WEIGHTED ENTROPY FUNCTION AS AN EXTENSION OF THE KOLMOGOROV-SINAI ENTROPY

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In this paper, the concept of weighted entropy function for dynamical systems on compact metric spaces is introduced using the generator notion. It is proved that this concept is an extension of the Kolmogorov-Sinai entropy. The independence of weighted entropy function of generators is proved. The persistence of weighted entropy function under a topological conjugate relation is deduced. A version of Jacobs Theorem concerning the entropy of a dynamical system is given. Moreover it is shown that weighted entropy function generates the Kolmogorov-Sinai entropy as a special case.

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1. Introduction

Ergodic theory today is a large and rapidly developing subject. It deals primarily with the complex behavior of all transformations that preserve the structure of measure spaces. Entropy was first introduced into the ergodic theory by Kolmogorov [3] and Sinai [10] via a measure theoretic approach. Kolmogorov-Sinai entropy measures the maximal loss of information for the iteration of finite partitions in a measure preserving transformation. Entropy has been studied from different viewpoints [2, 4, 5, 8, 11, 12, 13]. In all of this viewpoints, entropy is given as a non-negative extended real number. This paper is an attempt to present an extension of the Kolmogorov-Sinai entropy as a linear function rather than a non-negative number. In this article, a weight factor $f(x)$ to any point $x \in X$, where X denotes the base space of the system is assigned and the weighted entropy function for topological dynamical systems is defined using the generator notion. The weight factor can be cosidered as the local loss of information caused by the lack of experience of any intelligent point. The weighted entropy function concides with the Kolmogorov-Sinai entropy for dynamical systems when there is no weight factor in the middle.

2. Preliminary facts

This section is devoted to provide the prerequisites that are necessary for the next section. Let (X, β) denotes a $\sigma-$finite measure space, i.e. a set equiped with a $\sigma-$algebra β of subsets of X. Further let μ denote a probability measure on (X, β). Then (X, β, μ) is called a probability space. Let $T : X \to X$ be a measure preserving invertible transformation of probability space (X, β, μ). In particular $T(\beta) = \beta$ and $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \beta$. Then (X, β, μ, T) is called a dynamical system.
In this article the set of all probability measures on X preserving T is denoted by $M(X, T)$.

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We also write $E(X,T)$ for the set of all ergodic measures of T. Finally, for $\mu \in M(X,T)$, $h_\mu(T)$ denotes the Kolomogorov-Sinai entropy of T. In the following we recall some definitions and classical results that we need in the sequel.

Theorem 2.1. *(Birkhoff Ergodic Theorem[14]) Suppose* $T : (X,\beta,\mu) \to (X,\beta,\mu)$ *is measure preserving (where we allow* (X,β,μ) *to be* $\sigma-$*finite) and* $f \in L^1(\mu)$. *Then* $\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x))$ *convergence a.e. to a function* $f^* \in L^1(\mu)$. *Also* $f^*oT = f^*$ *a.e. and if* $\mu(X) < \infty$, *then* $\int_X f^* d\mu = \int_X f d\mu$.

Definition 2.1. *A partition* ξ *is a refinement of a partition* η, *if every element of* η *is a union of elements of* ξ. *If* ξ *is a refinement of* η, *we write* $\eta \prec \xi$.

Definition 2.2. *Given two partitions* ξ, η *we define their common refinement*

$$\xi \vee \eta = \{A_i \cap B_j; A_i \in \xi, B_j \in \eta\}.$$

Definition 2.3. *The entropy of the partition* $\xi = \{A_1, ..., A_n\}$ *of the probability space is defined by*

$$H_\mu(\xi) = -\sum_{i=1}^{n} \mu(A_i)\log\mu(A_i)$$

and the entropy of the partition of the dynamical system is given by

$$h_\mu(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi \vee T^{-1}\xi \vee ...T^{-n}\xi)$$

where $T^{-1}\xi = \{T^{-1}A_1, ..., T^{-1}A_n\}$. *Then the Kolmogrov- Sinai entropy of the automorphism* T *is defined by*

$$h_\mu(T) = \sup_\xi h_\mu(T,\xi)$$

where the supremum is over all finite partitions.

Definition 2.4. *We say that a partition* ξ *with* $H_\mu(\xi) < \infty$ *is called a generator for the probability space* (X,β,μ) *if* $\vee_{i=1}^{\infty} T^{-i}(\xi) = \beta$.

Theorem 2.2. *(Kolmogrov- Sinai Theorem [7]) If* ξ *is a generator then* $h_\mu(T) = h(T,\xi)$.

Theorem 2.3. *(Choquet [6]) Suppose that* Y *is a compact convex metrisable subset of a locally covex space* E, *and* $x_0 \in Y$. *Then, there exists a probability measure* τ *on* Y *which represents* x_0 *and is supported by the extreme points of* Y, *that is,* $\Phi(x_0) = \int_Y \Phi d\tau$ *for every continuous linear functional* Φ *on* E, *and* $\tau(ext(Y)) = 1$.

Let $\mu \in M(X,T)$ and $f : X \to \mathbb{R}$ be a bounded measurable function. As we know that $E(X,T)$ equals the extreme points of $M(X,T)$, applying the Choquets Theorem for $E = M(X)$, the space of finite regular Borel measures on X, and $Y = M(X,T)$, and using the linear functional $\Phi : M(X) \to \mathbb{R}$ given by $\Phi(\mu) = \int_X f d\mu$, we have the following corollary:

Corollary 2.1. *Suppose that* $T : X \to X$ *is a continuous map on the compact metric space* X. *Then, for each* $\mu \in M(X,T)$, *there is a unique measure* τ *on the Borel subsets of the compact metrsable space* $M(X,T)$, *such that* $\tau(E(X,T)) = 1$ *and*

$$\int_X f(x)d\mu(x) = \int_{E(X,T)} (\int_X f(x)dm(x))d\tau(m)$$

for every bounded measurable function $f : X \to \mathbb{R}$.

Under the assumptions of Corollary 2.8, we write $\mu = \int_{E(X,T)} m d\tau(m)$, called the ergodic decomposition of μ.

Theorem 2.4. *(Jacobs [14]) Let $T : X \to X$ be a continuous map on a compact metrisable space. If $\mu \in M(X,T)$ and $\mu = \int_{E(X,T)} m d\tau(m)$ is the ergodic decomposition of μ, then we have:*

(i) *If ξ is a finite Borel partition of X, then, $h_\mu(T,\xi) = \int_{E(X,T)} h_m(T,\xi)d\tau(m)$.*

(ii) *$h_\mu(T) = \int_{E(X,T)} h_m(T)d\tau(m)$ (both sides could be ∞).*

3. weighted entropy function of dynamical systems

In this section, the notion of weighted entropy function of dynamical systems is introduced and some of its ergodic properties is proved.

Definition 3.1. *Suppose that $T : X \to X$ is a continuous map on the topological space $X, x \in X$ and A a Borel subset of X. Then*

$$m_x^T(A) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i(x)).$$

Where χ_A is the characteristic function of A defined by

$$\chi_A(x) = \begin{cases} 1 \text{ if} & x \in A \\ 0 \text{ if} & x \in A^c \end{cases}$$

We write m_x for m_x^T where there is no confusion.
Now, let $x \in X$ and $\xi = \{A_1, A_2, ..., A_n\}$ be a finite Borel partition of X. We define

$$\rho(x,T,\xi) := -\sum_{i=1}^{n} m_x(A_i) \log m_x(A_i);$$

(We assume that $\log 0 = -\infty$ *and* $0 \times \infty = 0$).

It is clear $\rho(x,T,\xi) \geq 0$.

Definition 3.2. *Suppose that $T : X \to X$ is a continuous map on the topological space $X, x \in X$ and ξ be a finite Borel partition of X. The map $h(.,T,\xi) : X \to [0,\infty]$ is defined as*

$$h(x,T,\xi) = \limsup_{l\to\infty} \frac{1}{l} \rho(x,T,\vee_{i=0}^{l-1} T^{-i}\xi).$$

Definition 3.3. *Suppose that $T : X \to X$ is a continuous map on the topological space $X, x \in X$ and ξ be a finite Borel partition of X. We define the local entropy of T at x by*

$$h(x,T) = \sup_{\xi} h(x,T,\xi).$$

Definition 3.4. *Suppose that $T : X \to X$ is a continuous map on the compact metric space X, and ξ be a generator for the dynamical system (X,T). Let $\mu \in M(X,T)$ be such that $h_\mu(T) < \infty$. The weighted entropy function of T (with respect to μ), $L_T(.,\mu,\xi) : C(X) \to \mathbb{R}$, is defined as*

$$L_T(f,\mu,\xi) = \int_X f(x)h(x,T,\xi)d\mu(x)$$

for all $f \in C(X)$ (again $0 \times \infty := 0$).

In the following, we will prove the independence of weighted entropy function from the selection of the generator.

Theorem 3.1. *Definition* 3.4 *is independent of the choice of generator i.e if* ξ *and* η *are two generators of* T *then,*

$$L_T(f,\mu,\xi) = L_T(f,\mu,\eta).$$

for all $f \in C(X)$.

Proof : First, let $m \in E(X,T)$. For any Borel set $A \subset X$ and $x \in X$, applying Birkhoff Ergodic Theorem, we have

$$m_x(A) = m(A)$$

for almost all $x \in X$. Hence, if $\xi = \{A_1, A_2, ..., A_n\}$ is a finite Borel partition of X, then,

$$\rho(x,T,\xi) = -\sum_{i=1}^{n} m(A_i)\log m(A_i) = H_m(\xi)$$

for almost all $x \in X$. Thus,

$$\limsup_{l\to\infty} \frac{1}{l}\rho(x,T,\vee_{i=0}^{l-1}T^{-i}\xi) = h_m(T,\xi) \tag{1}$$

and

$$\limsup_{l\to\infty} \frac{1}{l}\rho(x,T,\vee_{i=0}^{l-1}T^{-i}\eta) = h_m(T,\eta) \tag{2}$$

Applying (1), (2), and Kolmogorov- Sinai Theorem, we have

$$\begin{aligned} h(x,T,\xi) &= \limsup_{l\to\infty} \frac{1}{l}\rho(x,T,\vee_{i=0}^{l-1}T^{-i}\xi) \\ &= h_m(T,\xi) = h_m(T) = h_m(T,\eta) \\ &= \limsup_{l\to\infty} \frac{1}{l}\rho(x,T,\vee_{i=0}^{l-1}T^{-i}\eta) = h(x,T,\eta) \end{aligned}$$

So, if $f \in C(X)$, then,

$$f(x)h(x,T,\xi) = f(x)h(x,T,\eta)$$

for all $x \in X$. Therefore,

$$L_T(f,m,\xi) = L_T(f,m,\eta).$$

Remark 3.1. *By Theorem* 3.5*, we conclude that the definition of weighted entropy function is independent of the selction of generators. Therefore, given any invariant measure* μ *and any generator* ξ*, we have the unique weighted entropy function. So, we can write* $L_T(f,\mu)$ *for* $L_T(f,\mu,\xi)$ *without confusion.*

Definition 3.5. *we say that two dynamical systems* (X,T_1) *and* (Y,T_2) *are conjugate if there exists a homeomorphism* $\varphi : X \to Y$ *such that* $\varphi o T_1(x) = T_2 o\varphi(x)$ *for all* $x \in X$.

Theorem 3.2. *Suppose that* $T : X \to X$ *is a continuous map on the compact metric space* X*. Then,*

(i) *Given any* $\mu \in M(X,T)$*, the weighted entropy function* $f \to L_T(f,\mu)$ *is linear.*
(ii) *Given any* $f \in C(X)$*, the map* $\mu \to L_T(f,\mu)$ *is affine.*
(iii) *If two dynamical systems* (X,T_1) *and* (Y,T_2) *are conjugate, and* $\mu \in M(X,T)$*, then,*

$$L_{T_1}(f,\mu) = L_{T_2}(f\varphi^{-1},\mu\varphi^{-1})$$

for all $f \in C(X)$.

Proof :

(i) and (ii) are trivial.

(iii) For $x \in X$ and the Borel set $A \subset X$, we have ${m_x}^{T_1}(A) = {m_{\varphi(x)}}^{T_2}(\varphi(A))$. Therefore, $\rho(x, T_1, \xi) = \rho(\varphi(x), T_2, \varphi(\xi))$ for any finite Borel partition ξ. By definition of $h(., T, \xi)$ we have $h(., T_1, \xi) = h(., T_2, \varphi(\xi)) o\varphi$. Note that $\varphi(\xi) = \{\varphi(A); A \in \xi\}$. Let $\mu \in M(X, T_1)$, and $f \in C(X)$. Then,

$$\begin{aligned} L_{T_1}(f, \mu) &= \int_X f(x)h(x, T_1, \xi)d\mu(x) \\ &= \int_X f(x)h(\varphi(x), T_2, \varphi(\xi))d\mu(x) \\ &= \int_Y f(\varphi^{-1}(x))h(x, T_2, \varphi(\xi))d(\mu\varphi^{-1})(x) \\ &= L_{T_2}(f\varphi^{-1}, \mu\varphi^{-1}). \end{aligned}$$

□

Now we can deduce the following version of Jacobs theorem.

Theorem 3.3. *Suppose that $T : X \to X$ is a continuous map on the compact metric space X. If $\mu \in M(X, T)$ and $\mu = \int_{E(X,T)} m d\tau(m)$ is the ergodic decomposition of μ, then,*

$$L_T(f, \mu) = \int_{E(X,T)} L_T(f, m)d\tau(m)$$

for all $f \in C(X)$.

Proof : Let ξ be a generator of dynamical system (X, T). Let $f \in C(X)$. Applying Corollary 2.7, we have

$$\begin{aligned} L_T(f, \mu, \xi) &= \int_X f(x)h(x, T, \xi)d\mu(x) \\ &= \int_{E(X,T)} (\int_X f(x)h(x, T, \xi)dm(x))d\tau(m) \\ &= \int_{E(X,T)} L_T(f, m)d\tau(m). \end{aligned}$$

□

In the following theorem, we extract the Kolmogorov-Sinai entropy from the weighted entropy function as a special case.

Theorem 3.4. *Suppose that $T : X \to X$ is a continuous map on the compact metric space X. Then $L_T(1, \mu) = h_\mu(T)$.*

Proof : Let ξ be a generator and $m \in E(X, T)$. As in the proof of Theorem 3.5, we have

$$h(x, T, \xi) = h_m(T).$$

Therefore,

$$L(1, m) = \int_X h(x, T, \xi)dm(x) = h_m(T).$$

Now, let $\mu \in M(X, T)$, and $\mu = \int_{E(X,T)} m d\tau(m)$ be the ergodic decomposition of μ. Applying Theorem 2.8 and Theorem 3.9 we have,

$$\begin{aligned} L_T(1, \mu) &= \int_{E(X,T)} L_T(1, m)d\tau(m) \\ &= \int_{E(X,T)} h_m(T)d\tau(m) \\ &= h_\mu(T). \end{aligned}$$

□

Theorem 3.5. *Suppose that $T : X \to X$ is a continuous map on the compact metric space X. Then, the weighted entropy function $f \to L_T(f, \mu)$ is a continuous linear function on $C(X)$, and $\|L_T(., \mu)\| = h_\mu(T)$.*

Proof : Let ξ be a generator. Let $f \in C(X)$, then,

$$\begin{aligned} |L_T(f, \mu)| &= |\int_X f(x)h(x, T, \xi)d\mu(x)| \leq \int_X |f(x)|h(x, T, \xi)d\mu(x) \\ &\leq \|f\|_\infty \int_X h(x, T, \xi)d\mu(x) = \|f\|_\infty L_T(1, \mu) = \|f\|_\infty h_\mu(T) \end{aligned}$$

Therefore, the weighted entropy function is a continuous function and $\|L_T(., \mu)\| \leq h_\mu(T)$. The equality holds by Theorem 3.10. □

4. **Concluding remarks and open problems**

In this paper, the notion of weighted entropy function for dynamical sytems on compact metric spaces is introduced. It is a continuous linear function on $C(X)$ such that its norm equals the Kolmogorov-Sinai entropy of T. Theorem 3.8 (ii) is the generalized form of the property that, the entropy map $\mu \to h_\mu(T)$ is affine. Theorem 3.8 (iii) generalizes the invariance of the entropy of a system, under topological conjugacy, to the weighted entropy function. Theorem 3.9 is the generalized Jacobs Theorem concerning the entropy of a dynamical system. Finally, $L_T(1, \mu)$ is the Kolmogorov-Sinai entropy of T.
An interesting open problem is to estabilish a proposition on existence of generators having finite weighted entropy function.

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