U.P.B. Sci. Bull., Series A, Vol. 80, Iss. 2, 2018 ISSN 1223-7027

COMPARISON OF STANDARD AND (EXTENDED) d-HOMOLOGIES

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In this article we compare the standard homology, d-homology and the extended d-homology functors with respect to a kernel transformation d. We also compare the homology and extended homology functors with respect to two kernel transformations.

Keywords: abelian category, standard homology, (extended) d-homology, category of R-modules.
MSC2010: 18E10, 55N20.

1. Introduction and Preliminaries

The definition of the standard homology functor has been extended from the category of R-modules to abelian categories in [9]. In [5] we have defined the homology with respect to a kernel transformation d, also called the d-homology, and in [7] have defined extended d-homology in more general categories. In this section we have given the definition of d-homology, extended d-homology and some of the results obtained in [5]. In Section 2, we have compared the standard homology as given in [9] and the d-homology, by giving a natural transformation from the standard homology functor to the d-homology functor. Standard homology and extended d-homology have been compared in [4]. In Section 3, we have compared the d- homology and the extended d-homology, by giving a natural transformation from the extended d- homology functor to the d-homology functor. Then we have considered conditions under which this natural transformation is a natural isomorphism. Also we have shown, in an abelian category, the d- homology is the extended d-homology with respect to a particular kernel transformation d. Some other results are also given at the end of this section. In Sections 4 and 5, respectively we have compared the extended homology functors and homology functors with respect to two kernel transformations. Throughout the manuscript we let R-mod be the category of R-modules over a commutative ring with unity.

To this end, for a pointed category $\mathcal{C}$, following the notation of [1, 5, 7], we recall:

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(i) for $f : A \to B$, the maps $K_f \xrightarrow{k_f} A$, $B \xrightarrow{c_f} C_f$ and $P_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A$ are respectively, the kernel, the cokernel and the kernel pair of f; so $Equ(f,g) \xrightarrow{equ(f,g)} A$ denotes the equalizer and $B \xrightarrow{coe(f,g)} Coe(f,g)$ the coequalizer of a pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$.

(ii) [2, 5]. The image I_f of f is the coequalizer of the kernel pair of f. Furthermore $f = m_f \circ e_f$ in which $e_f = coe(\pi_1, \pi_2)$.

(iii) [5]. Given the diagram below in which the squares are commutative and the rows are coequalizers, i is the unique map making the right square commute. Furthermore, i is a regular epi.

$$\begin{array}{ccccc} A & \underset{g}{\overset{f}{\rightrightarrows}} & B & \xrightarrow{q} & C \\ {\scriptstyle r}\downarrow & & \| & & \vdots\, {\scriptstyle i} \\ A' & \underset{g'}{\overset{f'}{\rightrightarrows}} & B & \xrightarrow[q']{} & C' \end{array}$$

For a pointed category $\mathcal{C}$ with pullbacks and pushouts, let $\bar{\mathcal{C}}$ be the arrow category and $\hat{\mathcal{C}}$ be the pair-chain category of $\mathcal{C}$. Pair-chains are composable pairs, (f,g), of morphisms of $\mathcal{C}$, such that $gf = 0$ and morphisms from (f,g) to (f',g') are triples (α,β,γ) making the following squares commutative:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' \end{array}$$

Let $K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be the kernel functor and $I : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be the image functor see[5]. Define $\mathbf{K} =: K \circ pr_2 : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ and $\mathbf{I} = I \circ pr_1 : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$.

(iv) The natural transformation $j : \mathbf{I} \longrightarrow \mathbf{K} : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ takes the object $(f,g) \in \hat{\mathcal{C}}$ to j_{fg} and the morphism $(\alpha,\beta,\gamma) : (f,g) \longrightarrow (f',g')$ to the following commutative square.

$$\begin{array}{ccc} I_f & \xrightarrow{j_{fg}} & K_g \\ {\scriptstyle I(\alpha,\beta)}\downarrow & & \downarrow{\scriptstyle K(\beta,\gamma)} \\ I_{f'} & \xrightarrow[j_{f'g'}]{} & K_{g'} \end{array}$$

In a pointed regular (homological, semiabelian or abelian) category j_{fg} is monic. See [2, 5].

(v) The homology functor H^s that takes the object $(f,g) \in \hat{\mathcal{C}}$ to $H^s_{fg} = Coker(j_{fg})$ is called standard homology functor. See [2, 8, 9]

(vi) Let S be the squaring functor. A kernel transformation in a $\mathcal{C}$ is a natural transformation $d : S \circ K \longrightarrow K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ such that for all (f,g) in $\hat{\mathcal{C}}$, the pullback, $j^*_{fg} : R_{fg} \longrightarrow K^2_g$, of j_{fg} along d_g and the coequalizer of the pair

$j_1 = pr_1 \circ j^*_{fg}$ and $j_2 = pr_2 \circ j^*_{fg}$ exist, where pr_1 and pr_2 are the projection maps.

Let R be a commutative ring with unity. Any kernel transformation in R-mod is of the form $d = rpr_1 + spr_2$, for some $r, s \in R$.

(vii) The d-homology functor $H^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ takes $(f, g) \in \hat{\mathcal{C}}$ to $H^d_{fg} = Coe(j_1, j_2)$ and the morphism (α, β, γ) to $H^d(\alpha, \beta, \gamma)$. We have the following commutative diagram.

$$\begin{array}{ccc} K_g & \xrightarrow{q_1} & H^d_{fg} \\ {\scriptstyle K(\beta,\gamma)}\downarrow & & \downarrow{\scriptstyle H^d(\alpha,\beta,\gamma)} \\ K_{g'} & \xrightarrow[q'_1]{} & H^d_{f'g'} \end{array}$$

(viii) Let $m : A \longrightarrow C$ and $j : B \longrightarrow C$ be two maps in $\mathcal{C}$. Define $A +_C B$ also denoted by $A+B$ by the pushout of the pair (α, γ) where $B \xleftarrow{\gamma} P_{jm} \xrightarrow{\alpha} A$ is the pullback of (j, m).

Let d be a natural transformation from $S \circ K$ to K, $(f, g) \in \hat{\mathcal{C}}$ and Δ be the diagonal map. we have the maps $m_{d_g\Delta_g} : I_{d_g\Delta_g} \longrightarrow K_g$ (such that $m_{d_g\Delta_g} e_{d_g\Delta_g} = d_g\Delta_g$) and $j_{fg} : I_f \longrightarrow K_g$. The sum $I_{d_g\Delta_g} + I_f$ is therefore obtained by the following diagrams:

$$\begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & I_{d_g\Delta_g} \\ {\scriptstyle\gamma}\downarrow & \text{pb} & \downarrow{\scriptstyle m_{d_g\Delta_g}} \\ I_f & \xrightarrow[j_{fg}]{} & K_g \end{array} \quad \text{and} \quad \begin{array}{ccc} P_{jm} & \xrightarrow{\alpha} & I_{d_g\Delta_g} \\ {\scriptstyle\gamma}\downarrow & \text{po} & \downarrow{\scriptstyle h} \\ I_f & \xrightarrow[i]{} & I_{d_g\Delta_g} + I_f \end{array}$$

Commutativity of the left diagram implies that there is a unique map $\beta : I_{d_g\Delta_g} + I_f \longrightarrow K_g$ making the following diagram commute.

$$\begin{array}{cccc} P_{jm} & \xrightarrow{\alpha} & I_{d_g\Delta_g} & \\ {\scriptstyle\gamma}\downarrow & \text{po} & \downarrow{\scriptstyle h} & \searrow{\scriptstyle m_{d_g\Delta_g}} \\ I_f & \xrightarrow[i]{} & I_{d_g\Delta_g} + I_f & \\ & \searrow{\scriptstyle j_{fg}} & & \overset{\beta}{\searrow} \ K_g \end{array}$$

With $\bar{H}^d_{fg} = C_\beta$, the cokernel of β, we have:

(ix) For each morphism $(\sigma, \delta, \zeta) : (f, g) \longrightarrow (f', g')$ in $\hat{\mathcal{C}}$, there is a unique map $\bar{H}^d(\sigma, \delta, \zeta) : \bar{H}^d_{fg} \longrightarrow \bar{H}^d_{f'g'}$, such that the following diagram commutes:

$$\begin{array}{ccc} K_g & \xrightarrow{c_\beta} & \bar{H}^d_{fg} \\ {\scriptstyle K(\delta,\zeta)}\downarrow & & \downarrow{\scriptstyle \bar{H}^d(\sigma,\delta,\zeta)} \\ K_{g'} & \xrightarrow[c_{\beta'}]{} & \bar{H}^d_{f'g'} \end{array}$$

The mapping $\bar{H}^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ that takes the object $(f,g) \in \hat{\mathcal{C}}$ to $\bar{H}^d_{fg}$ and the morphism (α, β, γ) to $\bar{H}^d(\alpha, \beta, \gamma)$ is a functor.
The functor $\bar{H}^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ is called the extended d-homology or the extended homology functor with respect to the kernel transformation d.

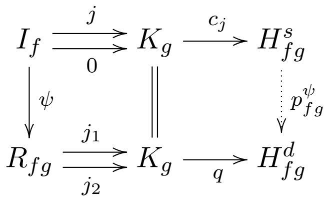
2. Standard homology versus d-homology

In this section, unless stated otherwise, we assume $\mathcal{C}$ is a pointed category with pullbacks, cokernels and coequalizer of kernel pairs, and we investigate the relation between the standard homology and the d-homology.

For two morphisms $f : C \to A$ and $g : C \to B$ there is a unique morphism $\langle f, g\rangle : C \to A \times B$ such that $pr_1 \circ \langle f, g\rangle = f$ and $pr_2 \circ \langle f, g\rangle = g$.

Theorem 2.1. [6]. *Let d be a kernel transformation in $\mathcal{C}$. There is a pointwise regular epi natural transformation $p : H^s \longrightarrow H^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$.*

Proof. Since $j \circ d_h \circ \langle 1, 0\rangle = d_g \circ \langle j, 0\rangle$ for $h : I_f \to 0$ and R_{fg} is a pullback of the pair (j_{fg}, d_g), there are unique maps ψ and p^{ψ}_{fg} such that $j^* \circ \psi = \langle j, 0\rangle$ and the following diagram commutes.



Furthermore by (iii) of 1, p_{fg} is regular epic. Simillarly there are unique maps ϕ and p^{ϕ}_{fg} such that $j_2 \circ \phi = j$ and $j_1 \circ \phi = 0$. Since $P^{\psi}_{fg} \circ c_j = q = p^{\phi}_{fg} \circ c_j$, $p^{\psi}_{fg} = p^{\phi}_{fg}$ and we denote it by p_{fg}. □ □

Lemma 2.1. *If for $(f,g) \in \hat{\mathcal{C}}$, $\psi : I_f \longrightarrow R_{fg}$ or $\phi : I_f \longrightarrow R_{fg}$ is epic, then $H^s_{fg} \cong H^d_{fg}$.*

Proof. $H^d_{fg} = Coe(j_1, j_2) \cong Coe(j_1 \circ \psi, j_2 \circ \psi) = Coker(j) = H^s_{fg}$. □

Theorem 2.2. [5]. *In an abelian category $\mathcal{C}$ if $d = pr_1 - pr_2$, then $H^d = H^s$.*

3. d-homology versus Extended d-homology

In this section we assume $\mathcal{C}$ is a pointed category with pullback and pushout and for a given natural transformation $d : S \circ K \to K$ we find relations between the d-homology H^d and the extended d-homology $\bar{H}^d$.

We know there are natural transformations $p : H^s \to H^d$ and $u : H^s \to \bar{H}^d$ which are pointwise regular epic. Furthermore for $(f,g) \in \hat{\mathcal{C}}$ there are $\psi : I_f \to R_{fg}$ and $i : I_f \to I_f + I_{d_g \Delta_g}$ such that $\beta \circ i = j$ and $j_1 \circ \psi = j$. Then we have:

Lemma 3.1. *If i is epic, then there is an epi $n_{fg} : \bar{H}^d_{fg} \to H^d_{fg}$.*

Proof. By Lemma 3.2 u_{fg} in [4] is an isomorphism. Define $n_{fg} = p_{fg} \circ u^{-1}_{fg}$. □

Theorem 3.1. *If for all $(f,g) \in \hat{\mathcal{C}}$, i is epic, then there is a pointwise regular epi natural transformation $n : \bar{H}^d \to H^d$.*

Proof. By hypothesis u is a natural isomorphism. Define $n = p \circ u^{-1}$. p is a regular epi, so is n. □

To the end of this section we assume $\mathcal{C}$ is an abelian category.

For $(f,g) \in \hat{\mathcal{C}}$, let j^* be the pullback of j along d_g and q be the coequalizer $(pr_1 j^*, pr_2 j^*)$. Then $q = coker(pr_1 j^* - pr_2 j^*) = coker(\delta j^*)$ in which $\delta = pr_1 - pr_2$. Factoring δj^* as $\delta j^* = me$, since e is epic, $q = coker(me) = coker(m)$.

Lemma 3.2. *Let $(f,g) \in \hat{\mathcal{C}}$. $q d_g \Delta_g = 0$ if and only if $d_g \Delta_g$ factors through m, i.e. there is $\alpha : K_g \to I_{\delta j^*}$ such that $d_g \Delta_g = m\alpha$.*

Proof. Since m is monic, $m = ker(coker(m)) = ker(q)$. The result then follows. □

Note: In the proof of theorem 2.1 there exists a unique map ψ, such that $j^*\psi = \langle j, 0\rangle$. Furthermore $j = \delta\langle j, 0\rangle = \delta j^* \psi$ so that $qj = q\delta j^*\psi = 0$.

Lemma 3.3. *Let $(f,g) \in \hat{\mathcal{C}}$. With the map $\beta : I_{d_g\Delta_g} + I_f \to K_g$ and $\delta j^* = me$, $q\beta = 0$ if and only if $d_g\Delta_g$ factors through m.*

Proof. Suppose $q\beta = 0$. Since $\beta h = m_{d_g\Delta_g}$ and $m_{d_g\Delta_g} e_{d_g\Delta_g} = d_g\Delta_g$, $q d_g \Delta_g = q m_{d_g\Delta_g} e_{d_g\Delta_g} = q\beta h e_{d_g\Delta_g} = 0$. By the above lemma $d_g\Delta_g$ factors through m.

Conversely suppose $d_g\Delta_g$ factors through m, so by the above lemma $q d_g \Delta_g = 0$. So $q m_{d_g\Delta_g} e_{d_g\Delta_g} = 0$. Since $e_{d_g\Delta_g}$ is epic, $q m_{d_g\Delta_g} = 0$. Since $m_{d_g\Delta_g} = \beta h$, $q\beta h = 0$. On the other hand since $qj = 0$ and $j = \beta i$, $q\beta i = 0$. The pushoutness of $I_{d_g\Delta_g} + I_f$ implies $q\beta = 0$. □

Proposition 3.1. *Let $(f,g) \in \hat{\mathcal{C}}$. If $d_g\Delta_g$ factors through m, then there is an epi $n_{fg} : \bar{H}^d_{fg} \to H^d_{fg}$. Furthermore $H^d_{fg} = C_{k_{n_{fg}}}$.*

Proof. By the above theorem $q\beta = 0$, so there is a unique map $n_{fg} : \bar{H}^d_{fg} \to H^d_{fg}$, such that $n_{fg} c_\beta = q$. Since q is epi, so is n_{fg}. The last assertion follows from the fact that C is an abelian category. □

Theorem 3.2. *If for all $(f,g) \in \hat{\mathcal{C}}$, $d_g\Delta_g$ factors through m, then there is a pointwise regular epi natural transformation $n : \bar{H}^d \to H^d$.*

Proof. By the above proposition, there is a unique $n_{fg} : \bar{H}^d_{fg} \to H^d_{fg}$ such that $n_{fg} c_\beta = q$. To show $n = \{n_{fg}\}$ is a natural transformation, given $(\sigma, \delta, \zeta) : (f,g) \to (f', g')$ in $\hat{\mathcal{C}}$, by (vii) of 1,
$H^d(\sigma,\delta,\zeta)q = q' K(\delta,\zeta)$, and by (ix) of 1, $\bar{H}^d(\sigma,\delta,\zeta)c_\beta = c_{\beta'} K(\delta,\zeta)$. So:

$$H^d(\sigma,\delta,\zeta) n_{fg} c_\beta = H^d(\sigma,\delta,\zeta) q = q' K(\delta,\zeta) = n_{f'g'} c_{\beta'} k(\delta,\zeta) = n_{f'g'} \bar{H}^d(\sigma,\delta,\zeta) c_\beta$$

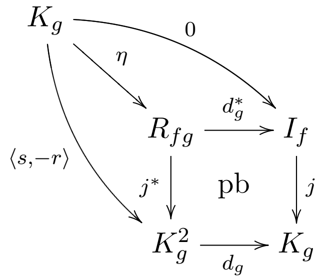
Since c_β is epic, the following diagram commutes.

$$\begin{array}{ccc} \bar{H}^d_{fg} & \xrightarrow{n_{fg}} & H^d_{fg} \\ {\scriptstyle \bar{H}^d(\sigma,\delta,\zeta)} \downarrow & & \downarrow {\scriptstyle H^d(\sigma,\delta,\zeta)} \\ \bar{H}^d_{f'g'} & \xrightarrow[n_{f'g'}]{} & H^d_{f'g'} \end{array}$$

and so $n : \bar{H}^d \to H^d : \hat{\mathcal{C}} \to \mathcal{C}$ is a natural transformation. $\square$

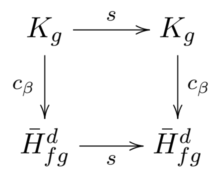
Corollary 3.1. *For $d = rpr_1 + spr_2$, with $r, s \in \mathbb{Z}$, there is a natural transformation $n : \bar{H}^d \to H^d$.*

Proof. Let $(f, g) \in \hat{\mathcal{C}}$. Since $d_g\langle s, -r\rangle = 0$, there is a unique morphism $\eta : K_g \longrightarrow R_{fg}$ such that the following triangles commute.



We have $q(r+s) = q\delta\langle s, -r\rangle = q\delta j^*\eta = 0\eta = 0$. But $d_g\Delta_g = r+s = m_{r+s}e_{r+s}$ and so $qm_{r+s}e_{r+s} = 0$. Since e_{r+s} is epic, $qm_{r+s} = 0$. But $m_{r+s} = m_{d_g\Delta_g} = \beta h$, hence $q\beta h = 0$. We know $q\beta i = qj = 0$ too. It follows that $q\beta = 0$. The result then follows from Lemma 3.3 and Theorem 3.2 $\square$

Let $d = rpr_1 + spr_2$ and $(f, g) \in \hat{\mathcal{C}}$. With d^* the pullback of d along j, we have:
$c_\beta d_g j^* = c_\beta j d_g^* = c_\beta \beta i d^* = 0 i d^* = 0$ and so $c_\beta rpr_1 j^* + c_\beta spr_2 j^* = 0$. On the other hand,
$c_\beta(r + s) = c_\beta m_{r+s}e_{r+s} = c_\beta \beta h e_{r+s} = 0 h e_{r+s} = 0$ and so $c_\beta rpr_1 j^* + c_\beta spr_1 j^* = 0$. It follows that $c_\beta s\delta j^* = 0$. Since the square



commutes, $sc_\beta\delta j^* = 0$. $\square$

Corollary 3.2. *Let $d = rpr_1 + spr_2$. If for all A, $s : A \to A$ is monic, then $n : \bar{H}^d \to H^d$ is an isomorphism.*

Proof. Let $(f, g) \in \hat{\mathcal{C}}$. Since s is monic, $c_\beta\delta j^* = 0$. Since q is a cokernel, there exists a unique map $n' : H^d \longrightarrow \bar{H}^d$ such that $n'q = c_\beta$. It then follows that $n'n = 1$ and $nn' = 1$. Hence $n_{fg} : \bar{H}^d_{fg} \cong H^d_{fg}$ is an isomorphism. $\square$

Remark 3.1. *For $\mathcal{C} = R$-mod Corollaries 3.1 and 3.2 can be generalized to the case $r, s \in R$. Also if for all R- module A, $s : A \to A$ by $s(a) = sa$ is monic, then $\bar{H}^d \cong H^d$.*

Example 3.1. *Let $\mathcal{C}$ be the category of R-modules and $(f, g) \in \hat{\mathcal{C}}$. For $d = rpr_1 + pr_2$ or $d = pr_1 + rpr_2$ with $r \in R$ we have:*

$$H^d_{fg} = \bar{H}^d_{fg} = \frac{K_g}{(1+r)K_g + I_f}$$

4. **Extended homologies with respect to Two Kernel Transformations**

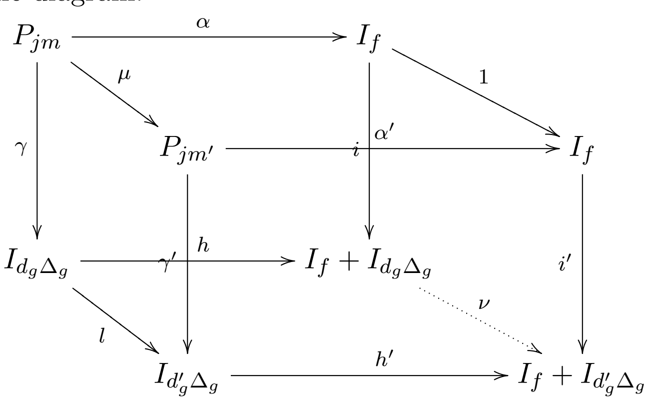
In this section we let $\mathcal{C}$ be a pointed with pullbacks and pushouts and we investigate the relation between extended homologies with respect to two kernel transformations.

Theorem 4.1. *Let d and d' be two kernel transformations. If for $(f, g) \in \hat{\mathcal{C}}$, $m_{d_g\Delta_g}$ factors through $m_{d'_g\Delta_g}$, then there is a unique morphism $p_{fg} : \bar{H}^d_{fg} \to \bar{H}^{d'}_{fg}$ such that $p_{fg}c_{\beta_d} = c_{\beta_{d'}}$. In addition p_{fg} is regular epic.*

Proof. By hypothesis, there is $l : I_{d_g\Delta_g} \to I_{d'_g\Delta_g}$ such that $m'l = m$, where $m = m_{d_g\Delta_g}$ and $m' = m_{d'_g\Delta_g}$. So we have the following pullbacks:

$$\begin{array}{ccccc} & & \overset{\alpha}{\frown} & & \\ & & /// & & \\ P_{jm} & \xrightarrow[\mu]{} & P_{jm'} & \xrightarrow[\alpha']{} & I_f \\ \gamma\downarrow & & \gamma'\downarrow & & \downarrow j \\ I_{d_g\Delta_g} & \xrightarrow{l} & I_{d'_g\Delta_g} & \xrightarrow{m'} & K_g \\ & & /// & & \\ & & \underset{m}{\smile} & & \end{array}$$

Since in the diagram:



(i)

the front and back squares are pushouts and the left and the top squares are commutative, there exists unique map ν such that the right and the bottom squares are commutative.

Since $\beta_d i = j = \beta_{d'} i' = \beta_{d'} \nu i$ and $\beta_d h = m = m'l = \beta_{d'} h' l = \beta_{d'} \nu h$, by pushoutness of the back square, $\beta_d = \beta_{d'} \nu$. Since $c_{\beta_{d'}} \beta_d = c_{\beta_{d'}} \beta_{d'} \nu = 0$, there is a unique map p_{fg} such that the following diagram commutes.

$$\begin{array}{ccccc} I_f + I_{d_g\Delta_g} & \xrightarrow{\beta_d} & K_g & \xrightarrow{c_{\beta_d}} & \bar{H}^d_{fg} \\ \nu\downarrow & & \| & & \downarrow p_{fg} \\ I_f + I_{d'_g\Delta_g} & \xrightarrow[\beta_{d'}]{} & K_g & \xrightarrow[c_{\beta_{d'}}]{} & \bar{H}^{d'}_{fg} \end{array}$$

By (iii) of 1 p_{fg} is regular epic. □

We know $m : I \to cod : \bar{\mathcal{C}} \to \mathcal{C}$ is a natural transformation. □

Theorem 4.2. *Let d and d' be two kernel transformations. If for all $g \in \hat{\mathcal{C}}$, $m_{d_g \Delta_g}$ factors through $m_{d'_g \Delta_g}$, then there is a pointwise regular epi natural transformation $p : \bar{H}^d \to \bar{H}^{d'}$.*

Proof. Let $(\sigma, \delta, \zeta) : (f, g) \to (f', g')$ be in $\hat{\mathcal{C}}$. By (ix) of 1, we have $\bar{H}^d(\sigma, \delta, \zeta) c_{\beta_d} = c_{\beta'_d} K(\delta, \zeta)$ and $\bar{H}^{d'}(\sigma, \delta, \zeta) c_{\beta_{d'}} = c_{\beta'_{d'}} K(\delta, \zeta)$. By Theorem 5.1, $p_{fg} c_{\beta_d} = c_{\beta_{d'}}$ and $p_{f'g'} c_{\beta'_d} = c_{\beta'_{d'}}$. So $p_{f'g'} \bar{H}^d(\sigma, \delta, \zeta) c_{\beta_d} = \bar{H}^{d'}(\sigma, \delta, \zeta) p_{fg} c_{\beta_d}$. Since c_{β_d} is epic, the following square commutes.

$$\begin{array}{ccc} \bar{H}^d_{fg} & \xrightarrow{p_{fg}} & \bar{H}^{d'}_{fg} \\ {\scriptstyle \bar{H}^d(\sigma,\delta,\zeta)}\Big\downarrow & & \Big\downarrow{\scriptstyle \bar{H}^{d'}(\sigma,\delta,\zeta)} \\ \bar{H}^d_{f'g'} & \xrightarrow[p_{f'g'}]{} & \bar{H}^{d'}_{f'g'} \end{array}$$

□

Corollary 4.1. *Let d and d' be two kernel transformations. If for all $g \in \hat{\mathcal{C}}$, $m_{d_g \Delta_g}$ factors through $m_{d'_g \Delta_g}$ by $l : I_{d_g \Delta_g} \to I_{d'_g \Delta_g}$ and l is epic, then $p : \bar{H}^d \to \bar{H}^{d'}$ is a natural isomorphism.*

Proof. Let $(f, g) \in \hat{\mathcal{C}}$. Since l is epic and the front square in the diagram (i) of Theorem 5.1 is a pushout, the morphism ν obtained in that diagram is epic. It follows that $c_{\beta_{d'}} = coker(\beta_{d'}) = coker(\beta_{d'} \nu) = coker(\beta_d) = c_{\beta_d}$ and so $p_{fg} : \bar{H}^d_{fg} \cong \bar{H}^{d'}_{fg}$ is an isomorphism. □

Example 4.1. *Let $\mathcal{C}$ be an abelian category, $d = rpr_1 + spr_2$ and $d' = r'pr_1 + s'pr_2$ such that $r + s = r' + s'$. Then $\bar{H}^d \cong \bar{H}^{d'}$.*

Proof. Since $d\Delta = r + s = r' + s' = d'\Delta$, the result follows. □

5. Homologies with respect to Two Kernel Transformations

In this section we let $\mathcal{C}$ be a pointed regular category with coequalizers and we investigate the relation between homologies with respect to two kernel transformations.

Recall [2] for each morphism $f : A \to B$ in the category $\mathcal{C}$, there is a functor $f^{-1} : Sub(B) \to Sub(A)$ in which $f^{-1}(m) : f^{-1}(M) \rightarrowtail A$ for a subobject $m : M \rightarrowtail B$, is the pulback of m along f. Since $\mathbf{K} : \hat{\mathcal{C}} \to \mathcal{C}$ is a functor, we can rewrite the natural transformation $d : S \circ K \to K : \bar{\mathcal{C}} \to \mathcal{C}$ to $d : S \circ \mathbf{K} \to \mathbf{K} : \hat{\mathcal{C}} \to \mathcal{C}$. We know $d^{-1}(j_{fg}) = j^*_{fg} : d^{-1}(I_f) = R_{fg} \to K^2_g$ for $(f, g) \in \hat{\mathcal{C}}$. Then the following diagram is a pullback in the category $Funct(\hat{\mathcal{C}}, \mathcal{C})$ of functors from $\hat{\mathcal{C}}$ to $\mathcal{C}$.

$$\begin{array}{ccc} d^{-1}(\mathbf{I}) & \xrightarrow{d^*} & \mathbf{I} \\ {\scriptstyle d^{-1}(j)}\downarrow & & \downarrow{\scriptstyle j} \\ S \circ \mathbf{K} & \xrightarrow[d]{} & \mathbf{K} \end{array}$$

Theorem 5.1. *Let d and d' be two kernel transformations and $(f, g) \in \hat{\mathcal{C}}$. If $j^* \leq j'^*$ i.e. $d^{-1}(j_{fg}) = j^*$ factors through $d'^{-1}(j_{fg}) = j'^*$, then there is a unique map $p_{fg} : H^d_{fg} \to H^{d'}_{fg}$ such that $p_{fg} \circ q = q'$. In addition p_{fg} is regular epic.*

Proof. By assumption there is a map ϕ such that $j'^* \circ \phi = j^*$. Then $j'_1 \circ \phi = j_1$ and $j'_2 \circ \phi = j_2$. Therefore there is a unique map $p_{fg} : H^d_{fg} \to H^{d'}_{fg}$ such that the following diagram commutes.

$$\begin{array}{ccccc} R_{fg} & \overset{j_1}{\underset{j_2}{\rightrightarrows}} & K_g & \xrightarrow{q} & H^d_{fg} \\ {\scriptstyle \phi}\downarrow & & \| & & \vdots\downarrow{\scriptstyle p_{fg}} \\ R'_{fg} & \overset{j'_1}{\underset{j'_2}{\rightrightarrows}} & K_g & \xrightarrow[q']{} & H^{d'}_{fg} \end{array}$$

□

Theorem 5.2. *Let d and d' be two kernel transformations. If $d^{-1}(j) \leq d'^{-1}(j)$, then there is a pointwise regular epi natural transformation $p : H^d \to H^{d'}$.*

Proof. Let $(\sigma, \delta, \zeta) : (f, g) \to (f', g')$ be in $\hat{\mathcal{C}}$. By (vii) of 1, we have $H^d(\sigma, \delta, \zeta)q_{fg} = q_{f'g'}K(\delta, \zeta)$ and $H^{d'}(\sigma, \delta, \zeta)q'_{fg} = q'_{f'g'}K(\delta, \zeta)$. By the above theorem, $p_{fg}q_{fg} = q'_{fg}$ and $p_{f'g'}q_{f'g'} = q'_{f'g'}$. So $H^{d'}(\sigma, \delta, \zeta)p_{fg}q_{fg} = p_{f'g'}H^d(\sigma, \delta, \zeta)q_{fg}$. Since q_{fg} is epic, the following square commutes.

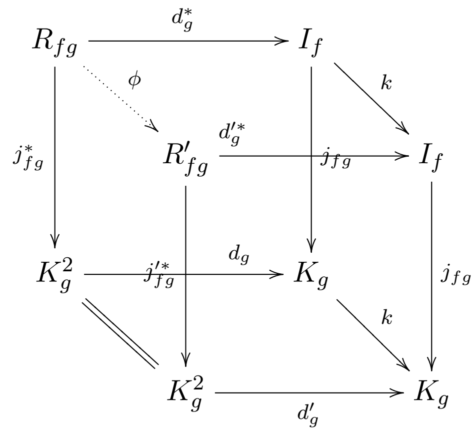
$$\begin{array}{ccc} H^d_{fg} & \xrightarrow{p_{fg}} & H^{d'}_{fg} \\ {\scriptstyle H^d(\sigma,\delta,\zeta)}\downarrow & & \downarrow{\scriptstyle H^{d'}(\sigma,\delta,\zeta)} \\ H^d_{f'g'} & \xrightarrow[p_{f'g'}]{} & H^{d'}_{f'g'} \end{array}$$

□

Corollary 5.1. *Let d and d' be two kernel transformations. If $d^{-1} \leq d'^{-1}$, then there is a pointwise regular epi natural transformation $p : H^d \to H^{d'}$.*

Corollary 5.2. *Let $\mathcal{C}$ be an abelian category and let k be an integer. If $d' = kd$, then there is a natural transformation $p : H^d \to H^{d'}$.*

Proof. Since for (f, g) in $\hat{\mathcal{C}}$, $kj_{fg} = j_{fg}k$, there is a unique ϕ such that the following diagram commutes.



So $j_{fg}^* \leq j_{fg}'^*$ and hence there is a unique map $p_{fg} : H_{fg}^d \to H_{fg}^{d'}$ which is regular epic. □

Corollary 5.3. *Let $\mathcal{C}$ = R-mod and $k \in R$. If $d' = kd$, then there is a natural transformation $p : H^d \to H^{d'}$.*

6. Conclusions

The d-homology and the extended d-homology functors are introduced and studied in [5, 6, 4] and [7]. In order to further study and investigate the properties of these homology functors, here we have done a comparison of the standard homology, the d-homology and the extended d-homology functors.

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