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APPROXIMATE IDEAL AMENABILITY OF BANACH ALGEBRAS

Madjid Eshaghi Gordji[1], Ali Jabbari[2]

In this paper, we study the approximate ideal amenability of several Banach algebras. Moreover, we show that approximate ideal amenability is different from approximate amenability and approximate weak amenability.

Keywords: Amenability; weak amenability; Approximate amenability; Ideal amenability.

MSC2010: Primary 46H25; Secondary 46H20.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. We can define the right and left actions of $\mathcal{A}$ on dual space X^* of X via

$$\langle x, \lambda.a\rangle = \langle a.x, \lambda\rangle, \qquad \textit{and} \qquad \langle x, a.\lambda\rangle = \langle x.a, \lambda\rangle,$$

for all $a \in \mathcal{A}$, $x \in X$ and $\lambda \in X^*$. Similarly, the second dual X^{**} of X becomes a Banach $\mathcal{A}$-bimodule (for more details see [3]). Thus, in particular, I is a Banach $\mathcal{A}$-bimodule and I^* is a dual $\mathcal{A}$-bimodule for every closed two-sided ideal I in $\mathcal{A}$. A derivation is a linear map $D : \mathcal{A} \longrightarrow X$ such that

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in \mathcal{A}).$$

From now on, we suppose that all derivations are bounded. The set of derivations from $\mathcal{A}$ into X is denoted by $Z^1(\mathcal{A}, X)$; which is a linear subspace of $\mathcal{L}(\mathcal{A}, X)$, the space of all bounded linear maps from $\mathcal{A}$ into X. For $x \in X$, set $D_x : \mathcal{A} \longrightarrow X$, $a \mapsto a.x - x.a, \mathcal{A} \longrightarrow X$. It is easy to see that D_x is a derivation for all $x \in X$. Derivations of this form are called inner derivations, and an inner derivation D_x is implemented by x. The set of inner derivations from $\mathcal{A}$ into X is a linear subspace $N^1(\mathcal{A}, X)$ of $Z^1(\mathcal{A}, X)$. We consider the quotient space $\mathcal{H}^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$, called first cohomology group of $\mathcal{A}$ with coefficients in X; clearly, $\mathcal{H}^1(\mathcal{A}, X) = \{0\}$ if and only if every derivation from $\mathcal{A}$ into X is inner.

The Banach algebra $\mathcal{A}$ is called amenable if $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for any $\mathcal{A}$-bimodule X. The concept of amenability for Banach algebras was introduced by Johnson in 1972 [17]. The Banach algebra $\mathcal{A}$ is weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. Of course, every amenable Banach algebra is weakly amenable; however the class

[1]Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Iran, e-mail: madjid.eshaghi@gmail.com

[2] Young Researchers Club, Ardabil Branch, Islamic Azad University, Ardabil, Iran, e-mail: jabbari_al@yahoo.com

of weakly amenable Banach algebras is considerably larger than that of amenable Banach algebras. For example, the group algebra $L^1(G)$ is weakly amenable for each locally compact group G. Examples of weakly amenable, but not amenable, Banach function algebras are given in [1], where it is noted that the commutative Banach algebra $\mathcal{A}$ is weakly amenable if and only if $\mathcal{H}^1(\mathcal{A}, X) = \{0\}$ for each Banach $\mathcal{A}$-module X.

The notion of approximate amenability of Banach algebras was introduced by Ghahramani and Loy in [9].

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. A derivation $D : \mathcal{A} \longrightarrow X$ is called approximately inner if there exists a net $(\xi_\alpha)_\alpha$ in X such that $D(a) = \lim_\alpha a.\xi_\alpha - \xi_\alpha.a$, for each $a \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is approximately amenable if for every Banach $\mathcal{A}$-bimodule X, every derivation $D : \mathcal{A} \longrightarrow X^*$ is approximately inner.

The concept of ideal amenability of Banach algebras was introduced by Gordji and Yazdanpanah in [14]. A Banach algebra $\mathcal{A}$ is called ideally amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{I}^*) = \{0\}$ for every closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$. Ideal amenability of group algebras $L^1(G)$, $L^\infty(G)$ and $M(G)$, where G is a locally compact group, are studied in [12]. Many new results for ideal amenability of Banach algebras are given in [11, 13, 15, 16, 24, 21], and [22].

Definition 1.1. Let $\mathcal{A}$ be a Banach algebra, and let I be a closed two-sided ideal in $\mathcal{A}$. $\mathcal{A}$ is said to be approximate I-weakly amenable if every derivation from $\mathcal{A}$ into I^* is approximately inner. If for every two-sided ideal I in $\mathcal{A}$, every derivation from $\mathcal{A}$ into I^* is approximately inner, then we say that $\mathcal{A}$ is approximate ideally amenable.

Recently, Mewomo in [23] studied the approximate ideal amenability of Banach algebras, and gave some results in this area. In this paper, we study the ideal amenability and approximate ideal amenability of several Banach algebras (see also [26]).

2. **Main Results**

In this section, we prove the main theorems of our paper.

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra, and I be a two-sided closed ideal of $\mathcal{A}$. If $\mathcal{A}$ is approximate I-weakly amenable then I is approximate weakly amenable.*

Proof. Let $D : I \longrightarrow I^*$ be a derivation and let $\jmath : \mathcal{A} \longrightarrow I$ be a natural projection. Then $D \circ \jmath : \mathcal{A} \longrightarrow I^*$ is a derivation. Since $\mathcal{A}$ is approximate I-weakly amenable, and $D \circ \jmath$ is a derivation from $\mathcal{A}$ into I^*, therefore $D \circ \jmath$ is approximately inner. Since $\jmath$ is natural projection, thus, $D \circ \jmath$ is approximately inner from I into I^*. This means that I is approximate weakly amenable. □

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule and Y be a closed submodule of X. By Theorem 1.6 of [14], if $\mathcal{H}^1(\mathcal{A}, Y^*) = \{0\}$, and $\mathcal{H}^1(\mathcal{A}, (X/Y)^*) = \{0\}$, then $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$. It follows that:

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra, X be a Banach $\mathcal{A}$-bimodule, and let Y be a closed submodule of X. If every derivation from $\mathcal{A}$ into Y^*, and every derivation* *from $\mathcal{A}$ into $(X/Y)^*$ are approximately inner, then every derivation from $\mathcal{A}$ into X^* is approximately inner.*

Proof. Let $D : \mathcal{A} \longrightarrow X^*$ be a derivation, and let $\imath : Y \longrightarrow X$ be the natural embedding $\mathcal{A}$-bimodule homomorphism. Then $\imath^* : X^* \longrightarrow Y^*$ is an $\mathcal{A}$-bimodule homomorphism, and $\imath^* \circ D : \mathcal{A} \longrightarrow Y^*$ is a derivation. From assumption, there is a net $(\xi_\alpha)_\alpha \subseteq Y^*$ such that

$$\imath^* \circ D(a) = \lim_\alpha a.\xi_\alpha - \xi_\alpha.a \qquad (a \in \mathcal{A}).$$

Set $D_\alpha(a) = a.\xi_\alpha - \xi_\alpha.a$, therefore $\imath^* \circ D(a) = \lim_\alpha D_\alpha$. Without loss of generality we can suppose that $(\xi_\alpha)_\alpha \subseteq X^*$ (by Hahn-Banach theorem). Now, define $d : \mathcal{A} \longrightarrow Y^\perp$ by $d = D - \lim_\alpha D_\alpha$. Therefore d is a derivation, and so there is a net $(\eta_\beta)_\beta \subseteq Y^\perp = (X/Y)^* \subseteq X^*$ such that

$$d(a) = \lim_\beta a.\eta_\beta - \eta_\beta.a \qquad (a \in \mathcal{A}).$$

Set $\zeta_{\alpha,\beta} = \xi_\alpha + \eta_\beta$ such that $\lim_\alpha \lim_\beta \zeta_{\alpha,\beta} = \lim_\alpha \xi_\alpha + \lim_\beta \eta_\beta$. Now by passing into suitable subnets and by iterated limit ([19]), there is a net $(\varepsilon_\gamma)_\gamma \subseteq X^*$ such that $\lim_\gamma \varepsilon_\gamma = \lim_\alpha \lim_\beta \zeta_{\alpha,\beta}$. Since $D = d + \lim_\alpha D_\alpha$, therefore we have

$$\begin{aligned} D(a) &= d(a) + \lim_\alpha D_\alpha = \lim_\beta a.\eta_\beta - \eta_\beta.a + \lim_\alpha a.\xi_\alpha - \xi_\alpha.a \\ &= \lim_\alpha \lim_\beta a.(\eta_\beta + \xi_\alpha) - (\eta_\beta + \xi_\alpha).a \\ &= \lim_\alpha \lim_\beta a.\zeta_\alpha - \zeta_\alpha.a = \lim_\gamma a.\varepsilon_\gamma - \varepsilon_\gamma.a \end{aligned}$$

for each $a \in \mathcal{A}$. This completes the proof. □

Corollary 2.3. *Let I and J be closed two-sided ideals in Banach algebra $\mathcal{A}$ such that $J \subseteq I$. If $\mathcal{A}$ is approximate J-weakly amenable, and every derivation from $\mathcal{A}$ into $(I/J)^*$ is approximately inner, then $\mathcal{A}$ is approximate I-weakly amenable.*

Corollary 2.4. *Let I be a closed two-sided ideals in Banach algebra $\mathcal{A}$. If every derivation from $\mathcal{A}$ into $(\mathcal{A}/I)^*$ is approximately inner, and $\mathcal{A}$ is approximate I-weakly amenable, then $\mathcal{A}$ is approximate weakly amenable.*

For finite-dimensional Banach algebras, approximate amenability and amenability are equivalent (see [20]). Now by a similar argument we have this equivalence relation, for weak amenability and approximate weak amenability.

Theorem 2.5. *For finite-dimensional Banach algebra $\mathcal{A}$ approximate weak amenability and weak amenability are equivalent.*

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a derivation, and let X be the subspace of A^* generated by $D(\mathcal{A}) + \mathcal{A}.D(\mathcal{A}) + D(\mathcal{A}).\mathcal{A} + \mathcal{A}.D(\mathcal{A}).\mathcal{A}$. Therefore X is finite-dimensional. Define $\widetilde{D} : \mathcal{A} \longrightarrow X$ by $\widetilde{D}(a) = D(a)$ for each $a \in \mathcal{A}$. Since $\mathcal{A}$ is approximate weakly amenable, thus there is a net $(\xi_\alpha)_\alpha \subseteq X$ such that

$$\widetilde{D}(a) = \lim_\alpha a.\xi_\alpha - \xi_\alpha.a \qquad (a \in \mathcal{A}).$$

Finite-dimensionality of $\mathcal{A}$ and X, conclude that $N^1(\mathcal{A}, X)$ is finite-dimension, and this means that it is closed subspace of $\mathcal{L}(\mathcal{A}, X)$ in the operator strong topology.

It follows that $\widetilde{D} \in N^1(\mathcal{A}, X)$, and hence there exists an element $\xi \in X \subseteq \mathcal{A}^*$ such that

$$\widetilde{D}(a) = a.\xi - \xi.a \qquad (a \in \mathcal{A}).$$

Therefore D is inner. □

Corollary 2.6. *Let $\mathcal{A}$ be a finite-dimensional Banach algebra, and let I be a closed two-sided ideal in $\mathcal{A}$. $\mathcal{A}$ is approximate I-weakly amenable if and only if $\mathcal{A}$ is I-weakly amenable.*

Corollary 2.7. *Let S be a finite semigroup. Then $\ell^1(S)$ is weakly amenable if and only if $\ell^1(S)$ is approximate weakly amenable.*

Let $\mathcal{A}$ be a Banach algebra and I be a closed two-sided ideal in $\mathcal{A}$. We say that I has the trace extension property, if every $m \in I^*$ such that $am = ma$ for each $a \in \mathcal{A}$, can be extended to $a^* \in \mathcal{A}$ such that $aa^* = a^*a$ for each $a \in \mathcal{A}$. If I has the trace extension property, and $\mathcal{A}$ is ideally amenable, then by Theorem 1.12 of [14], $\mathcal{A}/I$ is ideally amenable. For approximate ideal amenability we define approximate trace extension property as follows, and consider similar Theorem for approximate ideal amenability.

Definition 2.8. Let $\mathcal{A}$ be a Banach algebra, and let I be a closed two-sided ideal in $\mathcal{A}$. We say that I has the approximate trace extension property if for each $\lambda \in I^*$ with $a.\lambda = \lambda.a$, there is a net $(\tau_\alpha)_\alpha \subseteq \mathcal{A}^*$ such that $\tau_\alpha|_I = \lambda$ and $a.\tau_\alpha - \tau_\alpha.a \longrightarrow 0$, for all $a \in \mathcal{A}$ and all α.

Theorem 2.9. *Let $\mathcal{A}$ be an ideally amenable Banach algebra, and let I be a closed two-sided ideal in $\mathcal{A}$ with approximate trace extension property. Then $\mathcal{A}/I$ is approximate ideally amenable.*

Proof. Let J/I be a closed two-sided ideal in $\mathcal{A}/I$. Then J is a closed two-sided ideal in $\mathcal{A}$. We write $\pi : J \longrightarrow J/I$, $q : \mathcal{A} \longrightarrow \mathcal{A}/I$ for the natural quotient maps and π^* for the adjoint of π. Let $D : \mathcal{A}/I \longrightarrow (J/I)^*$ be a derivation. Then $d = \pi^* \circ D \circ q : \mathcal{A} \longrightarrow J^*$ is a derivation, so there is an element $\lambda \in J^*$ such that

$$d(a) = a.\lambda - \lambda.a \qquad (a \in \mathcal{A}).$$

Let m be the restriction of λ to I. Then $m \in I^*$ and for all $i \in I$ we have

$$\begin{aligned} \langle i, am - ma \rangle &= \langle ia - ai, m \rangle = \langle ia - ai, \lambda \rangle \\ &= \langle i, a.\lambda - \lambda.a \rangle = \langle i, \pi^* \circ D \circ q(a) \rangle \\ &= \langle \pi(i), D \circ q(a) \rangle = \langle I, D(a + I) \rangle = 0. \end{aligned}$$

Therefore, $am = ma$ for each $a \in \mathcal{A}$. Hence by assumption, there exists a net $(\kappa_\alpha)_\alpha \subseteq \mathcal{A}^*$ such that for any α, we have $\kappa_\alpha|I = m$ and $\lim_\alpha a.\kappa_\alpha - \kappa_\alpha.a = 0$ $(a \in \mathcal{A})$. Let τ_α be the restriction of κ_α to J, for every α. Then $(\tau_\alpha)_\alpha \subseteq J^*$ and

$\lambda - \tau_\alpha = 0$ on I. Therefore $\lambda - \tau_\alpha \in (J/I)^*$, and for each $j \in J$ and $a \in \mathcal{A}$ we have

$$\begin{aligned}\langle j + I, D(a + I)\rangle &= \langle \pi(j), D \circ q(a)\rangle = \langle j, \pi^* \circ D \circ q(a)\rangle \\ &= \langle j, d(a)\rangle = \langle j, a.\lambda - \lambda.a\rangle = \langle j, a.\lambda + \lim_\alpha(a.\tau_\alpha - \tau_\alpha.a) - \lambda.a\rangle \\ &= \lim_\alpha \langle j, a.\lambda + a.\tau_\alpha - \tau_\alpha.a - \lambda.a\rangle \\ &= \lim_\alpha \langle j, a.(\lambda - \tau_\alpha) - (\lambda - \tau_\alpha).a\rangle \\ &= \langle j + I, \lim_\alpha a.(\lambda - \tau_\alpha) - (\lambda - \tau_\alpha).a\rangle.\end{aligned}$$

Hence $D(a + I) = \lim_\alpha a.(\lambda - \tau_\alpha) - (\lambda - \tau_\alpha).a$. This means that $\mathcal{A}/I$ is approximate ideally amenable. □

Let $\mathcal{A}$ be a Banach algebra, and let I be a closed two-sided ideal in $\mathcal{A}$. If I is weakly amenable, then by Lemma 2.1 of [14], $\mathcal{A}$ is I-weakly amenable. We have the same result for approximate amenability as follows.

Theorem 2.10. *Let $\mathcal{A}$ be a Banach algebra, and let I be a closed two-sided ideal in $\mathcal{A}$. If I is approximately amenable, then $\mathcal{A}$ is approximate I-weakly amenable.*

Proof. Let $D : \mathcal{A} \longrightarrow I^*$ be a derivation, and let $\imath : I \longrightarrow \mathcal{A}$ be the embedding map. Then $D \circ \imath : I \longrightarrow I^*$ is a derivation, and so there is a net $(\xi_\alpha)_\alpha \subseteq I^*$ such that

$$D \circ \imath(a) = \lim_\alpha a.\xi_\alpha - \xi_\alpha.a \qquad (a \in I).$$

Since I is approximately amenable, therefore by Lemma 2.2 of [9], $\overline{I^2} = I$. Now, for each $i, j \in I$ we can write

$$\begin{aligned}\langle ij, D(a)\rangle &= \langle i, j.D(a)\rangle = \langle i, D(ja) - D(j).a\rangle = \langle i, D(ja)\rangle - \langle i, D(j).a\rangle \\ &= \langle i, \lim_\alpha ja.\xi_\alpha - \xi_\alpha.ja\rangle - \langle ai, \lim_\alpha j.\xi_\alpha - \xi.j\rangle \\ &= \lim_\alpha(\langle ija, \xi_\alpha\rangle - \langle aij, \xi_\alpha\rangle) = \lim_\alpha(\langle ij, a.\xi_\alpha - \xi_\alpha.a\rangle) \\ &= \langle ij, \lim_\alpha a.\xi_\alpha - \xi_\alpha.a\rangle \qquad (a \in \mathcal{A}).\end{aligned}$$

Therefore $D(a) = \lim_\alpha a.\xi_\alpha - \xi_\alpha.a$ for all $a \in \mathcal{A}$. Hence $\mathcal{A}$ is approximate I-weakly amenable. □

Corollary 2.11. *Let G be an amenable locally compact group. Then $M(G)$ is approximate $L^1(G)$-weakly amenable.*

Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. If $\mathcal{A}$ is an ideal of $\mathcal{A}^{**}$, and $\mathcal{A}^{**}$ is ideally amenable, then $\mathcal{A}$ is ideally amenable (Theorem 3.2 of [10]). If J is a closed two-sided ideal of $\mathcal{A}$ with a bounded approximate identity, and $\mathcal{A}$ is approximate ideally amenable, then J is approximate ideally amenable (Theorem 3.5 of [23]). One can prove the following Proposition similarly.

Proposition 2.12. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. If $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$, and $\mathcal{A}^{**}$ is approximate ideally amenable, then $\mathcal{A}$ is approximate ideally amenable.*

Proposition 2.13. *Let $\mathcal{A}$ be a commutative Banach algebra. If there exists a family (A_α) of closed subalgebras of $\mathcal{A}$ such that $\bigcup_\alpha \mathcal{A}_\alpha$ is dense in $\mathcal{A}$, and $Z^1(\mathcal{A}_\alpha, I^*) = \{0\}$, for all closed two-sided ideal I of $\mathcal{A}$, then $\mathcal{A}$ is ideally amenable.*

Proof. Let I be a closed two-sided ideal in $\mathcal{A}$, and let $D : \mathcal{A} \longrightarrow I^*$ be a derivation. Let $a \in A$ be an arbitrary element. Then there is $a_\alpha \in \mathcal{A}_\alpha$ such that $\|a - a_\alpha\| < \varepsilon/(\|D\|+1)$ for every $\varepsilon > 0$. Let D_α be the induced of derivation D on $\mathcal{A}_\alpha$. Therefore D_α is a derivation from $\mathcal{A}_\alpha$ into I^*. Since every derivation from $\mathcal{A}_\alpha$ into I^* is inner, then there is an element ξ in I^* such that $D_\alpha(b_\alpha) = b_\alpha.\xi - \xi.b_\alpha$ for all $b_\alpha \in \mathcal{A}_\alpha$. On the other hand I^* is a symmetric $\mathcal{A}$-bimodule, thus $b_\alpha.\xi - \xi.b_\alpha = 0$ for all $b_\alpha \in \mathcal{A}_\alpha$. Therefore $D_\alpha(b_\alpha) = 0$ for all $b_\alpha \in \mathcal{A}_\alpha$. We claim that $D = 0$. Because

$$\|D(a)\| = \|D(a) - D_\alpha(a_\alpha)\| = \|D(a - a_\alpha)\| \leq \|D\|\|a - a_\alpha\| < \varepsilon.$$

Since $a \in \mathcal{A}$ and $\varepsilon > 0$ are arbitrary, then $D = 0$. This means that $Z^1(\mathcal{A}, I^*) = \{0\}$. □

Note that in proof of above Proposition, commutativity of $\mathcal{A}$ is necessary, because assumed D_α must be zero on $\mathcal{A}$.

3. **Examples**

Let $\mathcal{A}$ be a Banach algebra. It is easy to see that the unitization $\mathcal{A}^\sharp$ of $\mathcal{A}$ is approximate ideally amenable if and only if $\mathcal{A}$ is approximate ideally amenable. By using this fact, we find the following important example:

Example 3.1. *Let B be the augmentation ideal $I_0(SL(2,\mathbb{R}))$ of the group algebra $L^1(SL(2,\mathbb{R}))$ of the locally compact group $SL(2,\mathbb{R})$. The group algebra $L^1(SL(2,\mathbb{R}))$ is weakly amenable and therefore it is approximate weakly amenable (Theorem 1 of* [7]*). In the other words, $\mathcal{A} = B^\sharp$ is approximate weakly amenable. On the other hand, it is known that B is not approximate weakly amenable (see Corollary 5 of* [2]*). Then B is not approximate ideally amenable. It follows that $A = B^\sharp$ is not approximate ideally amenable. Hence we have the following:*

The algebra $(I_0(SL(2,\mathbb{R})))^\sharp$ is approximate weakly amenable but it is not approximate ideally amenable.

There are several examples of approximate ideally amenable Banach algebras which are not approximately amenable. See the following

Example 3.2. *Let K be a compact metric space with metric d, and take α such that $0 < \alpha \leq 1$. Then the $Lip_\alpha K$ is the space of complex valued functions f on K such that*

$$p_\alpha(f) = \sup\{\frac{|f(x) - f(y)|}{d(x,y)^\alpha} : x, y \in K, x \neq y\}$$

is finite. For $f \in Lip_\alpha K$, set

$$\|f\|_\alpha = |f|_K + p_\alpha(f).$$

Then $(Lip_\alpha K, \|f\|_\alpha)$ is a Banach algebra. A function $f \in lip_\alpha K$ if

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \longrightarrow 0 \quad \text{as } d(x,y) \longrightarrow 0;$$

$lip_\alpha K$ is a closed subalgebra of $Lip_\alpha K$ (for more details see [27]*). Many results on amenability and weak amenability of Lipschitz algebras are given in* [1]*. If K is infinite, then $lip_\alpha(K)$ is not amenable (Theorem 3.9 of* [1]*) and by Theorem 3.4 of* [4]*, $lip_\alpha(K)$ is not approximately amenable. On the other hand, by Theorem 3.10 of* [1]*, and Proposition 3.4 of* [6]*, for $\alpha > 1/2$, $lip_\alpha(K)$ is approximate ideally amenable.*

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