ARENS REGULARITY OF QUASI-MULTIPLIERS

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Assume that A is a Banach algebra and $m : A \times A \to A$ is a quasi-multiplier on A. In this paper, we will study the relation between Arens regularities of m and A. Also, we define the quasi-multipliers on the dual of a Banach algebra A and we give a simple criterion for Arens regularity of a bounded quasi-multiplier of A^*. Also, we investigate those conditions under which the space of all quasi-multipliers of A^* is Arens regular.

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1. Introduction

The notion of a quasi-multiplier is a generalization of the notion of a multiplier on a Banach algebra and was introduced by Akemann and Pedersen [3] for C^*-algebras. McKennon [12] extended the definition to a general complex Banach algebra A with a bounded approximate identity (b.a.i., for brevity) as follows. A bilinear mapping $m: A \times A \to A$ is a quasi-multiplier on A if

$$m(ab, cd) = a\, m(b, c)\, d \qquad (a, b, c, d \in A).$$

Let $QM(A)$ denote the set of all separately continuous quasi-multipliers on A. It is showed in [12] that $QM(A)$ is a Banach space for the norm $\|m\| = \sup\{\|m(a, b)\|; a, b \in A, \|a\| = \|b\| = 1\}$.

Arens regularity of bilinear mappings have been extensively studied by many authors for example [4] , [5], [8], [13],...

In [2] we extended the notion of quasi-multipliers to the dual of a Banach algebra A whose second dual has a mixed identity. We considered algebras satisfying a weaker condition than Arens regularity.

In [1] we defined extended left (right) quasi-multipliers on the dual of a Banach algebra. We established some properties of $QM_{el}(A^*)$ of all bounded extended left quasi-multipliers of A^*. In particular, we characterized the $\gamma-$dual of $QM_{el}(A^*)$ and proved that $(QM_{el}(A^*), \gamma)^*$ under the topology of bounded convergence, is isomorphic to A^{***}.

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The aim of this paper is to present a few new statements on Arens regularity of quasi-multipliers.

Before we state our main results the basic notation is introduced. We mainly adopt the notation from the monograph [7]. The reader is referred to this book for some results used in this paper, as well.

For a Banach space X, let X^* be its topological dual. The pairing between X and X^* is denoted by $\langle\cdot,\cdot\rangle$. We always consider X naturally embedded into X^{**} through the mapping π, which is given by $\langle\pi(x),\xi\rangle = \langle\xi, x\rangle$ $(x \in X,\ \xi \in X^*)$.

Let A be a Banach algebra. It is well known that on the second dual A^{**} there are two algebra multiplications called the first and the second Arens product, respectively. Since in the paper we use mainly the first Arens product, we recall its definition. Let $a \in A$, $\xi \in A^*$, and $F,\ G \in A^{**}$ be arbitrary. Then one defines $\xi\cdot a$ and $G\cdot\xi$ as $\langle\xi\cdot a, b\rangle = \langle\xi, ab\rangle$ and $\langle G\cdot\xi, b\rangle = \langle G, \xi\cdot b\rangle$, where $b \in A$ is arbitrary. Now, the first Arens product of F and G is an element $F\circ G$ in A^{**} which is given by $\langle F\circ G,\xi\rangle = \langle F, G\cdot\xi\rangle$, where $\xi \in A^*$ is arbitrary. The second Arens product, which we donete by $\circ'$, is defined in a similar way.

Space A^{**} equipped with the first (or second) Arens product is a Banach algebra and A is a subalgebra of it. It is said that A is Arens regular if the equality $F\circ G = F\circ' G$ holds for all $F,\ G \in A^{**}$. For example, every C^*-algebra is Arens regular, see [6]. Note however, that $F\circ a = F\circ' a$ and $a\circ F = a\circ' F$ hold for any $a \in A$ and $F \in A^{**}$. An element E in the second dual A^{**} is said to be a mixed identity if it is a right identity for the first and a left identity for the second Arens product. Note that A^{**} has a mixed identity if and only if A has a b.a.i. By [7, Proposition 2.6.21], an element $E \in A^{**}$ is a mixed identity if and only if $E\cdot\xi = \xi = \xi\cdot E$, for every $\xi \in A^*$.

Let X, Y, Z be normed spaces. A bilinear mapping $m\ : X\times Y \to Z$ can be extended in a natural way to a bilinear map $X^{**}\times Y^{**} \to Z^{**}$; we outline the construction in stages, as follows:

$$\begin{aligned} m^* &: Z^*\times X \to Y^*, && \langle m^*(\xi,x),y\rangle = \langle\xi, m(x,y)\rangle,\\ m^{**} &: Y^{**}\times Z^* \to X^*, && \langle m^{**}(F,\xi),x\rangle = \langle F, m^*(\xi,x)\rangle,\\ m^{***} &: X^{**}\times Y^{**} \to Z^{**}, && \langle m^{***}(G,F),\xi\rangle = \langle G, m^{**}(F,\xi)\rangle. \end{aligned}$$

$(x \in X, y \in Y, \xi \in Z^*, G \in X^{**}, F \in Y^{**})$. The mapping m^{***} is the unique extention of m such that $G \to m^{***}(G,F)$ from X^{**} into Z^{**} is $weak^* - weak^*$ continuous for every $F \in Y^{**}$, but the mapping $F \to m^{***}(G,F)$ is not in general $weak^* - weak^*$ continuous from Y^{**} into Z^{**} unless $G \in X$. Hence the first topological center of m may be defined as following

$$Z_1(m) = \{G \in X^{**} : F \to m^{***}(G,F)\ is\ weak^* - weak^*\ continuous\}.$$

Let now $m^t : Y\times X \to Z$ be the transpose of m defined by $m^t(y,x) = m(x,y)$ for every $x \in X$ and $y \in Y$. Then m^t is a bilinear map from $Y\times X \to Z$ and so it may be extended as above to $m^{t***} : Y^{**}\times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**}\times Y^{**} \to Z^{**}$ in general is not equal to m^{***}. If $m^{***} = m^{t***t}$, then m is called Arens regular.

The mapping $F \to m^{t***t}(G,F)$ is $weak^* - weak^*$ continuous for every $F \in Y^{**}$, but the mapping $G \to m^{t***t}(G,F)$ from X^{**} into Z^{**} is not in general $weak^* - weak^*$ continuous for every $F \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{F \in Y^{**} : G \to m^{t***t}(G,F) \text{ is weak}^* - \text{weak}^* \text{ continuous}\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ and $Z_2(m) = Y^{**}$. Also, m is called strongly Arens irregular if and only if $Z_1(m) = X$ and $Z_2(m) = Y$. It is worthwile mentioning that in the case where π is the multiplication of a Banach algebra A, then π^{***} and π^{t***t} are actually the first and second Arens products which will be denote by $\circ$ and $\circ'$, respectively. We also say A is Arens regular if the multiplication π of A is Arens regular. This means that if we define

$$Z_1(A^{**}) = \{G \in A^{**} : F \to G \circ F \text{ is weak}^* - \text{weak}^* \text{ continuous}\}.$$

$$Z_2(A^{**}) = \{F \in A^{**} : G \to G \circ' F \text{ is weak}^* - \text{weak}^* \text{ continuous}\}.$$

Then a Banach algebra A is Arens regular if and only if $Z_1(A^{**}) = A^{**}$ and $Z_2(A^{**}) = A^{**}$. Also, A is called strongly Arens irregular if and only if $Z_1(A^{**}) = A$ and $Z_2(A^{**}) = A$.

2. Main results

Theorem 2.1. *Let A be a Banach algebra and $m : A \times A \to A$ be a quasi-multiplier on A.*
(i) If A^{**} has an identity and A is Arens regular, then m is Arens regular.
(ii) If m is onto and Arens regular, then A is Arens regular.

Proof. (i) Let E be an identity for A^{**}. It is easy to check that when m is a quasi-multiplier on A, then m^{***} is a quasi-multiplier on A^{**}. Then since $Z_1(A^{**}) = A^{**}$, the mapping

$$F \to m^{***}(G,E) \circ F = m^{***}(G, E \circ F) = m^{***}(G,F)$$

is $weak^* - weak^*$ continuous for all $F, G \in A^{**}$. Also since $Z_2(A^{**}) = A^{**}$, the mapping

$$G \to G \circ' m^{t***t}(E,F) = m^{t***t}(G \circ' E, F) = m^{t***t}(G,F)$$

is $weak^* - weak^*$ continuous for all $F, G \in A^{**}$. Therefore m is Arens regular.
(ii) Let $F \in A^{**}$ and $\{F''_\alpha\}_\alpha \subseteq A^{**}$ such that $F''_\alpha \to^{w^*} F$. Let $G \in A^{**}$. Since m is onto, m^{***} is onto as well. Let $H, K \in A^{**}$ such that $m^{***}(H,K) = G$. As m is Arens regular, we have

$$\begin{aligned} G \circ F = m^{***}(H,K) \circ F = m^{***}(H, K \circ F) = weak^* - \lim_\alpha m^{***}(H, K \circ F''_\alpha) \\ = weak^* - \lim_\alpha m^{***}(H,K) \circ F''_\alpha = weak^* - \lim_\alpha G \circ F_\alpha. \end{aligned}$$

Which means $Z_1(A^{**}) = A^{**}$. Similarly $Z_2(A^{**}) = A^{**}$. Thus A is Arens regular. □

Example 2.1. *Let* $X = [0,1]$ *and* $m : L_\infty(X) \times L_\infty(X) \to L_\infty(X)$ *be defined by* $m(f,g) = f * g$ *where* $*$ *is the convolution product which is given by*

$$f * g(x) = \int_0^x f(x-t)g(t)dt \text{ where } 0 \le x \le 1.$$

By [5], $L_\infty(X)$ *is an Arens regular Banach algebra whose second dual has an identity. So by Theorem 2.1, we conclude that* m *is Arens regular.*

Theorem 2.2. *Let* A *be a Banach algebra and* m *be a quasi-multiplier on* A. *If* m *is bijection and* A *is strongly Arens irregular. Then* m *is strongly Arens irregular.*

Proof. Since A is strongly Arens irregular, $Z_1(A^{**}) = Z_2(A^{**}) = A$. We show that $Z_1(m) = A$. Let $G \in Z_1(m)$. Assume that $\{H_\alpha\}_\alpha \subseteq A^{**}$ such that $H_\alpha \to^{w^*} H$. Then $F \circ H_\alpha \to F \circ H$ is $weak^* - weak^*$ continuous for all $F \in A^{**}$. Now, since $G \in Z_1(m)$, we have

$$\begin{aligned} m^{***}(G,F) \circ H = m^{***}(G, F \circ H) &= weak^* - \lim_\alpha m^{***}(G, F \circ H_\alpha) \\ &= weak^* - \lim_\alpha m^{***}(G,F) \circ H_\alpha. \end{aligned} \tag{1}$$

Consequently, the mapping $H \to m^{***}(G,F) \circ H$ is $weak^* - weak^*$ continuous for all $H \in A^{**}$. Which means that $m^{***}(G,F) \in Z_1(A^{**}) = A$. Since, m is a bijection, it follows that $G \in A$. □

Definition 2.1. *A bilinear map* $m : A^* \times A^{**} \to A^*$ *is a* right quasi-multiplier of A^* *if*

$$m(F \cdot \xi, G) = F \cdot m(\xi, G) \qquad \text{and} \qquad m(\xi, G \circ F) = m(\xi, G) \cdot F \tag{2}$$

hold for arbitrary $\xi \in A^*$ *and* $F, G \in A^{**}$.

Similarly, a bilinear map $m' : A^{**} \times A^* \to A^*$ *is a* left quasi-multiplier of A^* *if*

$$m'(F \circ G, \xi) = F \cdot m'(G, \xi) \qquad \text{and} \qquad m'(G, \xi \cdot F) = m'(G, \xi) \cdot F$$

hold for arbitrary $\xi \in A^*$ *and* $F, G \in A^{**}$.

Recall that $QMr(A^*)$ be the set of all separately continuous right quasi-multipliers of A^*. It is obvious that $QMr(A^*)$ is a linear space. Moreover, it is a Banach space with respect to the norm

$$||m|| = \sup\{||m(\xi, F)||; \quad \xi \in A^*, \ F \in A^{**}, \ ||\xi|| \le 1, \ ||F|| \le 1\}.$$

Of course, the same holds for $QMl(A^*)$, the set of all separately continuous left quasi-multipliers of A^*.

Theorem 2.3. *Let* A *be a commutative Banach algebra. If* m *is a quasi-multiplier on* A, *then* m^{**} *is a left quasi-multiplier of* A^*.

Proof. In order to prove that $m^{**} : A^{**} \times A^{*} \to A^{*}$ is a *left quasi-multiplier of* A^{*}, we show that for all $F, G \in A^{**}, \xi \in A^{*}$ and $x \in A$,

$$\langle m^{**}(F \circ G, \xi), x\rangle = \langle F \cdot m^{**}(G, \xi), x\rangle. \tag{3}$$

The following can be verified from the left side of (3)
(i) $\langle m^{**}(F \circ G, \xi), x\rangle = \langle F \circ G, m^{*}(\xi, x)\rangle = \langle F, G \cdot m^{*}(\xi, x)\rangle$
(ii) $\langle G \cdot m^{*}(\xi, x), y\rangle = \langle G, m^{*}(\xi, x) \cdot y\rangle$
(iii) $\langle m^{*}(\xi, x) \cdot y, z\rangle = \langle m^{*}(\xi, x), yz\rangle = \langle \xi, m(x, yz)\rangle$ (2.4)
From the right side of (3) we have:
(i′) $\langle F \cdot m^{**}(G, \xi), x\rangle = \langle F, m^{**}(G, \xi) \cdot x\rangle$
(ii′) $\langle m^{**}(G, \xi) \cdot x, y\rangle = \langle m^{**}(G, \xi), xy\rangle = \langle G, m^{*}(\xi, xy)\rangle$
(iii′) $\langle m^{*}(\xi, xy), z\rangle = \langle \xi, m(xy, z)\rangle$ (2.5)
Now, since m is a quasi-multiplier on A and A is commutative, by compairing (2.4) and (2.5) we obtain $m^{**}(F \circ G, \xi) = F \cdot m^{**}(G, \xi)$. □

Definition 2.2. *Let A be a general Banach algebra. Then a map $T : A^{*} \to A^{*}$ is called a right multiplier of A^{*} if $T(F \cdot \xi) = F \cdot T(\xi)$, for all $\xi \in A^{*}, F \in A^{**}$. With $Mr(A^{*})$ we denote the space of all bounded linear right multipliers on A^{*}.*

Remark 2.1. *It is obvious that for each $F \in A^{**}$ the right multiplication operator $R_F\xi = \xi \cdot F$ is a right multiplier on A^{*}. If A^{**} has a mixed identity, then each bounded linear right multiplier on A^{*} is a right multiplication operator. Indeed, let E be a mixed identity for A^{**} and $T \in Mr(A^{*})$ be arbitrary. Then equalities*

$$\langle T\xi, a\rangle = \langle E \circ a, T\xi\rangle = \langle E, T(a \cdot \xi)\rangle = \langle R_{T^{*}(E)}\xi, a\rangle$$

hold for all $a \in A$ and $\xi \in A^{*}$, which means $T = R_{T^{*}(E)}$.

Theorem 2.4. *If A^{**} has a mixed identity, then $\rho_T(\xi, F) = (T\xi) \cdot F$ ($T \in Mr(A^{*})$, $\xi \in A^{*}$, $F \in A^{**}$) defines an injective linear map $\rho : Mr(A^{*}) \to QMr(A^{*})$ with norm $\|\rho\| \le 1$. Moreover, ρ is onto if A^{**} has an identity. If A^{**} has a mixed identity with norm one, then ρ is an isometry.*

Proof. Let $T \in Mr(A^{*})$ be arbitrary. It is obvious that ρ_T is a bilinear map from $A^{*} \times A^{**}$ to A^{*} and that it is bounded with $\|T\|$. For $a \in A$, $\xi \in A^{*}$, and F, $G \in A^{**}$, we have $\rho_T(F \cdot \xi, G) = T(F \cdot \xi) \cdot G = (F \cdot T\xi) \cdot G = F \cdot (T\xi \cdot G) = F \cdot \rho_T(\xi, G)$ and $\rho_T(\xi, G \circ F) = (T\xi) \cdot (G \circ F) = (T\xi \cdot G) \cdot F = \rho_T(\xi, G) \cdot F$. Thus, $\rho_T \in QMr(A^{*})$. It follows from the definition that $\rho : Mr(A^{*}) \to QMr(A^{*})$ is linear. Obviously, $\|\rho_T\| \le \|T\|$, which gives $\|\rho\| \le 1$. Let $E \in A^{**}$ be a mixed identity. If $\rho_T = 0$, then we have $(T\xi) \cdot E = 0$ for every $\xi \in A^{*}$ and consequently $T = 0$. Assume that E is an identity for A^{**}. Let $m \in QMr(A^{*})$ be arbitrary. It is easily seen that $T\xi = m(\xi, E)$ ($\xi \in A^{*}$) defines a bounded right multiplier of A^{*}. Since equalities $\rho_T(\xi, F) = (T\xi) \cdot F = m(\xi, E) \cdot F = m(\xi, E \circ F) = m(\xi, F)$ hold for all $\xi \in A^{*}$ and $F \in A^{**}$ we conclude that ρ is onto.

At the end assume that E is mixed identity for A^{**} of norm one. Let $T \in Mr(A^{*})$ and $\varepsilon > 0$ be arbitrary. If $\xi \in A^{*}$ is such that $\|\xi\| \le 1$ and $\|T\| - \varepsilon < \|T\xi\|$, then $\|\rho_T\| \ge \|\rho_T(\xi, E)\| = \|T\xi\| > \|T\| - \varepsilon$. Thus, ρ is an isometry. □

Theorem 2.5. *If A is a Arens regular Banach algebra and A^{**} has an identity E. Then each bounded left quasi-multiplier of A^* is Arens regular.*

Proof. Let $m_1 \in QM_l(A^*)$. From Theorem 2.4 there exists a multiplier $T \in M_l(A^*)$ satisfying $m_1 = \rho_T$. By remark 2.1, $T = R_{T^*(E)}$ and then by [9, Theorem 2.1], T is weakly compact. Also, It is easily seen that $m_2 : A^{**} \times A^* \to A^*$ with $m_2(F, \xi) = F \cdot \xi$ $(F \in A^{**}, \xi \in A^*)$ defines a bounded bilinear map. And,

$$m_1(F, \xi) = \rho_T(F, \xi) = F \cdot T(\xi) = m_2(F, T(\xi)).$$

Therefore by [5, Theorem 2], m_1 is Arens regular. □

In our investigation in [2], we did not assume Arens regularity, we assumed that the given algebra satisfies the following weaker condition. We say a Banach algebra A satisfies condition (K) if $(F \cdot \xi) \cdot G = F \cdot (\xi \cdot G)$ $\quad (F,\ G \in A^{**},\ \xi \in A^*)$. Of course, every Arens regular Banach algebra satisfies condition (K). However, the class of Banach algebras satisfying (K) is larger. It contains, for instance, every Banach algebra A which is an ideal in its second dual.

Theorem 2.6. *Let A be a Banach algebra satisfying condition (K) and E be a mixed identity for A^{**}. Then the mapping $\mu : QM_l(A^*) \to A^{**}$, which is defined by $\mu(m) = m^*(E, E)$ is an isometric. If A^{**} has an identity, then μ is an isometric isomorphism from $QM_l(A^*)$ onto A^{**}.*

Proof. It is easy to see that μ is linear. Let us prove that for each $\xi \in A^*, F \in A^{**}, a \in A$, $\langle m^*(E, E),\ (\xi \cdot a) \cdot F\rangle = \langle m(F, \xi),\ a\rangle$. Now,

$$\begin{aligned}\langle m^*(E, E),\ (\xi.a) \cdot F\rangle &= \langle E, m(E, (\xi.a) \cdot F)\rangle = \langle E, m(E, \xi.a) \cdot F\rangle = \langle E \cdot m(E, \xi.a), F\rangle \\ &= \langle m(E, \xi.a), F\rangle = \langle F, m(E, \xi.a)\rangle = \langle F, m(E, \xi).a\rangle \\ &= \langle F.m(E, \xi), a\rangle = \langle m(F \circ E, \xi), a\rangle = \langle m(F, \xi), a\rangle\end{aligned}$$

Consider,

$$\begin{aligned}\|m\| &= \sup_{\substack{\|F\|\le 1\\ \|\xi\|\le 1}} \|m(F, \xi)\| = \sup_{\substack{\|F\|\le 1\\ \|\xi\|\le 1\\ \|a\|\le 1}} \|\langle m(F, \xi), a\rangle\| \\ &= \sup_{\substack{\|F\|\le 1\\ \|\xi\|\le 1\\ \|a\|\le 1}} \|\langle m^*(E, E), (\xi \cdot a) \cdot F\rangle\| \\ &\le \|m^*(E, E)\|\ \|\xi\|\ \|a\|\ \|F\| \le \|m^*(E, E)\| = \|\mu(m)\|\end{aligned}$$

Also, $\|\mu(m)\| = \|m^*(E, E)\| \le \|m^*\|\ \|E\| \le \|m\|$. Thus $\|\mu(m)\| = \|m\|$, i.e. μ is an isometry. Next we show that μ is onto. Assume that E is an identity for A^{**}. Let $G \in A^{**}$ be arbitrary. Since A satisfying the condition (K), the bilinear mapping $m(F, \xi) = (F \circ G) \cdot \xi$ $(F \in A^{**}, \xi \in A^*)$ denotes a bounded left quasi-multiplier of A^*. Moreover, since equalities

$$\begin{aligned}\langle \mu(m), \xi\rangle &= \langle m^*(E, E), \xi\rangle = \langle E, m(E, \xi)\rangle = \langle E, (E \circ G) \cdot \xi\rangle \\ &= \langle E, G \cdot \xi\rangle = \langle E \circ G, \xi\rangle = \langle G, \xi\rangle\end{aligned}$$

hold for all $\xi \in A^*$ we conclude that μ is onto. $\square$

The previous theorem holds, for instance, for every Arens regular Banach algebra with a b.a.i., in particular for every C^*-algebra.

Example 2.2. *Let H be a Hilbert space and let $A = K(H)$, the algebra of all compact operators on H. The dual of the space of compact operators is the space of all trace-class operators, $C_1(H)$. The second dual of A is $B(H)$. Since $K(H)$ is a C^*-algebra we have $QM_l(C_1(H)) \cong B(H)$.*

Now, we consider the group algebra of a compact group G. By [15], $L_1(G)$ is Arens regular if and only if G is finite. However, since $L_1(G)$ is a two-sided ideal in its second dual ([14]), it satisfies condition (K). Note that the dual $L_1(G)^*$ can be identified with $L_\infty(G)$. Let $M(G)$ be the convolution algebra of all bounded regular measures on G. Recall that the convolution product of $f \in L_1(G)$ and $\mu \in M(G)$ is given by

$$f * \mu(x) = \int_G f(xy^{-1})\, d\mu(y).$$

Of course, $L_\infty(G)$ is a Banach $L_1(G)^{**}$-bimodule. However, the space $L_\infty(G)$ has also a natural structure of a Banach $M(G)$-bimodule. The same holds for $L_\infty(G)^* = L_1(G)^{**}$. We will denote all these module multiplications by $*$.

Proposition 2.1. *Let G be a compact group and $A = L_1(G)$. Then the equation*

$$(\theta_\mu(\xi, F) := (\xi * \mu) * F \qquad (\mu \in M(G), \xi \in L_\infty(G),\ F \in L_1(G)^{**})$$

defines a linear isomorphism between $M(G)$ and a subspace of $QM_r(A^*)$.

Proof. Note that by the definition of module action $(\xi*\mu)*F = \xi*(\mu*F)$. From this and condition (K) we conclude that $\theta_\mu \in QMr(L_1(G)^*)$. Of course, $\theta : M(G) \to QMr(L_1(G)^*)$ is a bounded linear map. We claim that θ is injective. Indeed, suppose that $\theta_\mu = 0$. Then $(\xi * \mu) * F = 0$ for all $\xi \in L_\infty(G)$ and $F \in (L_\infty(G))^*$. Since $L_1(G)$ has a b.a.i. it follows $\xi \circ \mu = 0$. In particular, for each $\xi \in C_0(G)$, $\xi \circ \mu = 0$. Since the measure algebra $M(G)$ is the dual of $C_0(G)$ and it has a b.a.i., $\mu = 0$, as required. $\square$

Theorem 2.7. *Let A be a Banach algebra satisfying condition (K) and assume that A^{**} has an identity E. If A^{**} is Arens regular then the space $QM_r(A^*)$ is Arens regular.*

Proof. Define a map $\psi : A^{**} \to QM_r(A^*)$ by $\psi(H) = \rho_{R_H}$, where R_H is the right multiplication operator on A^* determined by $H \in A^{**}$. Then, for arbitrary $\xi \in A^*, F \in A^{**}$, $\psi(H)(\xi, F) = (\xi \cdot H) \cdot F$. We check only the multiplicativity of ψ since the linearity and continuity are evident. Let $H_1, H_2 \in A^{**}$. By Theorem 2.4, there exist $T_1, T_2 \in M_r(A^*)$ such that $\psi(H_1) = \rho_{T_1}$ and $\psi(H_2) = \rho_{T_2}$. Hence, for arbitrary $\xi \in A^*, F \in A^{**}$, we have $T_1(\xi) \cdot F = (\xi \cdot H_1) \cdot F$ and $T_2(\xi) \cdot F = (\xi \cdot H_2) \cdot F$. It follows

$$\begin{aligned}(\psi(H_1) \circ_\rho \psi(H_2))(\xi, F) = \rho_{T_2T_1}(\xi, F) = T_2(T_1(\xi)) \circ F = T_1\xi \cdot (H_2 \circ F)\\ = \xi \cdot (H_1 \circ H_2 \circ F) = \psi(H_1 \circ H_2)(\xi, F),\end{aligned}$$

which means ψ is a homomorphism. Now, let $m \in QM_r(A^*)$, then there exist $T \in M_r(A^*)$ such that $m = \rho_T = \rho_{R_{T^*(E)}} = \psi(T^*(E))$.

Thus, it is an onto homomorphism. Of course, $\psi^{**} : (A^{**})^{**} \to (QM_r(A^*))^{**}$ has the same property, as well. Let $\tilde{F},\ \tilde{G} \in (QM_r(A^*))^{**}$. Then there exist $F, G \in (A^{**})^{**}$ such that $\psi^{**}(F) = \tilde{F},\ \psi^{**}(G) = \tilde{G}$. Thus,

$$\tilde{F} \circ \tilde{G} = \psi^{**}(F) \circ \psi^{**}(G) = \psi^{**}(F \circ G) = \psi^{**}(F \circ' G) = \tilde{F} \circ' \tilde{G}. \qquad \square$$

Questions 2.1.

(1) *Under which conditions Theorem 2.1 is true if it would be extended to a bilinear mapping m from $A \times A$ to B?*

(2) *Suppose that A is a Banach algebra and m is a strongly Arens irregular quasi-multiplier on A. Under which conditions the algebra A is Arens irregular?*

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