lambda}{2}\right) \frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right) \frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations, while for $v = 1$, *equality holds if and only if* $q(z)$ *is a reciprocal of one of the functions such that the equality holds true in the case when* $v = 0$.

Theorem 2.6. *If* $f(z) \in k - \mathcal{MN}$, *then we have*

$$\frac{zf'(z)}{f(z)} \prec q_k(z) = \frac{1}{g_k(z)} \qquad (z \in \mathbb{U}), \tag{40}$$

where $g_k(z)$ *is as given by (37).*

Theorem 2.7. *Let* $0 \leq k < \infty$ *and let* f, *given by* (1), *be in the class* $k - \mathcal{MN}$. *Then*

$$|a_2| \leq \frac{4}{3(1+k)} \tag{41}$$

and

$$|a_3| \leq \begin{cases} \dfrac{29 + 9k}{18(1+k)^2} & \text{for } k < \frac{17}{3}, \\ \dfrac{2}{3(1+k)} & \text{for } k \geq \frac{17}{3}. \end{cases} \tag{42}$$

For any complex number μ

$$|a_3 - \mu a_2^2| \leq \frac{2}{3(1+k)} \max\left\{1; \left| \frac{13 + 9k}{12(1+k)} - \frac{4}{3(1+k)} + \frac{8\mu}{3(1+k)} \right| \right\}. \tag{43}$$

Proof. If $f \in k - \mathcal{MN}$, then exist a Schwarz function $\omega(z)$ is analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$, then

$$\frac{zf'(z)}{f(z)} = q_k(\omega(z)). \tag{44}$$

Define the function $\phi_1(z)$ is given by

$$\phi_1(z) := \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1 z + c_2 z^2 + \ldots. \tag{45}$$

Since $\omega(z)$ is a Schwarz function, we see that $\Re(\phi_1(z)) > 0$ and $\phi_1(0) = 1$. Define the function $p(z)$ is given by

$$p(z) := \frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3a_4)z^3 + \ldots \tag{46}$$

In view of above equations (44), (45) and (46), we have

$$p(z) = q_k\left(\frac{\phi_1(z) - 1}{\phi_1(z) + 1}\right). \tag{47}$$

Since

$$\frac{\phi_1(z) - 1}{\phi_1(z) + 1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + (c_3 + \frac{c_1^3}{4} - c_1c_2)z^3 + \cdots\right].$$

Therefore,

$$q_k\left(\frac{\phi_1(z) - 1}{\phi_1(z) + 1}\right) =$$

$$1 + \frac{2}{3(1+k)}c_1 z + \left[\frac{2}{3(1+k)}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{13+9k}{36(1+k)^2}c_1^2\right]z^2 + \cdots. \tag{48}$$

From (46) and (47), we get

$$a_2 = \frac{2}{3(1+k)}c_1, \tag{49}$$

and

$$2a_3 - a_2^2 = \frac{2}{3(1+k)}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{13+9k}{36(1+k)^2}c_1^2. \tag{50}$$

It is easily get

$$a_3 = \frac{1}{2}\left[\frac{2}{3(1+k)}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{13+9k}{36(1+k)^2}c_1^2 + \frac{4}{9(1+k)^2}c_1^2\right]. \tag{51}$$

Now applying the coefficient estimate $|c_k| \leq 2$ for $k = 1, 2, \ldots$, it is easily to get by Lemma 2.3,

$$|a_2| \leq \frac{2}{3(1+k)}|c_1| = \frac{4}{3(1+k)} \tag{52}$$

and

$$|a_3| \leq \frac{1}{2}\left[\frac{2}{3(1+k)}\left(2 - \frac{|c_1|^2}{2}\right) + \frac{13+9k}{36(1+k)^2}|c_1|^2 + \frac{4}{9(1+k)^2}|c_1|^2\right]. \tag{53}$$

That is

$$|a_3| \leq \frac{2}{3(1+k)} + \left[\frac{13+9k}{72(1+k)^2} + \frac{2}{9(1+k)^2} - \frac{1}{6(1+k)}\right]|c_1|^2. \tag{54}$$

This gives the bound for $|a_3|$ is given by (42).

$$a_3 - \mu a_2^2 = \frac{1}{3(1+k)}\left[c_2 - \sigma c_1^2\right]. \tag{55}$$

where

$$\sigma = \frac{1}{2}\left[1 + \frac{8\mu}{3(1+k)} - \frac{4}{3(1+k)} - \frac{13+9k}{12(1+k)}\right]. \tag{56}$$

This completes proof of the Theorem 2.7 by Lemma 2.3 . □

From Theorem 2.7, we get the following corollary.

Corollary 2.7. *Let $0 \le k < \infty$ and let f, given by (1), be in the class $k-\mathcal{MN}$. Then, for any real number μ*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{29+9k}{18(1+k)^2} - \dfrac{16\mu}{9(1+k)^2} & \text{if } \mu \le \sigma_1, \\ \dfrac{2}{3(1+k)} & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ -\dfrac{29+9k}{18(1+k)^2} + \dfrac{16\mu}{9(1+k)^2} & \text{if } \mu \ge \sigma_2, \end{cases} \tag{57}$$

where

$$\sigma_1 = \frac{9(1+k)}{32}\left(\frac{16}{9(1+k)} - \frac{4}{3} + \frac{13+9k}{9(1+k)}\right),$$
$$\sigma_2 = \frac{9(1+k)}{32}\left(\frac{16}{9(1+k)} + \frac{4}{3} + \frac{13+9k}{9(1+k)}\right).$$

The result is sharp.

Theorem 2.8. *Let $0 \le k < \infty$. Also, let $f(z) \in k-\mathcal{MN}$. Then $f(\mathbb{U})$ contains an open disk of radius*

$$\frac{3(1+k)}{10+6k}. \tag{58}$$

Proof. Let $\omega_0 \ne 0$ be a complex number such that $f(z) \ne \omega_0$ for $z \in \mathbb{U}$. Then

$$f_1(z) = \frac{\omega_0 f(z)}{\omega_0 - f(z)} = z + \left(a_2 + \frac{1}{\omega_0}\right) z^2 + \cdots . \tag{59}$$

Since f_1 is univalent in $\mathbb{U}$, so that

$$\left|a_2 + \frac{1}{\omega_0}\right| \le 2. \tag{60}$$

Now using Theorem 2.7, we have

$$\left|\frac{1}{\omega_0}\right| \le 2 + \frac{4}{3(1+k)}. \tag{61}$$

and hence

$$|\omega_0| \ge \frac{3(1+k)}{10+6k}. \tag{62}$$

□

Since, $f \in \mathcal{S}$, the inverse of $f(z)$ has a Maclaurin expansion in a disk of radius at least $\frac{1}{4}$, say

$$F(\omega) = f^{-1}(\omega) = \omega + d_2\omega^2 + d_3\omega^3 + \cdots .$$

In an earlier investigation, Libera and Złotkiewicz [9] obtained few earlier coefficients of the inverse of a regular convex function f. Now, we obtain the first two early coefficient estimates of the inverse when $f \in k-\mathcal{MN}$.

Theorem 2.9. *Let $f \in k-\mathcal{MN}$ and*

$$F(\omega) = f^{-1}(\omega) = \omega + d_2\omega^2 + d_3\omega^3 + \cdots .$$

Then

$$|d_2| \le \frac{4}{3(1+k)} \tag{63}$$

and

$$|d_3| \leq \frac{27+15k}{18(1+k)^2}. \tag{64}$$

Proof. As $F(f(z)) = z$, we have,

$$d_2 = -a_2, \quad d_3 = 2a_2^2 - a_3. \tag{65}$$

From (49) and (51), we have

$$d_2 = -\frac{2}{3(1+k)}c_1 \tag{66}$$

and

$$\frac{4}{9(1+k)^2}c_1^2 - \frac{1}{6}\left[\frac{2}{(1+k)}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{13+9k}{12(1+k)^2}c_1^2 + \frac{4}{3(1+k)^2}c_1^2\right]. \tag{67}$$

$$= \frac{1}{3(1+k)}c_2 + \left[\frac{2}{9(1+k)^2} - \frac{13+9k}{72(1+k)^2} + \frac{1}{6(1+k)}\right]c_1^2. \tag{68}$$

Now applying Lemma 2.3, the proof of Theorem 2.9 is completed. □

Concluding remarks and observations

Very similar to the interesting classes $\mathcal{MN}$ defined by Sokół and Nunokawa [19], and the class $k - \mathcal{MN}$ defined in this paper by the authors, one might think of the emerging classes that appear in (29),(30), (31)and (32) .

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