U.P.B. Sci. Bull., Series A, Vol. 80, Iss. 1, 2018 ISSN 1223-7027

G-FRAMES AND GREEDY APPROXIMATIONS IN HILBERT SPACES

Dongwei Li[1], Jinsong Leng[2], Thingzhu Huang[3]

In this paper, we introduce the greedy approximations for g-frames in Hilbert spaces. It is shown that g-frames satisfy the quasi greedy and almost greedy conditions. Moreover, we prove that g-Riesz bases satisfy the greedy condition.

Keywords: Frames, g-frames, Greedy approximation.

1. Introduction

The concept of frames was introduced by Duffine and Schaefer [9] to address some deep questions in non-harmonic Fourier series. It was made popular by Daubechies [6]. Today frame theory has applications in a variety of areas of mathematics, physics and engineering such as probability statistics[13], sigma-delta quantization [2], filter bank theory [5], signal and image processing [3], system modeling [20], wireless communications [17] and many other fields.

Sun in [18] introduced the concept of g-frame in Hilbert spaces. G-frames are natural generalizations of frames which cover many other recent generalizations of frames such as bounded quasi-projections [11], fusion frame [1] and pseudo-frames [14] and a class of time-frequency localization operators [8].

There is a long tradition for studying the approximations of frames. The approximation properties of frames in Banach spaces was studied in [4]. In [12], the authors introduced the symmetric approximation of frames by normalized tight frames. In [10], the authors studied nonlinear approximation properties of multivariate wavelet bi-frames. And the nonlinear approximations of frames in Hilbert spaces was studied in [15].

We know that g-frames and g-Riesz bases have properties similar to those of frames and Riesz bases, respectively. However, not all the properties are similar. For example, Riesz bases are equivalent to exact frames, but it is not the case for g-Riesz bases and exact g-frames [18]. So it is necessary to study the approximations of g-frames in Hilbert space. In this paper, we study the nonlinear approximations of g-frames by greedy approximations.

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $\{\mathcal{H}_n\}$ is a sequence of closed subspaces of $\mathcal{K}$ and $L(\mathcal{H}, \mathcal{H}_n)$ is the collection of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}_n$. For $T \in L(\mathcal{H})$, we denote $\mathcal{R}_T$ for the Range of T, and we denote $\mathcal{N}_T$ for the Ker of T.

[1]Dongwei Li, University of Electronic Science and Technology of China, P. R. China, e-mail: ldwmath@163.com

[2]Jinsong Leng, University of Electronic Science and Technology of China, P. R. China

[3]Thingzhu Huang, University of Electronic Science and Technology of China, P. R. China

Note that for any sequence $\{\mathcal{H}_n\}$, we can assume that there exists a Hilbert space $\mathcal{K}$ such that for all $n \in \mathbb{N}$, $\mathcal{H}_n \subset \mathcal{K}$ (for example $\mathcal{K} = \oplus_n \mathcal{H}_n$).

Definition 1.1. *A sequence $\{\Lambda_n\} \subset L(\mathcal{H}, \mathcal{H}_n)$ of bounded operators from $\mathcal{H}$ to $\mathcal{H}_n$ is said to be a generalized frame, or simply a g-frame, for $\mathcal{H}$ with respect to $\{\mathcal{H}_n\}$ if there are two positive constants A and B such that*

$$A\|x\|^2 \leq \sum_n \|\Lambda_n x\|^2 \leq B\|x\|^2, \text{ for all } x \in \mathcal{H}.$$

We call A and B the lower and upper g-frame bounds, respectively. We simply call $\{\Lambda_n\}$ a g-frame for $\mathcal{H}$ whenever the space sequence $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ is clear. We call $\{\Lambda_n\}_{n\in\mathbb{N}}$ a tight g-frame if $A = B$ and a Parseval g-frame if $A = B = 1$. If only the second inequality is required, we call it a Bessel g-sequence.

Note that the $\{\Lambda_n\}$ is a Bessel g-sequence if $\{\Lambda_n\}$ is a g-frame for $\{\Lambda_n^*(\mathcal{H}_n)\}$. And a g-basis is indeed a generalization of Schauder basis of a Hilbert space [16].

Let $\Lambda = \{\Lambda_n\}$ be a Bessel g-sequence, the synthesis operator for Λ is given by

$$T : (\sum_n \oplus \mathcal{H}_n)_{\ell_2} \longrightarrow \mathcal{H} \, : \, T\{c_n\} = \sum_n \Lambda_n^*(c_n).$$

The adjoint T^* of the synthesis operator is called the analysis operator. Then the frame operator S of Λ is defined as follows

$$Sx = \sum_n \Lambda_n^* \Lambda_n x, \text{ for all } x \in \mathcal{H}.$$

W.Sun proved in [18] showed that S is a well-defined, bounded and self-adjoint operator. Then the following reconstruction formula takes place for all $x \in \mathcal{H}$

$$x = SS^{-1}x = S^{-1}Sx = \sum_n \Lambda_n^* \widetilde{\Lambda}_n x = \sum_n (\widetilde{\Lambda}_n)^* \Lambda_n x.$$

We call $\{\widetilde{\Lambda}_n\} = \{\Lambda_n S^{-1}\}$ the canonical dual g-frame of Λ.

Definition 1.2. *If $\{\Lambda_n\}$ is g-complete and there are positive constants A and B such that for any finite subset $I \subset \mathbb{N}$ and $c_n \in \mathcal{H}_n$, $n \in I$,*

$$A \sum_{n\in I} \|c_n\|^2 \leq \| \sum_{n\in I} \Lambda_n^* c_n\|^2 \leq B \sum_{n\in I} \|c_n\|^2,$$

then we say that $\{\Lambda_n\}$ is a g-Riesz basis for $\mathcal{H}$.

In [18], the author has shown that every g-frame can be considered as a frame. More precisely, let $\{\Lambda_n\}$ be a g-frame for $\mathcal{H}$ and $\{e_{nj}\}_{j\in J_n}$ be an orthonormal basis for $\mathcal{H}_n$, where $J_n \subset \mathbb{N}$, then there exists a frame $\{u_{nj}\}_{n\in\mathbb{N}, j\in J_n}$ of $\mathcal{H}$ such that

$$u_{nj} = \Lambda_n^* e_{nj}. \tag{1}$$

We call $\{u_{nj}\}_{n\in\mathbb{N}, j\in J_n}$ the frame induced by $\{\Lambda_n\}$ with respect to $\{e_{nj}\}_{n\in\mathbb{N}, j\in J_n}$. The next lemma is a characterization of g-frame by a frame.

Lemma 1.1. [18] *Let $\{\Lambda_n\}$ be a family of linear operators and $\{u_{nj}\}$ be defined as in* (1). *Then $\{\Lambda_n\}$ is a g-frame (respectively Bessel g-sequence, g-Riesz basis) for $\mathcal{H}$ if and only if $\{u_{nj}\}_{n\in\mathbb{N}, j\in J_n}$ is a frame (respectively Bessel sequence, Riesz basis) for $\mathcal{H}$.*

The notion of N-term error of approximation and thresholding greedy algorithm of order N for Schauder basis in Banach space have been defined and studied in [7, 19].

Let $\mathcal{X}$ be a Banach space and (x_n, f_n) be a Schauder basis for $\mathcal{X}$, and let

$$\Sigma_N = \left\{ \sum_{n\in\sigma} a_n x_n : \sigma \subseteq \mathbb{N}, |\sigma| = N \in \mathbb{N},\ a_n \text{ are scalars} \right\}.$$

For $x \in \mathcal{X}$, $x = \sum_{n\in\mathbb{N}} f_n(x)x_n$ we define

$$\widetilde{\Sigma}_N = \left\{ \sum_{n\in\sigma} f_n(x)x_n : \sigma \subseteq \mathbb{N}\,, |\sigma| = N \in \mathbb{N} \right\}.$$

For each $x \in \mathcal{X}$ the N-term errors of approximation are defined by

$$\sigma_N(x) = \inf\left\{\|x-y\| : y \in \Sigma_N\right\},\ \widetilde{\sigma}_N(x) = \inf\left\{\|x-y\| : y \in \widetilde{\Sigma}_N\right\}.$$

Let $\delta = \{n_i\}$ be a permutation of natural numbers such that

$$|f_{n_1}(x)| \geq |f_{n_2}(x)| \geq |f_{n_3}(x)| \geq \cdots$$

The N-greedy approximant is given by $G_N(x) = \sum_{i=1}^N f_{n_i}(x)x_{n_i}$.

Definition 1.3. *A Schauder basis (x_n, f_n) is said to be quasi greedy if there exists a constant C such that $\|G_N(x)\| \leq C\|x\|$ for all $x \in \mathcal{X}$. It is said to be almost greedy if there exists a constant C such that $\|x - G_N(x)\| \leq C\widetilde{\sigma}_N$. If there exists a constant C such that $\|x - G_N(x)\| \leq C\sigma_N$ for all $x \in \mathcal{X}$, it is said to be greedy.*

2. G-frames and greedy approximation

Let $\Lambda = \{\Lambda_n\}$ be a g-frame for a Hilbert space $\mathcal{H}$ with canonical dual g-frame $\{\Lambda_n S^{-1}\}$. Let $x = \sum_n \Lambda_n^*\Lambda_n S^{-1}x$ for all $x \in \mathcal{H}$. We now define the nonlinear N-term approximation manifolds for g-frames, in the similar manner as we have defined for Schauder basis as follows.

$$\Sigma_N(\Lambda) = \left\{ \sum_{n\in\sigma} \Lambda_n^* c_n : \sigma \subseteq \mathbb{N}, |\sigma| = N,\ c_n \text{ are scalars} \right\},$$

$$\widetilde{\Sigma}_N(\Lambda) = \left\{ \sum_{n\in\sigma} \Lambda_n^*\Lambda_n S^{-1}x : \sigma \subseteq \mathbb{N}\,, |\sigma| = N \right\}.$$

We define the N-term approximation errors as

$$\sigma_N(x) = \inf\{\|x-y\| : y \in \Sigma_N(\Lambda)\},\ \widetilde{\sigma}_N(x) = \inf\{\|x-y\| : y \in \widetilde{\Sigma}_N(\Lambda)\}.$$

Next, we define $S_\sigma,\ Q_\sigma : \mathcal{H} \longrightarrow \mathcal{H}$ as

$$S_\sigma(x) = \sum_{n\in\sigma} \Lambda_n^*\Lambda_n x,\ \sigma \subseteq \mathbb{N}\,,\ |\sigma| = N,\ Q_\sigma(x) = S_\sigma(S^{-1}(x)) = \sum_{n\in\sigma} \Lambda_n^*\Lambda_n S^{-1}x.$$

Let $\gamma = \{n_i\}$ be a permutation of natural numbers such that

$$\|\Lambda_{n_1} S^{-1}x\| \geq \|\Lambda_{n_2} S^{-1}x\| \geq \|\Lambda_{n_3} S^{-1}x\| \geq \cdots$$

Now, define the N-greedy approximant for a g-frame $\{\Lambda_n\}$ as

$$G_N(x) = \sum_{i=1}^N \Lambda_{n_i}^*\Lambda_{n_i} S^{-1}x$$

for all $x \in \mathcal{H}$. We have $G_N(x) = Q_{\sigma_0}(x)$ for some $\sigma_0 \subseteq \mathbb{N}$, $|\sigma_0| = N$.

Lemma 2.1. *Let $\Lambda = \{\Lambda_n\}$ be a g-frame for a Hilbert space $\mathcal{H}$ with bounds A and B, and let $\sigma \subset \mathbb{N}$. Then $\|S_\sigma(x)\| \leq B\|x\|$ for all $x \in \mathcal{H}$.*

Proof. Let T_σ be the synthesis operator of $\{\Lambda_n\}_{n\in\sigma}$, for all $x \in \mathcal{H}$, we have

$$\|T_\sigma^*(x)\|^2 = \sum_{n\in\sigma} \|\Lambda_n x\|^2 \leq \sum_n \|\Lambda_n x\|^2 \leq B\|x\|^2.$$

Then we have

$$\|T_\sigma^*(x)\| \leq \sqrt{B}\|x\|$$

for all $x \in \mathcal{H}$. Thus, we get

$$\|S_\sigma(x)\| = \|T_\sigma T_\sigma^*(x)\| \leq \|T_\sigma\|\|T_\sigma^*(x)\| \leq B\|x\|, \ x \in \mathcal{H}.$$

□

The next result shows that g-frames satisfy the quasi greedy condition.

Theorem 2.1. *Let $\Lambda = \{\Lambda_n\}$ be a g-frame for $\mathcal{H}$ with bounds A and B. Then*

$$(1)\ \|G_N(x)\| \leq \frac{B}{A}\|x\| \ for\ all\ x \in \mathcal{H}, \ (2)\ \|x - G_N(x)\| \longrightarrow 0 \ as\ N \to \infty.$$

Proof. Let $G_N(x) = Q_{\sigma_0}(x)$ for some $\sigma_0 \subseteq \mathbb{N}$, $|\sigma_0| = N$, then we have

$$\|G_N(x)\| = \|Q_{\sigma_0}(x)\| = \|S_{\sigma_0} S^{-1}(x)\| \leq \|S_{\sigma_0}\| \cdot \|S^{-1}\| \cdot \|(x)\|.$$

By Lemma 2.1, $\|S_{\sigma_0}\| \leq B$ and $\|S^{-1}\| \leq A^{-1}$. Hence $\|G_N(x)\| \leq \frac{B}{A}\|x\|$.

We now prove (2).

$$\begin{aligned}
\|x - G_N(x)\|^2 &= \|\sum_{i>N} \Lambda_{n_i}^* \Lambda_{n_i} S^{-1}x\|^2 = \sup_{y\in\mathcal{H}, \|y\|=1} \left|\left\langle \sum_{i>N} \Lambda_{n_i}^* \Lambda_{n_i} S^{-1}x, y \right\rangle\right|^2 \\
&= \sup_{y\in\mathcal{H}, \|y\|=1} |\sum_{i>N} \langle \Lambda_{n_i} S^{-1}x, \Lambda_{n_i} y\rangle |^2 \\
&\leq \sum_{i>N} \|\Lambda_{n_i} S^{-1}x\|^2 \sup_{\|y\|=1} \sum_{i>N} \|\Lambda_{n_i} y\|^2 \leq B \sum_{i>N} \|\Lambda_{n_i} S^{-1}x\|^2.
\end{aligned}$$

By the definition of greedy algorithm, for any $\sigma \subseteq \mathbb{N}$ with $|\sigma| = N$, we have

$$\begin{aligned}
\sum_{i>N} \|\Lambda_{n_i} S^{-1}x\|^2 &\leq \sum_{n\in\mathbb{N}\setminus\sigma} \|\Lambda_n S^{-1}x\|^2 \\
&= \sum_i \|\Lambda_{n_i} S^{-1}x\|^2 - \sum_{i\in\sigma} \|\Lambda_{n_i} S^{-1}x\|^2 \to 0, \ as\ N \to \infty.
\end{aligned}$$

Thus, we obtain

$$\|x - G_N(x)\|^2 \leq B \sum_{n=N+1} \|\Lambda_n S^{-1}x\|^2 \to 0 \ as\ N \to \infty.$$

□

The next result shows that g-frames also satisfy the almost greedy condition. We need the following lemma.

Lemma 2.2. *Let $\Lambda = \{\Lambda_n\}$ be a g-frame for $\mathcal{H}$ with bounds A and B. Let T be its synthesis operator, for all $\{c_n\} \in \mathcal{N}_T^{\perp}$, we have*

$$A\sum_n |c_n|^2 \leq \left\| \sum_n \Lambda_n^* c_n \right\|^2 \leq B\sum_n |c_n|^2. \tag{2}$$

Proof. Since $\Lambda = \{\Lambda_n\}$ is a g-frame for $\mathcal{H}$, the $\Lambda = \{\Lambda_n\}$ is a Bessel g-sequence for $\mathcal{H}$ with bound B. Then

$$\left\| \sum_n \Lambda_n^* c_n \right\|^2 = \|T\{c_n\}\|^2 \leq B\sum_n |c_n|^2.$$

We now prove the left-hand inequality in (2). Assume that $\Lambda = \{\Lambda_n\}$ satisfies the lower frame condition with bound A. Note that $\mathcal{R}_{T^*}$ is closed because $\mathcal{R}_T$ is closed. Therefore

$$\mathcal{N}_T^{\perp} = \overline{\mathcal{R}_{T^*}} = \mathcal{R}_{T^*},$$

i.e., $\mathcal{N}_T^{\perp}$ consists of all sequences of the form $\{\Lambda_n x\}$, $x \in \mathcal{H}$. Now, given $x \in \mathcal{H}$,

$$\left(\sum_n \|\Lambda_n x\|^2\right)^2 = \| \langle Sx, x\rangle |^2 \leq \|Sx\|^2\|x\|^2 \leq \|Sx\|^2 \frac{1}{A}\sum_n \|\Lambda_n x\|^2.$$

This implies that

$$A\sum_n \|\Lambda_n x\|^2 \leq \|Sx\|^2 = \|T\{\Lambda_n x\}\|^2.$$

Let $\{c_n\} = \{\Lambda_n x\}$, then we get the following result. □

Theorem 2.2. *Let $\Lambda = \{\Lambda_n\}$ be a g-frame for $\mathcal{H}$ with bounds A and B. Then*

$$\|x - G_N(x)\| \leq \sqrt{\frac{B}{A}}\widetilde{\sigma}_N(x).$$

Proof. As in the proof of Theorem 2.1, we have

$$\|x - G_N(x)\|^2 \leq B\sum_{n\in\mathbb{N}\setminus\sigma} \|\Lambda_n S^{-1}x\|^2$$

for any $\sigma \subseteq \mathbb{N}$ with $|\sigma| = N$ and $x \in \mathcal{H}$. Also, by Lemma 2.2 we have

$$A\sum_n |c_n|^2 \leq \left\| \sum_n \Lambda_n^* c_n \right\|^2 \leq B\sum_n |c_n|^2$$

for all $\{c_n\} \in \mathcal{N}_T^{\perp}$. Moreover, $\{\Lambda_n S^{-1}x\} = T^*S^{-1}x \in \mathcal{R}_{T^*} = \mathcal{N}_T^{\perp}$. So, for any $\sigma \subseteq \mathbb{N}$ with $|\sigma| = N$ we obtain

$$\|x - G_N(x)\| \leq \frac{B}{A}\| \sum_{n\in\mathbb{N}\setminus\sigma} \Lambda_n^*\Lambda_n S^{-1}x\|^2$$

for all $x \in \mathcal{H}$. Now, let $y = \sum_{n\in\sigma} \Lambda_n^*\Lambda_n S^{-1}x \in \widetilde{\Sigma}_N(\Lambda)$. Then we have

$$\|x - y\|^2 = \| \sum_{n\in\mathbb{N}\setminus\sigma} \Lambda_n^*\Lambda_n S^{-1}x\|^2.$$

Thus,

$$\|x - G_N(x)\| \leq \frac{B}{A}\|x - y\|^2,$$

for any $y \in \widetilde{\Sigma}_N(\Lambda)$. Hence,

$$\|x - G_N(x)\| \leq \sqrt{\frac{B}{A}}\widetilde{\sigma}_N(x), \text{ for all } x \in \mathcal{H}.$$

$\square$

Corollary 2.1. *Let $\Lambda = \{\Lambda_n\}$ be a g-Riesz basis for $\mathcal{H}$, then it satisfies the quasi greedy (or almost greedy) condition.*

Proof. Since a g-Riesz basis for $\mathcal{H}$ is also a g-frame for $\mathcal{H}$, by Theorem 2.1 and Theorem 2.2, we get the following results. $\square$

Theorem 2.3. *Let $\Lambda = \{\Lambda_n\}$ be a g-frame for $\mathcal{H}$ and u_{nj} be defined as in* (1)*. Then $\Lambda = \{\Lambda_n\}$ satisfies the quasi greedy (or almost greedy) condition if and only if $\{u_{nj}\}$ satisfies the quasi greedy (or almost greedy) condition.*

Proof. The proof is straightforward. $\square$

It is easy to obtain the following result which is given by [15].

Corollary 2.2. *Let $\{x_n\}$ be a frame for $\mathcal{H}$, then it satisfies the quasi greedy (or almost greedy) condition.*

Finally, we prove the g-Riesz bases are greedy.

Theorem 2.4. *Let $\Lambda = \{\Lambda_n\}$ be a g-Riesz basis for Hilbert space $\mathcal{H}$ with bounds A and B. Then, for any $N \in \mathbb{N}$, we have*

$$\|x - G_N(x)\| \leq \sqrt{\frac{B}{A}}\sigma_N(x), \text{ for all } x \in \mathcal{H}.$$

Proof. Note that a g-Riesz basis is a g-frame for $\mathcal{H}$ with the same bounds. Since A and B are the bounds of the g-Riesz basis $\{\Lambda_n\}$, by the g-frame inequality we have

$$A\|x\|^2 \leq \sum_n \|\Lambda_n f\|^2 \leq B\|x\|^2, \text{ for all } x \in \mathcal{H}.$$

As in the proof of Theorem 2.1, we have

$$\|x - G_N(x)\|^2 \leq B \sum_{n \in \mathbb{N} \setminus \sigma} \|\Lambda_n S^{-1} x\|^2$$

for any $\sigma \subseteq \mathbb{N}$ with $|\sigma| = N$ and $x \in \mathcal{H}$. Let S_σ be the frame operator for the g-frame $\{\Lambda_n\}_{n\in\sigma}$ of $\mathcal{H}_\sigma = \{\Lambda_n^*(\mathcal{H}_n)\}_{n\in\sigma}$. Let P_σ be the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_\sigma$, given by $P_\sigma(x) = \sum_{n\in\sigma} \Lambda_n^* \Lambda_n S_\sigma^{-1} x$. By the inequalities of canonical dual g-frame of $\{\Lambda_n\}$, we have

$$\begin{aligned}
\frac{1}{A}\|x - P_\sigma(x)\|^2 \geq \sum_{n\in\mathbb{N}} \|\Lambda_n S^{-1}(x - P_\sigma(x))\|^2 &\geq \sum_{n\in\mathbb{N}\setminus\sigma} \|\Lambda_n S^{-1} x - \Lambda_n S^{-1} P_\sigma(x)\|^2 \\
&= \sum_{n\in\mathbb{N}\setminus\sigma} \|\Lambda_n S^{-1} x - \sum_{i\in\sigma} \Lambda_n S^{-1} \Lambda_i^* \Lambda_i S_\sigma^{-1} x\|^2 \\
&= \sum_{n\in\mathbb{N}\setminus\sigma} \|\Lambda_n S^{-1} x - \sum_{i\in\sigma} \widetilde{\Lambda}_n \Lambda_i^* \Lambda_i S_\sigma^{-1} x\|^2 \\
&= \sum_{n\in\mathbb{N}\setminus\sigma} \|\Lambda_n S^{-1} x\|^2.
\end{aligned}$$

Thus, for any $\sigma \subseteq \mathbb{N}$ with $|\sigma| = N$, we obtain

$$\|x - G_N(x)\|^2 \leq \frac{B}{A}\|x - P_\sigma(x)\|^2, \text{ for all } x \in \mathcal{H}. \tag{3}$$

For any $x \in \mathcal{H}$ we have

$$\|x - P_\sigma(x)\| = \operatorname{dist}(x, \mathcal{H}_\sigma) = \inf\{\|x - y\| : y \in \mathcal{H}_\sigma\}.$$

So, for any $y \in \mathcal{H}_\sigma$ we have $\|x - P_\sigma(x)\| \leq \|x - y\|$. Hence

$$\sigma_N(x) = \inf_\sigma\{\|x - P_\sigma(x)\| : \sigma \subseteq \mathbb{N},\ |\sigma| = N\}. \tag{4}$$

By (3) and (4), we have

$$\|x - G_N(x)\|^2 \leq \frac{B}{A}\|x - P_\sigma(x)\|^2 \leq \frac{B}{A}\|x - y\|^2 = \frac{B}{A}\sigma_N(x),$$

for all $x \in \mathcal{H}$. $\square$

Corollary 2.3. *Let $\{x_n\}$ be a Riesz basis for $\mathcal{H}$, then it satisfies the greedy condition.*

Acknowledgements

The research is supported by the National Natural Science Foundation of China (11271001) and the National Natural Science Foundation of China (61370147).

REFERENCES

[1] *M. S. Asgari and K. Amir*, Frames and bases of subspaces in Hilbert spaces, Math. Anal. Appl., **308**(2004), No. 2, 541-553.

[2] *J. J. Benedetto, A. M. Powell and Ö. Ylmaz*, Second-order sigma-delta ($\Sigma\Delta$) quantization of finite frame expansions, Appl. Comput. Harmon. Anal., **20**(2006), No. 1, 126-148.

[3] *E. J. Cands and D. L. Donoho*, New tight frames of curvelets and optimal representations of objects with piecewise C2 singularities, Comm. Pure Appl. Math., **57**(2004), No. 2, 219-266.

[4] *P. G. Casazza, D. Han and D. R. Larson*, Frames for Banach spaces, Contemp. Math., **247**(1999), 149-182.

[5] *L. Cai, J. Zhang, C. Zhang and E. Mosca*, Frame-theory-based analysis and design of oversampled filter banks: direct computational method, IEEE Trans. Signal Process, **55**(2007), No. 2, 507-519.

[6] *I. Daubechies*, Ten lectures on wavelets, SIAM, 1992.

[7] *R. A. DeVore*, Nonlinear approximation, Acta Numer., **7**(1998), 51-150.

[8] *M. Ðorfler, H. G. Feichtinger and K. Gröchenig*, Time-frequency partitions for the gelfand triple (s0, l 2,), Math. Scand., (2006), 81-96.

[9] *R. J. Duffin and A. C. Schaeffer*, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., **72**(1952), 341-366.

[10] *M. Ehler*, Nonlinear approximation schemes associated with nonseparable wavelet bi-frames, J. Approx. Theory, **161**(2009), No. 1, 292-313.

[11] *M. Fornasier*, Quasi-orthogonal decompositions of structured frames, J. Math. Anal. Appl., **289**(2004), No. 1, 180-199.

[12] *M. Frank, V. Paulsen and T. Tiballi*, Symmetric approximation of frames and bases in Hilbert spaces, Trans. Amer. Math. Soc., **354**(2002), No. 2, 777-793.

[13] *D. Li, J. Leng, T. Huang and Y. Xu*, Some equalities and inequalities for probabilistic frames, J. Inequal. Appl., (2016), No. 1, 245.

[14] *S. Li and H. Ogawa*, Pseudoframes for subspaces with applications, J. Fourier Anal. Appl., **10**(2004), No. 4, 409-431.

[15] *K. T. Poumai and S. K. Kaushik*, Frames and non linear approximations in Hilbert spaces, Izv. Nats. Akad. Nauk Armenii Mat., **51**(2016), No. 1, 41-49.

[16] *S. M. Ramezani and A. Nazari*, g-orthonormal bases, g-Riesz bases and g-dual of g-frames, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **78**(2016), No. 1, 91-98.

[17] *T. Strohmer and R. W. Heath*, Grassmannian frames with applications to coding and communication, Politehn. Appl. Comput. Harmon. Anal., **14**(2003), No. 3, 257-275.

[18] *W. Sun*, G-frames and g-Riesz bases, J. Math. Anal. Appl., **322**(2006), No. 1, 437-452.

[19] *V. N. Temlyakov*, The best m-term approximation and greedy algorithms, Adv. Comput. Math., **8**(1998), No. 3, 249-265.

[20] *N. F. D. Ward and J. R. Partington*, A Construction of Rational Wavelets and Frames in Hardy-Sobolev Spaces With Applications to System Modeling, SIAM J. Control Optim., **36**(1998), No. 2, 654-679.