U.P.B. Sci. Bull., Series A, Vol. 77, Iss. 4, 2015 ISSN 1223-7027

THE NUMBER OF CHAINS OF SUBGROUPS OF A FINITE ELEMENTARY ABELIAN p-GROUP

Marius Tărnăuceanu[1]

In this short note we give a formula for the number of chains of subgroups of a finite elementary abelian p-group. This completes our previous work [5].

Keywords: chains of subgroups, fuzzy subgroups, finite elementary abelian p-groups, recurrence relations.

MSC2010: Primary 20N25, 03E72; Secondary 20K01, 20D30.

1. Introduction

Let G be a group. A *chain of subgroups* of G is a set of subgroups of G totally ordered by set inclusion. A chain of subgroups of G is called *rooted* (more exactly *G-rooted*) if it contains G; otherwise, it is called *unrooted*. A *fuzzy subgroup* of G is a fuzzy subset $\mu : G \longrightarrow [0,1]$ satisfying the following two conditions:

a) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in G$;
b) $\mu(x^{-1}) \geq \mu(x)$, for any $x \in G$.

The fuzzy subgroups of G can be classified up to some natural equivalence relations on the set of all fuzzy subsets of G. One of them is defined by

$$\mu \sim \eta \text{ iff } (\mu(x) > \mu(y) \Longleftrightarrow \eta(x) > \eta(y), \text{ for all } x, y \in G),$$

and two fuzzy subgroups μ, η of G are said to be *distinct* if $\mu \not\sim \eta$. Notice that there is a bijection between the set of *G-rooted* chains of subgroups of G and the set of distinct fuzzy subgroups of G (see e.g. [5]), which is used to solve many computational problems in fuzzy group theory.

The starting point for our discussion is given by the paper [5], where a formula for the number of rooted chains of subgroups of a finite cyclic group is obtained. This leads in [3] to precise expression of the well-known central Delannoy numbers in an arbitrary dimension and has been simplified in [2]. Some steps in order to determine the number of rooted chains of subgroups of a finite elementary abelian p-group are also made in [5]. Moreover, this counting problem has been naturally extended to non-abelian groups in other works, such as [1, 4]. The purpose of the current note is to improve the results of [5], by indicating an explicit formula for the number of rooted chains of subgroups of a finite elementary abelian p-group.

[1]Associate Professor, Faculty of Mathematics, Al. I. Cuza University of Iaşi, Romania, e-mail: tarnauc@uaic.ro

Given a finite group G, we will denote by $\mathcal{C}(G)$, $\mathcal{D}(G)$ and $\mathcal{F}(G)$ the collection of all chains of subgroups of G, of unrooted chains of subgroups of G and of G-rooted chains of subgroups of G, respectively. Put $C(G) = |\mathcal{C}(G)|$, $D(G) = |\mathcal{D}(G)|$ and $F(G) = |\mathcal{F}(G)|$. The connections between these numbers have been established in [2], namely:

Theorem 1. *Let G be a finite group. Then*

$$F(G) = D(G) + 1 \text{ and } C(G) = F(G) + D(G) = 2F(G) - 1\,.$$

In the following let p be a prime, n be a positive integer and $\mathbb{Z}_p^n$ be an elementary abelian p-group of rank n (that is, a direct product of n copies of $\mathbb{Z}_p$). First of all, we recall a well-known group theoretical result that gives the number $a_{n,p}(k)$ of subgroups of order p^k in $\mathbb{Z}_p^n$, $k = 0, 1, ..., n$.

Theorem 2. *For every $k = 0, 1, ..., n$, we have*

$$a_{n,p}(k) = \frac{(p^n - 1)\cdots(p-1)}{(p^k-1)\cdots(p-1)(p^{n-k}-1)\cdots(p-1)}\,.$$

Our main result is the following.

Theorem 3. *The number of rooted chains of subgroups of the elementary abelian p-group $\mathbb{Z}_p^n$ is*

$$F(\mathbb{Z}_p^n) = 2+2f(n)\sum_{k=1}^{n-1}\sum_{1\le i_1<i_2<...<i_k\le n-1}\frac{1}{f(n-i_k)f(i_k-i_{k-1})\cdots f(i_2-i_1)f(i_1)}\,,$$

where $f : \mathbb{N} \longrightarrow \mathbb{N}$ is the function defined by $f(0) = 1$ and $f(r) = \prod_{s=1}^{r}(p^s - 1)$ for all $r \in \mathbb{N}^*$.

Obviously, explicit formulas for $C(\mathbb{Z}_p^n)$ and $D(\mathbb{Z}_p^n)$ also follow from Theorems 1 and 2. By using a computer algebra program, we are now able to calculate the first terms of the chain $f_n = F(\mathbb{Z}_p^n)$, $n \in \mathbb{N}$, namely:

- $f_0 = 1$;
- $f_1 = 2$;
- $f_2 = 2p + 4$;
- $f_3 = 2p^3 + 8p^2 + 8p + 8$;
- $f_4 = 2p^6 + 12p^5 + 24p^4 + 36p^3 + 36p^2 + 24p + 16$.

Finally, we remark that the above f_3 is in fact the number $a_{3,p}$ obtained by a direct computation in Corollary 10 of [5].

2. **Proof of Theorem 3**

We observe first that every rooted chain of subgroups of $\mathbb{Z}_p^n$ are of one of the following types:

$$G_1 \subset G_2 \subset \ldots \subset G_m = \mathbb{Z}_p^n \text{ with } G_1 \neq 1 \tag{1}$$

and

$$1 \subset G_2 \subset \ldots \subset G_m = \mathbb{Z}_p^n\,. \tag{2}$$

It is clear that the numbers of chains of types (1) and (2) are equal. So

$$f_n = 2x_n\,, \tag{3}$$

where x_n denotes the number of chains of type (2). On the other hand, such a chain is obtained by adding $\mathbb{Z}_p^n$ to the chain

$$1 \subset G_2 \subset \ldots \subset G_{m-1},$$

where G_{m-1} runs over all subgroups of $\mathbb{Z}_p^n$. Moreover, G_{m-1} is also an elementary abelian p-group, say $G_{m-1} \cong \mathbb{Z}_p^k$ with $0 \le k \le n$. These show that the chain x_n, $n \in \mathbb{N}$, satisfies the following recurrence relation

$$x_n = \sum_{k=0}^{n-1} a_{n,p}(k) x_k\,, \tag{4}$$

which is more facile than the recurrence relation founded by applying the Inclusion-Exclusion Principle in Theorem 9 of [5].

Next we prove that the solution of (4) is given by

$$x_n = 1 + \sum_{k=1}^{n-1} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n-1} a_{n,p}(i_k) a_{i_k,p}(i_{k-1}) \cdots a_{i_2,p}(i_1)\,. \tag{5}$$

We will proceed by induction on n. Clearly, (5) is trivial for $n = 1$. Assume that it holds for all $k < n$. One obtains

$$x_n = \sum_{k=0}^{n-1} a_{n,p}(k) x_k = 1 + \sum_{k=1}^{n-1} a_{n,p}(k) x_k$$

$$= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) \left(1 + \sum_{r=1}^{k-1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le k-1} a_{k,p}(i_r) a_{i_r,p}(i_{r-1}) \cdots a_{i_2,p}(i_1) \right)$$

$$= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) + \sum_{k=1}^{n-1} a_{n,p}(k) \sum_{r=1}^{k-1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le k-1} a_{k,p}(i_r) a_{i_r,p}(i_{r-1}) \cdots a_{i_2,p}(i_1)$$

$$= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) + \sum_{k=1}^{n-1} a_{n,p}(k) \sum_{r=1}^{n-2} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le k-1} a_{k,p}(i_r) a_{i_r,p}(i_{r-1}) \cdots a_{i_2,p}(i_1)$$

$$= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) + \sum_{k=1}^{n-1} a_{n,p}(k) \sum_{r=2}^{n-1} \sum_{1 \le i_1 < i_2 < \ldots < i_{r-1} \le k-1} a_{k,p}(i_{r-1}) a_{i_{r-1},p}(i_{r-2}) \cdots a_{i_2,p}(i_1)$$

$$= 1+ \sum_{1\le i_1\le n-1} a_{n,p}(i_1) + \sum_{r=2}^{n-1}\sum_{k=1}^{n-1} a_{n,p}(k) \sum_{1\le i_1<i_2<\ldots<i_{r-1}\le k-1} a_{k,p}(i_{r-1})a_{i_{r-1},p}(i_{r-2})\cdots a_{i_2,p}(i_1)$$

$$= 1+ \sum_{1\le i_1\le n-1} a_{n,p}(i_1) + \sum_{r=2}^{n-1} \sum_{1\le i_1<i_2<\ldots<i_r\le n-1} a_{n,p}(i_r)a_{i_r,p}(i_{r-1})\cdots a_{i_2,p}(i_1)$$

$$= 1 + \sum_{r=1}^{n-1} \sum_{1\le i_1<i_2<\ldots<i_r\le n-1} a_{n,p}(i_r)a_{i_r,p}(i_{r-1})\cdots a_{i_2,p}(i_1)\,,$$

as desired.

Since by Theorem 2

$$a_{n,p}(k) = \frac{(p^n-1)\cdots(p-1)}{(p^k-1)\cdots(p-1)(p^{n-k}-1)\cdots(p-1)} = \frac{f(n)}{f(k)f(n-k)}\,, \forall\, 0\le k\le n\,,$$

the equalities (3) and (5) imply that

$$f_n = 2 + 2f(n)\sum_{k=1}^{n-1} \sum_{1\le i_1<i_2<\ldots<i_k\le n-1} \frac{1}{f(n-i_k)f(i_k-i_{k-1})\cdots f(i_2-i_1)f(i_1)}\,,$$

completing the proof. □

Acknowledgements. The author is grateful to the reviewer for its remarks which improve the previous version of the paper.

REFERENCES

[1] *B. Davvaz and R.K. Ardekani*, Counting fuzzy subgroups of a special class of non-abelian groups of order p^3, Ars Combin. **103** (2012), 175-179.

[2] *J.M. Oh*, The number of chains of subgroups of a finite cyclic group, European J. Combin. **33** (2012), 259-266.

[3] *M. Tărnăuceanu*, The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers, European J. Combin. **30** (2009), 283-287, doi: 10.1016/j.ejc.2007.12.005.

[4] *M. Tărnăuceanu*, Classifying fuzzy subgroups of finite nonabelian groups, Iran. J. Fuzzy Syst. **9** (2012), 33-43.

[5] *M. Tărnăuceanu and L. Bentea*, On the number of fuzzy subgroups of finite abelian groups, Fuzzy Sets and Systems **159** (2008), 1084-1096, doi: 10.1016/j.fss.2007.11.014.