THE NUMBER OF CHAINS OF SUBGROUPS OF A FINITE ELEMENTARY ABELIAN \( p \)-GROUP

Marius Tărnăuceanu

In this short note we give a formula for the number of chains of subgroups of a finite elementary abelian \( p \)-group. This completes our previous work [5].

Keywords: chains of subgroups, fuzzy subgroups, finite elementary abelian \( p \)-groups, recurrence relations.

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1. Introduction

Let \( G \) be a group. A chain of subgroups of \( G \) is a set of subgroups of \( G \) totally ordered by set inclusion. A chain of subgroups of \( G \) is called rooted (more exactly \( G \)-rooted) if it contains \( G \); otherwise, it is called unrooted. A fuzzy subgroup of \( G \) is a fuzzy subset \( \mu : G \rightarrow [0, 1] \) satisfying the following two conditions:

a) \( \mu(xy) \geq \min\{\mu(x), \mu(y)\} \), for all \( x, y \in G \);

b) \( \mu(x^{-1}) \geq \mu(x) \), for any \( x \in G \).

The fuzzy subgroups of \( G \) can be classified up to some natural equivalence relations on the set of all fuzzy subsets of \( G \). One of them is defined by

\[ \mu \sim \eta \quad \text{iff} \quad (\mu(x) > \mu(y) \iff \eta(x) > \eta(y)) \quad \text{for all} \quad x, y \in G, \]

and two fuzzy subgroups \( \mu, \eta \) of \( G \) are said to be distinct if \( \mu \not\sim \eta \). Notice that there is a bijection between the set of \( G \)-rooted chains of subgroups of \( G \) and the set of distinct fuzzy subgroups of \( G \) (see e.g. [5]), which is used to solve many computational problems in fuzzy group theory.

The starting point for our discussion is given by the paper [5], where a formula for the number of rooted chains of subgroups of a finite cyclic group is obtained. This leads in [3] to precise expression of the well-known central Delannoy numbers in an arbitrary dimension and has been simplified in [2]. Some steps in order to determine the number of rooted chains of subgroups of a finite elementary abelian \( p \)-group are also made in [5]. Moreover, this counting problem has been naturally extended to non-abelian groups in other works, such as [1, 4]. The purpose of the current note is to improve the results of [5], by indicating an explicit formula for the number of rooted chains of subgroups of a finite elementary abelian \( p \)-group.

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\(^1\)Associate Professor, Faculty of Mathematics, Al. I. Cuza University of Iaşi, Romania, e-mail: tarnauc@uaic.ro
Given a finite group $G$, we will denote by $C(G)$, $D(G)$ and $F(G)$ the collection of all chains of subgroups of $G$, of unrooted chains of subgroups of $G$ and of $G$-rooted chains of subgroups of $G$, respectively. Put $C(G) = |C(G)|$, $D(G) = |D(G)|$ and $F(G) = |F(G)|$. The connections between these numbers have been established in [2], namely:

**Theorem 1.** Let $G$ be a finite group. Then

$$F(G) = D(G) + 1 \quad \text{and} \quad C(G) = F(G) + D(G) = 2F(G) - 1.$$ 

In the following let $p$ be a prime, $n$ be a positive integer and $\mathbb{Z}_p^n$ be an elementary abelian $p$-group of rank $n$ (that is, a direct product of $n$ copies of $\mathbb{Z}_p$). First of all, we recall a well-known group theoretical result that gives the number $a_{n,p}(k)$ of subgroups of order $p^k$ in $\mathbb{Z}_p^n$, $k = 0, 1, \ldots, n$.

**Theorem 2.** For every $k = 0, 1, \ldots, n$, we have

$$a_{n,p}(k) = \frac{(p^n - 1) \cdots (p - 1)}{(p^k - 1) \cdots (p - 1)(p^{n-k} - 1) \cdots (p - 1)}.$$ 

Our main result is the following.

**Theorem 3.** The number of rooted chains of subgroups of the elementary abelian $p$-group $\mathbb{Z}_p^n$ is

$$F(\mathbb{Z}_p^n) = 2 + 2f(n) \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n-1} \frac{1}{f(n-i_k)f(i_k-i_{k-1}) \cdots f(i_2-i_1)f(i_1)},$$

where $f : \mathbb{N} \to \mathbb{N}$ is the function defined by $f(0) = 1$ and $f(r) = \prod_{s=1}^{r} (p^s - 1)$ for all $r \in \mathbb{N}^*$.Obviously, explicit formulas for $C(\mathbb{Z}_p^n)$ and $D(\mathbb{Z}_p^n)$ also follow from Theorems 1 and 2. By using a computer algebra program, we are now able to calculate the first terms of the chain $f_n = F(\mathbb{Z}_p^n)$, $n \in \mathbb{N}$, namely:

- $f_0 = 1$;
- $f_1 = 2$;
- $f_2 = 2p + 4$;
- $f_3 = 2p^3 + 8p^2 + 8p + 8$;

Finally, we remark that the above $f_3$ is in fact the number $a_{3,p}$ obtained by a direct computation in Corollary 10 of [5].
2. Proof of Theorem 3

We observe first that every rooted chain of subgroups of \( \mathbb{Z}_p^n \) are of one of the following types:

(i) \( G_1 \subset G_2 \subset \ldots \subset G_m = \mathbb{Z}_p^n \) with \( G_1 \neq 1 \)

and

(ii) \( 1 \subset G_2 \subset \ldots \subset G_m = \mathbb{Z}_p^n \).

It is clear that the numbers of chains of types (1) and (2) are equal. So

\[
\begin{align*}
  f_n &= 2x_n, \\
  \text{where } x_n \text{ denotes the number of chains of type (2). On the other hand, such a chain is obtained by adding } \mathbb{Z}_p^n \text{ to the chain } \\
  1 &\subset G_2 \subset \ldots \subset G_{m-1},
\end{align*}
\]

where \( G_{m-1} \) runs over all subgroups of \( \mathbb{Z}_p^n \). Moreover, \( G_{m-1} \) is also an elementary abelian \( p \)-group, say \( G_{m-1} \cong \mathbb{Z}_p^k \) with \( 0 \leq k \leq n \). These show that the chain \( x_n \), \( n \in \mathbb{N} \), satisfies the following recurrence relation

\[
x_n = \sum_{k=0}^{n-1} a_{n,p}(k)x_k,
\]

which is more facile than the recurrence relation founded by applying the Inclusion-Exclusion Principle in Theorem 9 of [5].

Next we prove that the solution of (4) is given by

\[
x_n = 1 + \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n-1} a_{n,p}(i_k) a_{i_{k-1},p}(i_{k-1}) \cdots a_{i_2,p}(i_1).
\]

We will proceed by induction on \( n \). Clearly, (5) is trivial for \( n = 1 \). Assume that it holds for all \( k < n \). One obtains

\[
x_n = \sum_{k=0}^{n-1} a_{n,p}(k)x_k = 1 + \sum_{k=1}^{n-1} a_{n,p}(k)x_k
\]

\[
= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) \left( 1 + \sum_{r=1}^{k-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k-1} a_{k,p}(i_r) a_{i_{r-1},p}(i_{r-1}) \cdots a_{i_2,p}(i_1) \right)
\]

\[
= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) + \sum_{k=1}^{n-1} a_{n,p}(k) \sum_{r=1}^{k-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k-1} a_{k,p}(i_r) a_{i_{r-1},p}(i_{r-1}) \cdots a_{i_2,p}(i_1)
\]

\[
= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) + \sum_{k=1}^{n-1} a_{n,p}(k) \sum_{r=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n-1} a_{k,p}(i_r) a_{i_{r-1},p}(i_{r-1}) \cdots a_{i_2,p}(i_1)
\]

\[
= 1 + \sum_{k=1}^{n-1} a_{n,p}(k) + \sum_{k=1}^{n-1} a_{n,p}(k) \sum_{r=2}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-1} \leq k-1} a_{k,p}(i_r) a_{i_{r-2},p}(i_{r-2}) \cdots a_{i_2,p}(i_1)
\]
= 1 + \sum_{1 \leq i_1 \leq n-1} a_{n,p}(i_1) + \sum_{r=2}^{n-1} \sum_{k=1}^{n-1} a_{n,p}(k) \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-1} \leq k-1} a_{k,p}(i_{r-1})a_{i_{r-1},p}(i_{r-2}) \cdots a_{i_2,p}(i_1) \\
= 1 + \sum_{1 \leq i_1 \leq n-1} a_{n,p}(i_1) + \sum_{r=2}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-1} \leq n-1} a_{n,p}(i_r)a_{i_r,p}(i_{r-1}) \cdots a_{i_2,p}(i_1) \\
= 1 + \sum_{r=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n-1} a_{n,p}(i_r)a_{i_r,p}(i_{r-1}) \cdots a_{i_2,p}(i_1),
\text{as desired.}

Since by Theorem 2
\[ a_{n,p}(k) = \frac{(p^n - 1) \cdots (p - 1)}{(p^k - 1) \cdots (p - 1)(p^{n-k} - 1) \cdots (p - 1)} = \frac{f(n)}{f(k)f(n-k)}, \forall 0 \leq k \leq n, \]
the equalities (3) and (5) imply that
\[ f_n = 2 + 2f(n) \sum_{k=1}^{n-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n-1} \frac{1}{f(n-i_k)f(i_k-i_{k-1}) \cdots f(i_2-i_1)f(i_1)}, \]
completing the proof. □

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**REFERENCES**


