SOME FIXED POINT THEOREMS IN EXTENDED b-METRIC SPACES

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In this paper, we introduce a new class of functions denoted by Ψ_s. We utilize our new class to formulate and prove fixed point theorems in the setting of extended b-metric space, in sense of Kamran et al.

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1. Introduction

In 1922, Banach [1] proved a fundamental result in fixed point theory which is one of the important tools in the field of nonlinear analysis and its applications. Many authors extended Banach Theorem in different metric spaces; for example see [20]-[29]. In 1989, Bakhtin [2] generalized Banach's contraction principle. Actually, he introduced the concept of a b-metric space and proved some fixed point theorems for some contractive mappings in b-metric spaces. Later, in 1993, Czerwik [15] extended the results of b-metric spaces. In 2012, Samet [5] introduced the $\alpha-\psi$-contraction on the Banach contraction principle. Recently, many research was conducted on b-metric space under different contraction conditions, [7]-[14]. After that many authors used $\alpha-\psi$-contraction mapping on different metric spaces [16]-[18]. The notion of extended b-metric space has been introduced recently by Kamran et al [6]. In this paper, we generalize the results of Mehmet [19] and Mukheimer [4] by introducing the $\alpha-\psi$-contractive mapping on some fixed point theorems defined on extended b-metric spaces.

Definition 1.1 ([3]). *Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:*
(i) $d(x, y) = 0$ *if and only if* $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space.

Definition 1.2 ([6]). *Let X be a non-empty set and $\theta : X \times X \to [1,\infty]$ be a mapping. We define the extended b-metric to be the function $d_\theta : X \times X \to [0,\infty]$ that satisfies the following properties:*

(1) $d_\theta(x, y) = 0$ *if and only if* $x = y$;
(2) $d_\theta(x, y) = d_\theta(y, x)$;
(3) $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The space (X, d_θ) is called an extended b-metric space.

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Note that every b-metric space is an extended b-metric space by taking $\theta(x, y) = s \geq 1$ to be a constant function.

Example 1.1. Consider the set $X = \{2, 1, -1\}$, define the function θ on $X \times X$ to be the function $\theta(x, y) = |x| + |y|$. We define the function $d_\theta(x, y)$ as follows:
$d_\theta(2, 2) = d_\theta(1, 1) = d_\theta(-1, -1) = 0$, $d_\theta(1, 2) = 1/2 = d_\theta(2, 1)$, and $d_\theta(1, -1) = d_\theta(-1, 1) = d_\theta(2, -1) = d_\theta(-1, 2) = 1/3$. Then it is clear that $d_\theta(x, y)$ satisfies the first two conditions of Definition 1.2. We need to verify the last condition:
$d_\theta(1, 2) = 1/2 \leq 3[1/3 + 1/3] = \theta(1, 2)[d_\theta(1, -1) + d_\theta(-1, 2)]$.
$d_\theta(1, -1) = 1/3 \leq 2[1/2 + 1/3] = \theta(1, -1)[d_\theta(1, 2) + d_\theta(2, -1)]$.
$d_\theta(-1, 2) = 1/3 \leq 3[1/3 + 1/2] = \theta(-1, 2)[d_\theta(-1, 1) + d_\theta(1, 2)]$.
Therefore, $d_\theta(x, y)$ satisfies the last condition of the definition and hence (X, d_θ) is an extended b-metric space.

Definition 1.3 ([6]). *Let (X, d_θ) be an extended b-metric space. A sequence $\{x_n\}_{n=1}^\infty$ in X is said to be convergent to $x \in X$ if $\lim_{n\to\infty} d_\theta(x_n, x) = 0$. Also, the sequence is said to be a Cauchy sequence if $\lim_{n,m\to\infty} d_\theta(x_n, x_m) = 0$. If every Cauchy sequence is convergent, then we call the space (X, d_θ) complete extended b-metric space.*

2. Main Results.

Let Ψ denoted to the set of all functions $\psi : [0, \infty) \to [0, \infty)$ such that

(1) ψ is decreasing,
(2) $\sum_{n=1}^\infty \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ.

Now, we recall the following lemma.

Lemma 2.1 ([5]). *If $\psi \in \Psi$, then $\psi(t) < t$, for all $t \in [0, +\infty)$.*

To facility our subsequent arguments, we introduce the following definition:

Definition 2.1. *Let X be a set and $\theta : X \times X \to [1, +\infty)$ be a mapping. A function $\psi : [0, \infty) \to [0, \infty)$ is said to be an extended comparison functions if ψ satisfies the following conditions:*

(1) *ψ is decreasing.*
(2) *$\sum_{n=1}^\infty \psi^n(t) \prod_{i=1}^n \theta(x_i, x_m) < +\infty$, any sequence $\{x_n\}_{n=1}^\infty$ in X, for all $t > 0$ and $m \in \mathbb{N}$ where ψ^n is the n-th iterate of ψ.*

The set of all extended comparison functions is denoted by Ψ_s.

Note that if $\psi \in \Psi_s$ then we have $\sum_{n=1}^\infty \psi^n(t) < \infty$, since $\psi^n(t) \prod_{i=1}^n \theta(x_i, x_m) \geq \psi^n(t)$, for all $t > 0$. Hence, by Lemma 2.1, we have $\psi(t) < t$.
Also, note that the family Ψ_s is a non-empty set, as it can be shown in the following examples.

Example 2.1. (1) Consider the extended b-metric space (X, d_θ) that was defined in Example 1.1. Define the mapping $\psi(t) = \frac{kt}{4}$, where $k < 1$. Note that $\theta(x, y) \leq 4$. Then we have $\psi^n(t) \prod_{i=1}^n \theta(x_i, x) \leq \frac{k^n t}{4^n}.4^n = k^n t$. Therefore, $\sum_{n=1}^\infty \psi^n(t) \prod_{i=1}^n \theta(x_i, x) \leq \sum_{n=1}^\infty k^n t < \infty$.
(2) Consider the extended b-metric space (X, d_θ), where $X = [1, \infty)$ and $\theta(x, y) = 1 + \frac{1}{1+\ln(x+y)}$. Define the mapping $\psi(t) = \frac{kt}{2}$. Note that $1 + \frac{1}{1+\ln(x+y)} \leq 2$. So we have $\psi^n(t) \prod_{i=1}^n \theta(x_i, x) \leq \frac{k^n t}{2^n}.2^n = k^n t$. Therefore, $\sum_{n=1}^\infty \psi^n(t) \prod_{i=1}^n \theta(x_i, x) < \infty$ and hence Ψ_s is a non-empty set.

Now we introduce a generalization of α-ψ-contractive mapping on extended b-metric spaces.

Definition 2.2. *Let (X, d_θ) be an extended b-metric space and T be a self mapping on X. We say that T is an α-ψ-contractive mapping if there exist two functions $\psi \in \Psi_s$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$\alpha(x, y) d_\theta(Tx, Ty) \le \psi(d_\theta(x, y)) \quad for \quad all \quad x, y \in X. \tag{1}$$

Also, T is said to be α-admissible if $\alpha(Tx, Ty) \ge 1$, for all $x, y \in X$ with $\alpha(x, y) \ge 1$.

Theorem 2.1. *Let (X, d_θ) be a complete extended b-metric space and $T : X \to X$ be an α-ψ-contractive mapping for some $\psi \in \Psi_s$. Suppose that the following conditions hold:*

(1) *T is α-admissible;*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
(3) *T is continuous;*

Then, T has a fixed point.
Moreover, suppose that there exists $z \in X$ such that $\alpha(x, z) \ge 1$ for any $x \in X$ with $Tx = x$. Then the fixed point is unique.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$. Define the sequence $x_n = Tx_{n-1}$. Note that if there exists n such that $x_{n+1} = x_n$, then we obtain a fixed point $x = x_n$. Assume that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}$. This implies that $d_\theta(x_{n+1}, x_n) > 0$. The second condition of the Theorem implies that $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \ge 1$. So by induction on n, we have $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$. Now we apply the contractive condition 1.

$$\begin{aligned} d_\theta(x_n, x_{n+1}) &\le \alpha(x_{n-1}, x_n) d_\theta(x_n, x_{n+1}) \\ &= \alpha(x_{n-1}, x_n) d_\theta(Tx_{n-1}, Tx_n) \\ &\le \psi(d_\theta(x_{n-1}, x_n)) \\ &\le \psi(\alpha(x_{n-2}, x_{n-1}) d_\theta(x_{n-1}, x_n)) \\ &= \psi(\alpha(x_{n-2}, x_{n-1}) d_\theta(Tx_{n-2}, Tx_{n-1})) \\ &\le \psi^2(d_\theta(x_{n-2}, x_{n-1})) \\ &\ \ \vdots \\ &\le \psi^n(d_\theta(x_0, x_1)). \end{aligned}$$

Now for any $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d_\theta(x_n, x_m) &\le \theta(x_n, x_m)\,[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_m)] \\ &= \theta(x_n, x_m) d_\theta(x_n, x_{n+1}) + \theta(x_n, x_m) d_\theta(x_{n+1}, x_m) \\ &\le \theta(x_n, x_m)\psi^n(d_\theta(x_0, x_1)) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\,[d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2}, x_m)] \\ &\le \theta(x_n, x_m)\psi^n(d_\theta(x_0, x_1)) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\psi^{n+1}(d_\theta(x_0, x_1)) \\ &\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m) d_\theta(x_{n+2}, x_m) \\ &\ \ \vdots \\ &\le \psi^n(d_\theta(x_0, x_1))\theta(x_n, x_m) + \psi^{n+1}(d_\theta(x_0, x_1))\theta(x_n, x_m)\theta(x_{n+1}, x_m) \\ &\quad + \cdots + \psi^{m-1}(d_\theta(x_0, x_1))\theta(x_n, x_m)\theta(x_{n+1}, x_m) \cdots \theta(x_{m-1}, x_m) \\ &= \sum_{j=n}^{m-1} \psi^j(d_\theta(x_0, x_1)) \prod_{i=n}^{j} \theta(x_i, x_m) \\ &= \sum_{j=1}^{m-1} \psi^j(d_\theta(x_0, x_1)) \prod_{i=n}^{j} \theta(x_i, x_m) - \sum_{j=1}^{n-1} \psi^j(d_\theta(x_0, x_1)) \prod_{i=n}^{j} \theta(x_i, x_m) \\ &= S_{m-1} - S_{n-1}, \end{aligned}$$

where $S_{m-1} = \sum_{j=1}^{m-1} \psi^j(d_\theta(x_0, x_1)) \prod_{i=n}^{j} \theta(x_i, x_m)$. Since $\psi \in \Psi_s$,

$$\lim_{n,m\to\infty} [S_{m-1} - S_{n-1}] = 0.$$

Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. The completeness of the space (X, d_θ) implies that the sequence is convergent to a point $x \in X$.
Now, we show that x is a fixed point for T.

$$\begin{aligned} d_\theta(x, Tx) &\leq \theta(x, Tx)\,[d_\theta(x, x_n) + d_\theta(x_n, Tx)] \\ &\leq \theta(x, Tx)\,[0 + 0]\,, \end{aligned}$$

since (x_n) converges to x and T is continuous. Therefore, T has a fixed point x.
Finally we show that the uniqueness of the fixed point. Suppose that T has two fixed points $u, v \in X$. The last assumption of the theorem implies that $\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$. Since T is α-admissible, we have $\alpha(T^n u, T^n z) = \alpha(u, T^n z) \geq 1$ and $\alpha(T^n v, T^n z) = \alpha(v, T^n z) \geq 1$, for all $n \in \mathbb{N}$. Using the α-ψ-contractive condition on T, we obtain

$$\begin{aligned} d(u, T^n z) &= d(Tu, T(T^{n-1}z)) \\ &\leq \alpha(u, T^{n-1}z) d(Tu, T(T^{n-1}z)) \\ &\leq \psi(d(u, T^{n-1}z)) \\ &\ \ \vdots \\ &\leq \psi^n(d(u, z)), \end{aligned}$$

Since $\psi \in \Psi_s$, the sequence $\{\psi^n(d(u, z))\}$ converges to 0. Therefore, $T^n z$ converges to u. Similarly, we can show that $T^n z$ converges to v. The uniqueness of the limit implies that $u = v$. □

Example 2.2. Let (X, d_θ) be the extended b-metric space that was defined in Example 1.1. Define the function $\alpha : X \times X \to \mathbb{R}$ such that

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) = (1, 1), \\ \frac{1}{20} & \text{if } (x, y) \neq (1, 1). \end{cases}$$

Define the self mapping T on $X = \{1, -1, 2\}$ by $T(1) = 1, T(-1) = 2, T(2) = 1$ and the function $\psi(t) = \frac{1}{8}t$.
We want to verify that T satisfies the conditions of Theorem 2.1. It is clear that T is continuous and for $x_0 = 1$ we have $\alpha(1, T(1)) = \alpha(1, 1) = 1 \geq 1$. So T is α-admissible. Now we verify that T is $\alpha - \psi$-contractive mapping. Note that $\alpha(x, y) = \alpha(y, x)$.

- $\alpha(x, x) d_\theta(Tx, Tx) = 0 \leq \psi(d_\theta(x, x)) = \psi(0) = 0$, for all $x \in X$,
- $\alpha(1, 2) d_\theta(T1, T2) = \frac{1}{20} d_\theta(1, 1) = 0 \leq \psi(d_\theta(1, 2)) = \psi(1/2) = 1/16$,
- $\alpha(1, -1) d_\theta(T1, T-1) = (1/20) d_\theta(1, 2) = 1/40 \leq \psi(d_\theta(1, -1)) = \psi(1/3) = 1/24$,
- $\alpha(2, -1) d_\theta(T2, T-1) = (1/20) d_\theta(1, 2) = 1/40 \leq \psi(d_\theta(2, -1)) = \psi(1/3) = 1/24$

Therefore, T satisfies the conditions in Theorem 2.1 and hence it has a unique fixed point $x = 1$.

Corollary 2.1. *Let (X, d_θ) be a complete extended b-metric space and $T : X \to X$ be a mapping satisfying the following conditions:*

1. *T is continuous;*
2. *there exists $\psi \in \Psi_s$ such that $d_\theta(Tx, Ty) \leq \psi(d_\theta(x, y))$ holds for all $x, y \in X$.*

Then, T has a unique fixed point.

Proof. Define the function $\alpha : X \times X \to [0, +\infty)$ via $\alpha(x, y) = 1$. Note that T is $\alpha-$admissible. Moreover, T satisfies all the conditions of Theorem 2.1. So T has a unique fixed point. □

Corollary 2.2. *Let (X, d_θ) be a complete extended b-metric space and $\alpha : X \times X \to [0, +\infty)$ be an $\alpha-$admissible function. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(1) *T is continuous;*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(3) *there exists $k \in [0, 1)$ such that for any sequence $\{x_n\}_{n=1}^{\infty}$ in X and $x \in X$, we have $\sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(x_i, x) < +\infty$;*
(4) *for any $x, y \in X$, T satisfies $\alpha(x, y)d_\theta(Tx, Ty) \leq kd_\theta(x, y)$.*

Then, T has a fixed point. Moreover, suppose that there exists $z \in X$ such that $\alpha(x, z) \geq 1$ for any $x \in X$ with $Tx = x$. Then the fixed point is unique.

Proof. Define $\psi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ by $\psi(t) = kt$. Note that T satisfies all the conditions of Theorem 2.1. So T has a unique fixed point. □

Corollary 2.3. *Let (X, d_θ) be a complete extended b-metric space and $\alpha : X \times X \to [0, +\infty)$ be an $\alpha-$admissible function. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(1) *T is continuous;*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(3) *there exists $k \in [0, 1)$ be such that for any sequence $\{x_n\}_{n=1}^{\infty}$ in X and $x \in X$; $\lim_{n\to+\infty} \theta(x_n, x) < \frac{1}{k}$;*
(4) *for any $x, y \in X$, T satisfies $\alpha(x, y)d_\theta(Tx, Ty) \leq kd_\theta(x, y)$.*

Then, T has a fixed point. Moreover, suppose that there exists $z \in X$ such that $\alpha(x, z) \geq 1$ for any $x \in X$ with $Tx = x$. Then the fixed point is unique.

Proof. Consider a sequence $\{x_n\}$ in X and $x \in X$ be such that $\lim_{n\to+\infty} \theta(x_i, x) < \frac{1}{k}$. Note that

$$\lim_{n\to\infty} \frac{\prod_{i=1}^{n+1} \theta(x_i, x_m)k^{n+1}}{\prod_{i=1}^{n} \theta(x_i, x_m)k^n} = \lim_{n\to\infty} \theta(x_{n+1}, x_m)k < 1.$$

Thus, by the ratio test we conclude that for any sequence $\{x_n\}_{n=1}^{\infty}$ in X and $x \in X$ we have $\sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(x_i, x) < +\infty$. Moreover, note that T satisfies the conditions in Corollary 2.2. Therefore T has a unique fixed point. □

The following result can be obtained from Theorem 2.1, Corollary 2.1 and Corollary 2.2 by considering the constant mapping $\theta(x, y) = s \geq 1$.

Corollary 2.4. *Let (X, d) be a complete b-metric space and T be a self mapping on X that satisfies the conditions in Theorem 2.1. Then T has a unique fixed point.*

Corollary 2.5. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a mapping satisfying the following conditions:*

(1) *T is continuous;*
(2) *there exists $\psi \in \Psi$ such that $d(Tx, Ty) \leq \psi(d(x, y))$ holds for all $x, y \in X$.*

Then, T has a unique fixed point.

Corollary 2.6. *Let (X, d) be a complete b-metric space and $\alpha : X \times X \to [0, +\infty)$ be an $\alpha-$admissible function. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(1) *T is continuous;*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(3) *there exist $k \in [0, \frac{1}{s})$, such that $\alpha(x, y)d(Tx, Ty) \leq kd(x, y)$ holds for all $x, y \in X$.*

Then, T has a fixed point. Moreover, suppose that there exists $z \in X$ such that $\alpha(x, z) \geq 1$ for any $x \in X$ with $Tx = x$. Then the fixed point is unique.

In the following result we can replace the continuity of the mapping T with another condition.

Theorem 2.2. *Let (X, d_θ) be a complete, extended b-metric space and $T : X \to X$ be an α-ψ-contractive mapping for some $\psi \in \Psi_s$. Suppose that the following conditions hold:*

1. *T is α-admissible;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
3. *If $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then, T has a fixed point.

Proof. In proving the result, we follow the same steps as in the proof of Theorem 2.1 to construct a sequence $\{x_n\}_{n=1}^{\infty}$ that converges to a point $x \in X$. The constructed sequence has the property $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. The last assumption of the result implies that $\alpha(x_n, x) \geq 1$. We finally show that x is a fixed point for T. From the triangle inequality, we obtain:

$$\begin{aligned} d_\theta(x, Tx) &\leq \theta(x, Tx)\left[d_\theta(x, x_{n+1}) + d_\theta(x_{n+1}, Tx)\right] \\ &= \theta(x, Tx)d_\theta(x, x_{n+1}) + \theta(x, Tx)d_\theta(x_{n+1}, Tx). \end{aligned}$$

Note that the first term of the inequality, $\theta(x, Tx)d_\theta(x, x_{n+1})$, converges to 0, since $\{x_n\}$ converges to 0. Also, the second term will be

$$\begin{aligned} \theta(x, Tx)d_\theta(x_{n+1}, Tx) &\leq \theta(x, Tx)d_\theta(Tx_n, Tx)\alpha(x_n, x) \\ &\leq \theta(x, Tx)\psi(d_\theta(x_n, x)) \\ &\leq \theta(x, Tx)d_\theta(x_n, x), \end{aligned}$$

which converges to 0. Therefore, $d_\theta(x, Tx) \leq 0$ and hence x is a fixed point for T.

□

Remark 2.1. The uniqueness of the fixed point in Theorem 2.2 can be proved considering the fourth condition of Theorem 2.1. The uniqueness proof is similar to the proof of Theorem 2.1.

From Theorem 2.2 we can easily prove the following result by taking $\alpha(x, y) = 1$.

Corollary 2.7. *Let (X, d_θ) be a complete extended b-metric space and $T : X \to X$ be a mapping. If there exists $\psi \in \Psi_s$ such that $d_\theta(Tx, Ty) \leq \psi(d_\theta(x, y))$ holds for all $x, y \in X$, then T has a unique fixed point.*

Corollary 2.8. *Let (X, d_θ) be a complete extended b-metric space and $\alpha : X \times X \to [0, +\infty)$ be an $\alpha-$admissible function. Let $T : X \to X$ be a mapping satisfying the following conditions:*

1. *If $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
3. *there exists $k \in [0, 1)$ be such that for any sequence $\{x_n\}_{n=1}^{\infty}$ in X and $x \in X$, we have $\sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(x_i, x) < +\infty$;*
4. *for any $x, y \in X$, T satisfies $\alpha(x, y)d_\theta(Tx, Ty) \leq kd_\theta(x, y)$.*

Then, T has a fixed point. Moreover, suppose that there exists $z \in X$ such that $\alpha(x, z) \geq 1$ for any $x \in X$ with $Tx = x$. Then the fixed point is unique.

Corollary 2.9. *Let (X, d_θ) be a complete extended b-metric space and $\alpha : X \times X \to [0, +\infty)$ be an $\alpha-$admissible function. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(1) *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* n*;*
(2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
(3) *there exists* $k \in [0, 1)$ *be such that for any sequence* $\{x_n\}_{n=1}^{\infty}$ *in* X *and* $x \in X$*;* $\lim_{n\to+\infty} \theta(x_n, x) < \frac{1}{k}$*;*
(4) *for any* $x, y \in X$, T *satisfies* $\alpha(x, y)d_\theta(Tx, Ty) \leq kd_\theta(x, y)$.

Then, T *has a fixed point. Moreover, suppose that there exists* $z \in X$ *such that* $\alpha(x, z) \geq 1$ *for any* $x \in X$ *with* $Tx = x$*. Then the fixed point is unique.*

The following result can be obtained from Theorem 2.2, Corollary 2.7 and Corollary 2.8 by considering the constant mapping $\theta(x, y) = s \geq 1$.

Corollary 2.10. *Let* (X, d) *be a* b*-metric space and* T *be a self mapping on* X *that satisfies the conditions in Theorem 2.2. Then* T *has a fixed point.*

Corollary 2.11. *Let* (X, d) *be a complete* b*-metric space and* $T : X \to X$ *be a mapping satisfying the following conditions:*

(1) *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* n*;*
(2) *there exists* $\psi \in \Psi$ *such that* $d(Tx, Ty) \leq \psi(d(x, y))$ *holds for all* $x, y \in X$.

Then, T *has a unique fixed point.*

Corollary 2.12. *Let* (X, d) *be a complete* b*-metric space and* $\alpha : X \times X \to [0, +\infty)$ *be an* $\alpha-$*admissible function. Let* $T : X \to X$ *be a mapping satisfying the following conditions:*

(1) *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* n*;*
(2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*
(3) *there exist* $k \in [0, \frac{1}{s})$, *such that* $\alpha(x, y)d(Tx, Ty) \leq kd(x, y)$ *holds for all* $x, y \in X$.

Then, T *has a fixed point. Moreover, suppose that there exists* $z \in X$ *such that* $\alpha(x, z) \geq 1$ *for any* $x \in X$ *with* $Tx = x$*. Then the fixed point is unique.*

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