PROJECTIVE COVARIANT φ-MAPS

Tania-Luminiţa Costache[1]

In this paper we construct a projective covariant representation associated with a φ-map and a projective covariant quasi-representation associated with a projective (u, u')-covariant φ-map. We gave a projective version of a result in [13] *as a Stinespring's representation theorem for pairs of completely positive, symmetric, invariant, multilinear maps.*

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1. Introduction

Asadi [2] and Bhat, Ramesh and Sumesh [5] provided a representation theorem for a class of maps on Hilbert C^*-modules, as a generalization of Stinespring's representation theorem for completely positive maps on C^*-algebras.

Trivedi [20] gave a Stinespring type theorem for τ-maps in the context of von Neumann algebras. He proved a decomposition of τ-maps in terms of quasi-representations, which generalize the notion of representations of Hilbert C^*-modules on Hilbert spaces and a covariant version of this result, by defining a covariant τ-maps using the notion of C^*-correspondence.

In this paper we construct a projective covariant representation associated with a φ-map and a projective covariant quasi-representation associated with a projective (u, u')-covariant φ-map. We gave a projective version of a result in [13] as a Stinespring's representation theorem for pairs of completely positive, symmetric, invariant, multilinear maps.

Definition 1.1. *(*[17]*) A* ***pre-Hilbert*** A***-module*** *is a complex vector space* E *which is also a right* A*-module, compatible with the complex algebra structure, equipped with an* A*-valued inner product* $\langle\cdot,\cdot\rangle : E \times E \to A$ *which is* $\mathbb{C}$ *-and* A*-linear in its second variable and satisfies the following relations:*
(1) $\langle\xi, \eta\rangle^* = \langle\eta, \xi\rangle$ *for every* $\xi, \eta \in E$;
(2) $\langle\xi, \xi\rangle \geq 0$ *for every* $\xi \in E$;
(3) $\langle\xi, \xi\rangle = 0$ *if and only if* $\xi = 0$.

[1]Lecturer, University "Politehnica" of Bucharest, Faculty of Applied Sciences, Department of Mathematical Methods and Models, Spl. Independenţei 313, Bucharest, Postal Code 060042, ROMANIA, e-mail: `lumycos1@yahoo.com`

*We say that E is a **Hilbert A-module** if E is complete with respect to the topology determined by the norm $\|\cdot\|$ given by $\|\xi\| = \sqrt{\|\langle \xi, \xi \rangle\|}$.*

Definition 1.2. *([19], [1]) Let A and B be two C^*-algebras and let $M_n(A)$, respectively $M_n(B)$ denote the $*$-algebra of all $n \times n$ matrices over A, respectively B with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A, respectively B. A linear map $\varphi\colon A \to B$ is **completely positive** if the linear map $\varphi^{(n)}\colon M_n(A) \to M_n(B)$, defined by $\varphi^{(n)}([a_{ij}]_{i,j=1}^n) = [\varphi(a_{ij})]_{i,j=1}^n$ is positive for all positive integers n.*

Definition 1.3. *([16]) Let G be a locally compact group with identity e and let $\mathbb{T}$ be the group of complex numbers of modulus one. A **multiplier** ω of G is a function $\omega\colon G \times G \to \mathbb{T}$ with the properties :*
i) $\omega(x, e) = \omega(e, x) = 1$ *for all* $x \in G$;
ii) $\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z)$ *for all* $x, y, z \in G$.

Definition 1.4. *([15]) A multiplier is **normalized** if $\omega(x, x^{-1}) = 1$ for all $x \in G$.*

Definition 1.5. *([3], [4]) Let H be a Hilbert space and G a locally compact group with the identity e. A **projective representation** of G with multiplier ω is a map $u\colon G \to \mathcal{U}(H)$ such that*
i) $u_{gg'} = \omega(g, g')u_g u_{g'}$ *for all* $g, g' \in G$;
ii) $u_e = I_H$, *where I_H is the identity operator on H.*

Definition 1.6. *([21]) A C^***-dynamical system** is a triple (G, A, α), where G is a locally compact group, A is a C^*- algebra and α is a continuous action of G on A, i.e. a continuous homomorphism $\alpha\colon G \to Aut(A)$, where $Aut(A)$ is the group of automorphism of A.*

Definition 1.7. *([7]) Let (G, A, α) be a C^*-dynamical system and let u be a projective unitary representation of G on a Hilbert space H. We say that a completely positive linear map $\varphi\colon A \to \mathcal{L}(H)$ is **projective u-covariant** with respect to the C^*-dynamical system (G, A, α) if $\varphi(\alpha_g(a)) = u_g\varphi(a)u_g^*$ for all $a \in A$ and $g \in G$.*

Definition 1.8. *([5]) Let E be a Hilbert C^*-module over a C^*-algebra A and let H_1, H_2 be Hilbert spaces . Let $\varphi\colon A \to \mathcal{L}(H_1)$ be a linear map. A map $\Phi\colon E \to \mathcal{L}(H_1, H_2)$ is called :*
*i) φ**-map** if $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$, for all $x, y \in E$;*
*ii) φ**-morphism** if Φ is a φ-map and φ is a morphism;*
*iii) φ**-representation** if Φ is a φ-morphism and φ is a representation.*

2. Main results

Following the results in [5] and [8], we generalize them in the projective covariant case.

Theorem 2.1. *Let E be a Hilbert B-module, let (G, A, α) be a unital C^*-dynamical system and let H_1, H_2 be two Hilbert spaces. If u is a projective unitary representation of G on H_1 with the normalized multiplier ω and $\varphi\colon A \to \mathcal{L}(H_1)$ is a unital projective u-covariant completely positive linear map and $\Phi\colon E \to \mathcal{L}(H_1, H_2)$ a $\varphi -$ map, then there are :*

(i) *K_1, K_2 Hilbert spaces;*
(ii) *a unital representation $\pi\colon A \to \mathcal{B}(K_1)$ and a π-representation $\psi\colon E \to \mathcal{L}(K_1, K_2)$;*
(iii) *a projective unitary representation v of G on K_1 with the multiplier ω;*
(iv) *$V\colon H_1 \to K_1$ an isometry and $W\colon H_2 \to K_2$ a co-isometry such that for all $a \in A$, $g \in G$ and $x \in E$,*
(a) *$\varphi(a) = V^*\pi(a)V$,*
(b) *$u_g = V^* v_g V$,*
(c) *π is projective v-covariant,*
(d) *$\Phi(x) = W^*\psi(x)V$.*

Proof. Following the proof of Stinespring's Theorem (Theorem 1.1.1, [1]), we form the algebraic tensor product $A \otimes_{alg} H_1$ and endow it with a pre-inner product by setting $\langle a \otimes \xi, b \otimes \zeta \rangle_{A\otimes_{alg}H_1} = (\varphi(b^*a)\xi|\zeta)_{H_1}$. To obtain K_1 we divide $A \otimes_{alg} H_1$ by the kernel $N = \left\{z \in A \otimes_{alg} H_1 | \langle z, z\rangle_{A\otimes_{alg}H_1} = 0\right\}$ of $\langle \cdot, \cdot \rangle_{A\otimes_{alg}H_1}$ and complete. K_1 becomes a Hilbert space with respect to the inner product given by $\langle z_1 + N, z_2 + N\rangle_{K_1} = \langle z_1, z_2\rangle_{A\otimes_{alg}H_1}$, $z_1, z_2 \in A \otimes_{alg} H_1$.

The isometry $V\colon H_1 \to K_1$ is defined by $V\xi = 1_A \otimes \xi + N$ for all $\xi \in H_1$. It is easy to check that $V^*\colon K_1 \to H_1$ is given by $V^*(a \otimes \xi + N) = \varphi(a)\xi$.

The representation π of A on K_1 is defined by $\pi(a)(b \otimes \xi + N) = (ab) \otimes \xi + N$ for all $\xi \in H_1, a, b \in A$.

We define $v\colon G \to \mathcal{L}(K_1)$ by setting $v_g(a \otimes \xi + N) = \alpha_g(a) \otimes u_g\xi + N$ for all $a \in A, g \in G, \xi \in H_1$.

Since $\langle v_g(a \otimes \xi + N), v_g(b \otimes \zeta + N)\rangle_{K_1} =$
$\langle \alpha_g(a) \otimes u_g\xi + N, \alpha_g(b) \otimes u_g\zeta + N\rangle_{K_1} = \langle \alpha_g(a) \otimes u_g\xi, \alpha_g(b) \otimes u_g\zeta\rangle_{A\otimes_{alg}H_1} =$
$(\varphi(\alpha_g(b)^*\alpha_g(a))u_g\xi|u_g\zeta)_{H_1} = (\varphi(\alpha_g(b^*)\alpha_g(a))u_g\xi|u_g\zeta)_{H_1} =$
$(\varphi(\alpha_g(b^*a))u_g\xi|u_g\zeta)_{H_1} = (u_g\varphi(b^*a)u_g^*u_g\xi|u_g\zeta)_{H_1} = (\varphi(b^*a)\xi|\zeta)_{H_1} =$
$\langle a \otimes \xi, b \otimes \zeta\rangle_{A\otimes_{alg}H_1} = \langle a \otimes \xi + N, b \otimes \zeta + N\rangle_{K_1}$, for all $g \in G, a, b \in A, \xi, \zeta \in H_1$, v_g extends linearly to an isometry on K_1. It can be easily verified that v_g is a unitary operator on K_1.

We show now that v is a projective representation with the multiplier ω. Let $a \in A, g_1, g_2 \in G, \xi \in H_1$. Since α is a group homomorphism and u is a projective representation with the multiplier ω, we have

$$\begin{aligned} v_{g_1g_2}(a \otimes \xi + N) &= \alpha_{g_1g_2}(a) \otimes u_{g_1g_2}\xi + N = \alpha_{g_1}(a)\alpha_{g_2}(a) \otimes \omega(g_1, g_2)u_{g_1}u_{g_2}\xi + N \\ &= \omega(g_1, g_2)\alpha_{g_1}(\alpha_{g_2}(a)) \otimes u_{g_1}(u_{g_2}\xi) + N = \omega(g_1, g_2)v_{g_1}(\alpha_{g_2}(a) \otimes u_{g_2}\xi + N) \\ &= \omega(g_1, g_2)v_{g_1}v_{g_2}(a \otimes \xi + N). \end{aligned}$$

So we proved that v is a projective representation with the multiplier ω.

Let $a, b, x, y \in A$ and $\xi, \zeta \in H_1$. We have

$(\varphi(y^*b^*ax)\xi|\zeta)_{H_1} = (\varphi((by)^*ax)\xi|\zeta)_{H_1} = \langle ax \otimes \xi, by \otimes \zeta\rangle_{A\otimes_{alg}H_1}$
$= \langle \pi(a)(x \otimes \xi), \pi(b)(y \otimes \zeta)\rangle_{A\otimes_{alg}H_1} = \langle \pi(b)^*\pi(a)(x \otimes \xi), y \otimes \zeta\rangle_{A\otimes_{alg}H_1}$
$= \langle \pi(b^*)\rho(a)(x \otimes \xi), y \otimes \zeta\rangle_{A\otimes_{alg}H_1} = \langle \pi(b^*a)(x \otimes \xi), y \otimes \zeta\rangle_{A\otimes_{alg}H_1}$
$= \langle (b^*ax) \otimes \xi, y \otimes \zeta\rangle_{A\otimes_{alg}H_1} = \langle (y^*b^*ax) \otimes \xi, 1_A \otimes \zeta\rangle_{A\otimes_{alg}H_1}$
$= \langle (y^*b^*ax1_A) \otimes \xi, 1_A \otimes \zeta\rangle_{A\otimes_{alg}H_1} = \langle \pi(y^*b^*ax)(1_A \otimes \xi), 1_A \otimes \zeta\rangle_{A\otimes_{alg}H_1}$
$= \langle \pi(y^*b^*ax)V\xi, V\zeta\rangle_{A\otimes_{alg}H_1} = (V^*\pi(y^*b^*ax)V\xi|\zeta)_{H_1}$.

Hence $V^*\pi(c)V = \varphi(c), \forall c \in A$, so condition (a) is verified.

We verify now condition (b). Let $g \in G$ and $\xi \in H_1$. We have

$V^*v_gV\xi = V^*v_g(1_A \otimes \xi + N) = V^*(\alpha_g(1_A) \otimes u_g\xi + N) = V^*(1_A \otimes u_g\xi + N) = \varphi(1_A)u_g\xi = I_Hu_g\xi = u_g\xi$, because φ is unital.

We prove condition (c). Let $a, b \in A, g \in G, \xi \in H_1$.

Then $v_g\pi(a)v_g^*(b \otimes \xi + N) = v_g\pi(a)v_{g^{-1}}(b \otimes \xi + N)$
$= v_g\pi(a)(\alpha_{g^{-1}}(b) \otimes u_{g^{-1}}\xi + N) = v_g(a\alpha_{g^{-1}}(b) \otimes u_{g^{-1}}\xi + N)$
$= \alpha_g(a\alpha_{g^{-1}}(b)) \otimes (u_gu_{g^{-1}}\xi) + N = \alpha_g(a)\alpha_g(\alpha_{g^{-1}}(b)) \otimes \overline{\omega(g, g^{-1})}u_{gg^{-1}}\xi + N$
$= \alpha_g(a)\alpha_{gg^{-1}}(b) \otimes I_H\xi + N = \alpha_g(a)b \otimes \xi + N = \pi(\alpha_g(a))(b \otimes \xi + N)$, so π is projective v-covariant.

Let $K_2 = [\Phi(E)H_1]$. We define $\psi\colon E \to \mathcal{L}(K_1, K_2)$ by

$$\psi(x)(\pi(a)V\xi) = \Phi(xa)\xi,$$

for all $a \in A, \xi \in H_1, x \in E$.

We show that $\psi(x)$ is well defined and bounded.

$\|\psi(x)(\pi(a)V\xi)\|^2 = \|\Phi(xa)\xi\|^2 = \langle \Phi(xa)\xi, \Phi(xa)\xi\rangle = \langle \xi, (\Phi(xa))^*\Phi(xa)\xi\rangle$
$= \langle \xi, \varphi(a^* \langle x, x\rangle a)V\xi\rangle = \langle \xi, V^*\pi(a^* \langle x, x\rangle a)V\xi\rangle = \langle \pi(a)V\xi, \pi(\langle x, x\rangle)\pi(a)V\xi\rangle$
$\leq \|\pi(\langle x, x\rangle)\|\|\pi(a)V\xi\|^2 \leq \|x\|^2\|\pi(a)V\xi\|^2$.

Hence, $\psi(x)$ can be extended to K_1.

We prove that ψ is a π-morphism. Let $x, y \in E, a, b \in A, , \xi, \zeta \in H_1$.

$\langle \psi(x)^*\psi(y)(\pi(b)V\xi), \pi(a)V\zeta\rangle = \langle \Phi(yb)\xi, \Phi(xa)\zeta\rangle = \langle (\Phi(xa))^*\Phi(yb)\xi, \zeta\rangle$
$= \langle \varphi(\langle xa, yb\rangle)\xi, \zeta\rangle = \langle V^*\pi(a)^*\pi(\langle x, y\rangle)\pi(b)V\zeta\rangle = \langle \pi(\langle x, y\rangle)(\pi(b)V\xi), \pi(b)V\zeta\rangle$

Thus, $\psi(x)^*\psi(y) = \pi(\langle x, y\rangle)$.

Let W be the orthogonal projection onto K_2. Then $W^*\colon K_2 \to H_2$ is the inclusion map (because, obviously, $K_2 \subseteq H_2$). Hence $WW^* = I_{K_2}$, that means W is a coisometry.

For $x \in E, \xi \in H_1$, we have

$W^*\psi(x)V\xi = \psi(x)V\xi = \psi(x)(\pi(1_A)V\xi) = \Phi(x)\xi$, so (d) holds. □

Remark 2.1. *The pair of triples $((\pi, V, K_1), (\psi, W, K_2))$ is a projective covariant Stinespring representation of (φ, Φ) if conditions $(i) - (iv)$ of Theorem 2.1 are satisfied.*

Definition 2.1. *(*[20]*) Let A be a C^*-algebra, E a Hilbert A-module and let H, K be two Hilbert spaces. A map $\psi\colon E \to \mathcal{L}(H, K)$ is called **quasi-representation** if there is a $*$-homomorphism $\pi\colon A \to \mathcal{L}(H)$ such that*

$$\langle \psi(y)f_1, \psi(x)f_2\rangle = \langle \pi(\langle x, y\rangle)f_1, f_2\rangle$$

for all $x, y \in E$, $f_1, f_2 \in H$.

We say that π is associated to ψ.

Quasi-representations generalize the notion of representations of Hilbert C^*-modules on Hilbert spaces.

Definition 2.2. *([14]) Let G be a locally compact group. A continuous action of G on a full Hilbert A-module E is a group morphism $\eta\colon G \to Aut(E)$, where $Aut(E)$ is the the group of all isomorphisms of Hilbert C^*-modules from E to E, such that the map $(t, x) \longmapsto \eta_t(x)$ from $G \times E$ to E is continuous. The triple (G, E, η) is called a* ***dynamical system on Hilbert C^*-modules****.*

Remark 2.2. *([14]) Any C^*-dynamical system (G, A, α) can be regarded as a dynamical system on Hilbert C^*-modules.*

Any continuous action η of G on E induces a unique continuous action α^η of G on A such that $\alpha_t^\eta(\langle x, y\rangle) = \langle \eta_t(x), \eta_t(y)\rangle$ for all $x, y \in E, t \in G$.

Moreover, for all $x \in E$ and $a \in A$, we have $\eta_t(xa) = \eta_t(x)\alpha_t^\eta(a)$.

Definition 2.3. *Let E be a Hilbert C^*-module over a C^*-algebra A, (G, η, E) a dynamical system and H, K two Hilbert spaces, $v\colon G \to \mathcal{U}(H)$ and $w\colon G \to \mathcal{U}(K)$ two projective unitary representations. A quasi-representation $\psi\colon E \to \mathcal{L}(H, K)$ is called* ***projective*** *(w, v)****-covariant*** *with respect to (G, η, E) if*

$$\psi(\eta_t(\xi)) = w_t \psi(\xi) v_t^*$$

for all $\xi \in E, t \in G$. Then (ψ, v, w, H, K) is called a ***projective covariant quasi-representation*** *of (G, η, E).*

Definition 2.4. *Let E be a Hilbert C^*-module over a C^*-algebra A, (G, η, E) a dynamical system and H, K two Hilbert spaces, $u\colon G \to \mathcal{U}(H)$ and $u'\colon G \to \mathcal{U}(K)$ two projective unitary representations. A φ-map $\Phi\colon E \to \mathcal{L}(H, K)$ is called* ***projective*** *(u', u)****-covariant*** *with respect to (G, η, E) if*

$$\Phi(\eta_t(\xi)) = u'_t \Phi(\xi) u_t^*$$

for all $\xi \in E, t \in G$.

Remark 2.3. *If E is full and $\Phi\colon E \to F$ is a φ-map which is projective (u', u)-covariant with respect to (G, η, E), then the map φ is projective u-covariant with respect to the induced C^*-dynamical system (G, α^η, A).*

Theorem 2.2. *Let E be a full Hilbert C^*-module over a C^*-algebra A, (G, η, E) a dynamical system and H, K two Hilbert spaces, $u\colon G \to \mathcal{U}(H)$ and $u'\colon G \to \mathcal{U}(K)$ two projective unitary representations. If $\varphi\colon A \to \mathcal{L}(H)$ is completely positive and $\Phi\colon E \to \mathcal{L}(H, K)$ is a φ-map which is (u', u)-covariant with respect to (G, η, E) , then there are*

1) *a) a Hilbert space X with a projective covariant representation (π, v) of (G, α^η, A)*

   b) an isometry $V : H \to X$ such that

   i) *$\varphi(a)\xi = V^*\pi(a)V\xi$ for all $a \in A, \xi \in H$*

ii) $v_t V\xi = V u_t \xi$ *for all* $t \in G, \xi \in H$

2) *a Hilbert space* Y *and a projective covariant quasi-representation* (ψ, v, w, X, Y) *of* (G, η, E) *such that* π *is associated to* ψ

3) *a coisometry* $S\colon K \to Y$ *such that*

a) $\Phi(x)\xi = S^*\psi(x)V\xi$ *for all* $x \in E, \xi \in H$

b) $w_t S l = S u'_t l$ *for all* $t \in G, l \in K$

Proof. 1) a) Let $\langle \cdot, \cdot \rangle$ be a A-valued positive semi-inner product on $A \otimes_{alg} H$ defined by

$$\langle a \otimes \xi, b \otimes \zeta \rangle_{A \otimes_{alg} H} = (\varphi(a^* b)\xi | \zeta)_H$$

for all $a, b \in A, \xi, \zeta \in H$.

Let $N = \{ z \in A \otimes_{alg} H | \langle z, z \rangle_{A \otimes_{alg} H} = 0 \}$.

$\langle \cdot, \cdot \rangle$ extends naturally on the quotient $A \otimes_{alg} H/N$. To obtain X we complete $A \otimes_{alg} H/N$.

Let $\pi\colon A \to \mathcal{L}(X)$ defined by

$$\pi(a)(b \otimes \xi + N) = ab \otimes \xi + N$$

for all $a, b \in A, \xi \in H$

and $v\colon G \to \mathcal{U}(X)$ defined by

$$v_t(a \otimes \xi + N) = \alpha_t^\eta(a) \otimes u_t(\xi) + N$$

for all $a \in A, \xi \in H, t \in G$.

Since $\langle v_t(a \otimes \xi + N), v_t(b \otimes \zeta + N) \rangle =$ $\langle \alpha_t^\eta(a) \otimes u_t(\xi) + N, \alpha_t^\eta(b) \otimes u_t(\zeta) + N \rangle = (\varphi((\alpha_t^\eta(a))^* \alpha_t^\eta(b)) u_t(\xi) | u_t(\zeta))_H =$ $(\varphi(\alpha_t^\eta(a^*)\alpha_t^\eta(b)) u_t(\xi) | u_t(\zeta))_H = (u_t^* \varphi(\alpha_t^\eta(a^* b)) u_t(\xi) | \zeta)_H =$ $(\varphi(a^* b)\xi | \zeta)_H = \langle a \otimes \xi + N, b \otimes \zeta + N \rangle$, by the covariance of φ, v_t extends to an isometry on X.

We verify that v is a projective representation with the multiplier ω. Let $a \in A, \xi \in H, t_1, t_2 \in G$. We have

$v_{t_1 t_2}(a \otimes \xi + N) = \alpha_{t_1 t_2}^\eta(a) \otimes u_{t_1 t_2}(\xi) + N = \alpha_{t_1}^\eta \alpha_{t_2}^\eta(a) \otimes \omega(t_1, t_2) u_{t_1} u_{t_2}(\xi) +$ $N = \omega(t_1, t_2) v_{t_1}(\alpha_{t_2}^\eta(a) \otimes u_{t_2}(\xi) + N) = \omega(t_1, t_2) v_{t_1} v_{t_2}(a \otimes \xi + N)$

We prove that π is a covariant representation.

$v_t \pi(a) v_t^*(b \otimes \zeta + N) = v_t \pi(a)(\alpha_{t^{-1}}^\eta(b) \otimes u_{t^{-1}}(\zeta) + N) =$ $v_t(a \alpha_{t^{-1}}^\eta(b) \otimes u_{t^{-1}}(\zeta) + N) = \alpha_t^\eta(a) \alpha_t^\eta(\alpha_{t^{-1}}^\eta(b)) \otimes u_t u_{t^{-1}} \zeta + N =$ $\alpha_t^\eta(a) b \otimes \omega(t, t^{-1}) u_{tt^{-1}} \zeta + N = \alpha_t^\eta(a) b \otimes \zeta + N = \pi(\alpha_t^\eta(a))(b \otimes \zeta + N)$

b) Let $V\colon H \to X$ defined by

$$V\xi = 1_A \otimes \xi + N$$

for all $\xi \in H$. It can be easily checked that $V^*\colon X \to H$,

$$V^*(a \otimes \xi + N) = \varphi(a)\xi.$$

i) This condition is verified as in the proof of Theorem 2.1 (iv) (a).

ii) $v_t V\xi = v_t(1_A \otimes \xi + N) = \alpha_t^\eta(1_A) \otimes u_t(\xi) + N = 1_A \otimes u_t(\xi) + N = V u_t \xi$

2) Let $Y = [\Phi(E)H]$.

Let $w_t = u'_t/Y$ for all $t \in G$.Then $t \longmapsto w_t$ is a projective representation of G.

We define $\psi\colon E \to \mathcal{L}(X, Y)$ by $\psi(x)(\pi(a)V\xi) = \Phi(xa)\xi$ for all $a \in A, \xi \in H, x \in E$.

We prove that ψ is a quasi-representation:

$\langle\psi(y)(\pi(a)V\xi), \psi(x)(\pi(b)V\zeta)\rangle = \langle\Phi(ya)\xi, \Phi(xb)\zeta\rangle = \langle\varphi(\langle ya, xb\rangle)\xi, \zeta\rangle = \langle V^*\pi(\langle ya, xb\rangle)V\xi, \zeta\rangle = \langle\pi(\langle ya, xb\rangle)V\xi, V\zeta\rangle = \langle\pi(b)^*\pi(\langle x, y\rangle)\pi(a)V\xi, V\zeta\rangle = \langle\pi(\langle x, y\rangle)\pi(a)V\xi, \pi(b)V\zeta\rangle$, for all $a \in A, x, y \in E, \xi, \zeta \in H$.

We prove that ψ is projective (w, v)-covariant.

For all $a \in A, t \in G, x \in E, \xi \in H$, we have

$\psi(\eta_t(x))(\pi(a)V\xi) = \Phi(\eta_t(x)a)\xi = \Phi(\eta_t(x)\alpha_t(\alpha^\eta_{t^{-1}}(a)))\xi = \Phi(\eta_t(x\alpha^\eta_{t^{-1}}(a))\xi = u'_t\Phi(x\alpha^\eta_{t^{-1}}(a))u^*_t\xi$

On the other hand, by 1) a) and 1) b) ii), $w_t\psi(x)v^*_t(\pi(a)V\xi) = w_t\psi(x)v_{t^{-1}}(\pi(a)V\xi) = w_t\psi(x)\pi(\alpha^\eta_{t^{-1}}(a))v_{t^{-1}}V\xi = w_t\psi(x)\pi(\alpha^\eta_{t^{-1}}(a))Vu_{t^{-1}}\xi = w_t\psi(x)\pi(\alpha^\eta_{t^{-1}}(a))Vu^*_t\xi = w_t\Phi(x\alpha^\eta_{t^{-1}}(a))u^*_t\xi = u'_t\Phi(x\alpha^\eta_{t^{-1}}(a))u^*_t\xi$

3) By Theorem 5.2, [18], there is an orthogonal projection S from K into Y.

a)$S^*\psi(x)V\xi = \psi(x)V\xi = \psi(x)(\pi(1_A)V\xi) = \Phi(x1_A)\xi = \Phi(x)\xi$

b) It is clear.

□

Definition 2.5. *(*[6], [13]*) Let A be a C^*-algebra, H be a Hilbert space and k be a positive integer.*

*A k-linear map $\varphi\colon A^k \to \mathcal{L}(H)$ is called **symmetric** if $\varphi = \varphi^*$, where $\varphi^*\colon A^k \to \mathcal{L}(H)$ is the k-linear map given by $\varphi^*(a_1, a_2, \ldots, a_k) = \varphi(a^*_k, \ldots, a^*_2, a^*_1)^*$.*

*A k-linear map $\varphi\colon A^k \to \mathcal{L}(H)$ is called **completely bounded** if*

$$\|\varphi\|_{cb} = \sup_n \|\varphi_n\| < \infty,$$

where $\varphi_n\colon M_n(A) \to \mathcal{L}(H^n)$, $\varphi_n(A_1, A_2, \ldots A_k) = \left[\Sigma^n_{l,r,\ldots,t=1}\varphi(a_{1il}, a_{2lr}, \ldots, a_{ktj})\right]^n_{i,j=1}$, for $A_l = [a_{lij}]^n_{i,j=1} \in M_n(A), l = \overline{1,k}$ and $\|\varphi_n\| = \sup\left\{\|\varphi_n(A_1, A_2, \ldots, A_k)\|; A_l \in M_n(A), \|A_l\| \le 1, l = \overline{1,k}\right\}$.

*A k-linear map $\varphi\colon A^k \to \mathcal{L}(H)$ is called **completely positive** if*

$$\varphi_n(A_1, A_2, \ldots, A_k) \ge 0$$

for all $(A_1, A_2, \ldots, A_k) \in M_n(A)^k$ with $(A_1, A_2, \ldots, A_k) = (A^*_k, \ldots\ldots, A^*_2, A^*_1)$ and $A_m \ge 0$ if kis odd and $m = \left[\frac{k+1}{2}\right]$ and for all $n \in \mathbb{N}$.

Definition 2.6. *(*[11], [12], [13]*) A k-linear map $\varphi\colon A^k \to \mathcal{L}(H)$ is called **invariant** if :*

(a) for odd $k = 2m - 1$,

$$\varphi(a_1c_1, \ldots, a_{m-1}c_{m-1}, a_m, a_{m+1}, \ldots, a_k) = \varphi(a_1, \ldots, a_{m-1}, a_m, c_ma_{m+1}, \ldots, c_1a_k)$$

for all $a_1, \ldots, a_k, c_1, \ldots, c_m \in A$, *(b) for even $k = 2m$,*

$$\varphi(a_1c_1, \dots, a_mc_m, a_{m+1}, \dots, a_k) = \varphi(a_1, \dots, a_{m-1}, a_m, c_{m-1}a_{m+1}, \dots, c_1a_k)$$

for all $a_1, \dots, a_k, c_1, \dots, c_{m-1} \in A$.

Definition 2.7. *([13]) A **k-representation** of A on H is a k-linear map $\pi\colon A^k \to \mathcal{L}(H)$ with the properties:*

(i) for each $l \in \{1, \dots, k\}$, the map $\pi_l\colon A \to \mathcal{L}(H)$ defined by $\pi_l(a) = \pi(1_A, \dots, 1_A, a, 1_A, \dots, 1_A)$ is a representation of A on H, where a is on the l-th position;

(ii) $\pi(a_1, \dots, a_k) = \pi_1(a_1) \cdots \pi_k(a_k)$ for all $a_1, \dots, a_k \in A$.

Definition 2.8. *([13]) Let A be a C^*-algebra, H, K be two Hilbert spaces, E be a Hilbert A-module and k be a positive integer. Let $\varphi\colon A^k \to \mathcal{L}(H)$ be a k-linear map and $\Phi\colon E^k \to \mathcal{L}(H, K)$ a map. Then*

*(1) Φ is called a **φ-map** if*

$$\Phi(x_1, \dots, x_k)^*\Phi(y_1, \dots, y_k) = \varphi(\langle x_1, y_k\rangle, \dots, \langle x_k, y_1\rangle)$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E$.

*(2) Φ is called a **φ-representation** of E if Φ is a φ-map and φ is a k-representation of A on H. In this case we say that the pair (Φ, φ) is a **k-representation** of E on H and K.*

*We say that a φ-map Φ is **symmetric** (respectively, **invariant**, **completely bounded**, **completely positive**) if the corresponding map φ is symmetric (respectively, invariant, completely bounded, completely positive). Similarly we can define symmetric, invariant, completely bounded, completely positive φ-representations of E.*

Let (G, A, α) be a C^*-dynamical system. The action α naturally induces the action $\widetilde{\alpha}\colon G \to \mathrm{Aut}(A^k)$ by $\widetilde{\alpha}_t(a_1, \dots, a_k) = (\alpha_t(a_1), \dots, \alpha_t(a_k))$ for all $a_1, \dots, a_k \in A$ and $t \in G$. ([11])

Following the definition in [11], [10], we introduce the notions of projective u-covariant k-linear map and projective $(\widetilde{\tau}, v, u)$-covariant k-linear map.

Definition 2.9. *Let $u\colon G \to \mathcal{L}(H)$ be a projective unitary representation of G on H. A k-linear map $\varphi\colon A^k \to \mathcal{L}(H)$ is called **projective u-covariant** if*

$$\varphi(\widetilde{\alpha}_t(a_1, \dots, a_k)) = \varphi(\alpha_t(a_1), \dots, \alpha_t(a_k)) = u_t\varphi(a_1, \dots, a_k)u_t^*$$

for all $a_1, \dots, a_k \in A$ and $t \in G$.

Definition 2.10. *Let E be a Hilbert A-module, H, K be two Hilbert spaces and $u\colon G \to \mathcal{L}(H)$ be a projective unitary representation. For any map $\tau_t\colon E \to E$ we define $\widetilde{\tau}_t\colon E^k \to E^k$ by $\widetilde{\tau}_t(x_1, \dots, x_k) = (\tau_t(x_1), \dots, \tau_t(x_k))$. A k-linear map $\Phi\colon E^k \to \mathcal{L}(H, K)$ is called **projective $(\widetilde{\tau}, v, u)$-covariant** if there is a map $\tau\colon G \to \mathcal{B}_A(E)$ and a projective unitary representation $v\colon G \to \mathcal{L}(K)$ such that*

$$\Phi(\widetilde{\tau}_t(x_1, \dots, x_k)) = \Phi(\tau_t(x_1), \dots, \tau_t(x_k)) = v_t\Phi(x_1, \dots, x_k)u_t^*$$

for all $x_1, \ldots, x_k \in E$ *and* $t \in G$.

We prove now the multilinear projective version of the Stinespring's representation theorem for a pair of two k-linear maps. ([5], [11], [12], [13])

Theorem 2.3. *Let* (G, A, α) *be a unital* C^**-dynamical system,* $u\colon G \to \mathcal{L}(H)$ *be a projective unitary representation of* G *on a Hilbert space* H *with the multiplier* ω. *Let* $\varphi\colon A^k \to \mathcal{L}(H)$ *be an invariant, symmetric, completely positive* k*-linear map,* K *be a Hilbert space,* E *be a Hilbert* A*-module and* $\Phi\colon E^k \to \mathcal{L}(H, K)$ *be a* φ*-map. If* φ *is projective* u*-covariant and* Φ *is projective* $(\widetilde{\tau}, v, u)$*-covariant, then there are* H_φ, K_Φ *two Hilbert spaces, an invariant, symmetric* k*-representation* (Π_Φ, π_φ) *of* E *on* H_φ *and* K_Φ*, a projective unitary representation* σ *of* G *on* H_φ*, a bounded linear operator* $V_\varphi \in \mathcal{L}(H, H_\varphi)$ *and a coisometry* $W_\Phi\colon K \to K_\Phi$ *such that:*

a) *i)* $H_\varphi = \overline{sp}\left\{\pi_\varphi(a_1, \ldots, a_k)V_\varphi\xi \quad a_1, \ldots, a_k \in A, \xi \in H\right\}$
   ii) $K_\Phi = \overline{sp}\left\{\Pi_\Phi(x_1, \ldots, x_k)V_\varphi\xi \ \ x_1, \ldots, x_k \in E, \xi \in H\right\}$

b) $\varphi(a_1, \ldots, a_k) = V_\varphi^*\pi_\varphi(a_1, \ldots, a_k)V_\varphi$ *for all* $a_1, \ldots, a_k \in A$;

c) π_φ *is projective* σ*-covariant :* $\pi_\varphi(\widetilde{\alpha}_t(a_1, \ldots, a_k)) = \sigma_t\pi_\varphi(a_1, \ldots, a_k)\sigma_t^*$ *for all* $a_1, \ldots, a_k \in A, t \in G$;

d) $V_\varphi u_t = \sigma_t V_\varphi$ *for all* $t \in G$;

e) Π_Φ *is projective* $(\widetilde{\tau}, v, \widetilde{u})$*-covariant, where* $\widetilde{u}_t = id_{A^{\otimes m}} \otimes u_t$;

f) $v_t\Pi_\Phi(x_1, \ldots, x_k)\sigma_{t^{-1}} = \Pi_\Phi(\widetilde{\tau}_t(x_1, \ldots, x_k))(\alpha_{t^{-1}}^{\bigotimes^m} \otimes id_H)$ *for all* $x_1, \ldots, x_k \in E, t \in G$.

The triple of pairs $((\Pi_\Phi, \pi_\varphi), (H_\varphi, K_\Phi), (V_\varphi, W_\Phi))$ is called the projective covariant Stinespring's representation associated to a k-linear φ-map.

Proof. By [10], [9], there is $(\pi_\varphi, H_\varphi, V_\varphi)$ the minimal Stinespring's representation associated to φ such that $a)i)$ and $b)$ hold,

There are $(\Pi_\Phi, K_\Phi, W_\Phi)$ as in Theorem 3.3 ([13]) and $a)ii)$ holds.

We define the unitary representation σ of G on H_φ as in the proof of Proposition 4.1, [13].

We prove that σ is a projective representation with the multiplier ω.

Let $a_1, \ldots, a_k \in A, \xi \in H, t_1, t_2 \in G$. We have

$$\sigma_{t_1t_2}(\pi_\varphi(a_1, \ldots, a_k)V_\varphi\xi) = \pi_\varphi(\widetilde{\alpha}_{t_1t_2}(a_1, \ldots, a_k))V_\varphi u_{t_1t_2}\xi =$$
$$\pi_\varphi(\widetilde{\alpha}_{t_1}(\widetilde{\alpha}_{t_2}(a_1, \ldots, a_k)))V_\varphi\omega(t_1, t_2)u_{t_1}u_{t_2}\xi =$$
$$\omega(t_1, t_2)\pi_\varphi(\widetilde{\alpha}_{t_1}(\widetilde{\alpha}_{t_2}(a_1, \ldots, a_k)))V_\varphi u_{t_1}u_{t_2}\xi =$$
$$\omega(t_1, t_2)\sigma_{t_1}(\pi_\varphi(\alpha_{t_2}(a_1, \ldots, a_k))V_\varphi u_{t_2}\xi) = \omega(t_1, t_2)\sigma_{t_1}\sigma_{t_2}(\pi_\varphi(a_1, \ldots, a_k)V_\varphi\xi)$$

By Proposition 4.1 and Theorem 4.4 ([13]), $c)$, $d)$, $e)$ and $f)$ are satisfied. □

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REFERENCES

[1] *W. Arveson*, Subalgebra of C^*-algebras, Acta Math. **123** (1969), 141-224.

[2] *Mohammad B. Asadi*, Stinespring theorem for Hilbert C*-modules, J. Operator Theory **62**, No. 2 (2009), 235-238.

[3] *B. Bagchi, G. Misra*, A note on the multipliers and projective representations of semi-simple Lie groups, The Indian Journal of Statistics, Vol. **62**, Series A, Pt. 3 (2000).

[4] *E. Bedos, R. Conti*, On infinite tensor products of projective unitary representations, Rocky Mountain J. Math. **34**, (2004), 467-493.

[5] *B.V. Rajarama Bhat, G. Ramesh, K. Sumesh*, Stinespring's theorem for maps on Hilbert C*-modules, J. Operator Theory **68**, No. 1 (2012), 173-178.

[6] *E. Christensen, A. Sinclair*, Representations of completely bounded multilinear operators, J. Funct. Anal. **72** (1987), 151-181.

[7] *T.-L. Costache*, Projective version of Stinespring type theorems, Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie, Vol. **53**, No.2 (2010), 109-124.

[8] *T.-L. Costache*, Th KSGNS construction associated with a projective u-covariant completely positive linear map, UPB Sci. Bull., Series A, Vol. **75**, Iss.2 (2013), 11-20.

[9] *T.-L. Costache*, Multilinear Completely Bounded Projective u-Covariant Maps Extended on Twisted Crossed Products , Recent Advances in Mathematics and Computer in Business, Economics, Biology and Chemistry, Proceedings of "the 11th WSEAS International Conference on Mathematics and Computers in Biology and Chemistry (MCBC'10), "G. Enescu" University, Iasi, Romania, June 13-15, 2010" (2010), 54-61.

[10] *T.-L. Costache*, Extensions on twisted crossed products of completely positive invariant projective u-covariant multi-linear maps, BSG Proceedings, Vol. **17** (2010), 56-67.

[11] *J. Heo*, Representations of invariant multilinear maps and Hilbert C^*-modules, Israel J. Math. **118** (2000), 125-146.

[12] *J. Heo*, A Radon-Nikodym theorem for completely positive invariant multilinear maps and its applications, Math. Proc. Comb. Phil. Soc. **132** (2002), 181-192.

[13] *J. Heo, M. Joiţa*, A Stinespring type theorem for completely positive multilinear maps on Hilbert C^*-modules, Linear and Multilinear Algebra (2017)

[14] *M. Joiţa*, Crossed products of locally C^*-algebras and strong Morita equivalence, Mediterr. J.Math. 5 (2008), 4, 467-492.

[15] *A. Kleppner*, Continuity and measurability of multiplier and projective representations, J. Functional Analysis **17** (1974), 214-226.

[16] *A. Kleppner*, The structure of some induced representations, Duke Math. J. **29** (1962), 555-572.

[17] *E. C. Lance*, Hilbert C^*-modules. A toolkit for operator algebraists, London Math. Soc. Lect. Notes Ser. 210, Cambridge University Press, Cambridge (1995).

[18] *M. Skeide*, Generalised matrix C^*-algebras and representations of Hilbert modules, Math. Proc. R. Ir. Acad. 100 A (2000), no. 1, 11-38.

[19] *W. F. Stinespring*, Positive functions on C*-algebras, Proc. Amer. Math. Soc. **6** (1955), 211-216.

[20] *H. Trivedi*, A covariant Stinespring type theorem for τ-maps, Surveys in Mathematics and its Applications, Vol. **5** (2010)....

[21] *D. Williams*, Crossed Products of C^*- algebras, Mathematical Surveys and Monographs, Vol.134, Amer. Math. Soc., Providence, RI, 2007.