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WEAK AMENABILITY OF FRÉCHET ALGEBRAS

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Let $\mathcal{A}$ be a Fréchet algebra. We introduce and study the notion of weak amenability of $\mathcal{A}$. In fact we show that some results in the field of weak amenability of Banach algebras can be generalized for Fréchet algebras. For example we prove that if $\mathcal{A}$ is weakly amenable then $\mathcal{A}$ is essential. Moreover if I is a quasinormable closed ideal in the Fréchet algebra $\mathcal{A}$ such that $\frac{\mathcal{A}}{I}$ and I are weakly amenable, then $\mathcal{A}$ is weakly amenable, as well.

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1. Introduction and Preliminaries

The ground work for amenability of Banach algebras was laid by Johnson in [13]. Then various notions of amenability have been introduced and studied. In [4] and [5], some generalized notions of amenability have been introduced and investigated. Moreover in [6], pseudo-amenable and pseudo-contractible Banach algebras were defined and investigated. Also in the recent work of the first author [1], pseudo-contractibility of weighted L^p-algebras was studied.

Weakly amenable Banach algebras were introduced in [2]. The general definition is due to B. E. Johnson. Moreover Grønbæk in [8] and [9], presented some results about weakly amenable Banach algebras. Also he characterized weakly amenable, commutative Banach algebras [10]. We also refer to [2] and [14], for more results in the field. We refer to [3], as a standard reference in this field. The notion of amenability of a Fréchet algebra was studied by A. Yu. Pirkovskii [17]. He generalized some theorems about amenability of Banach algebras such as strictly flat Banach A-bimodule, virtual diagonal and approximate diagonal of Banach algebras to Fréchet algebras. Also in [15], P. Lawson and C. J. Read introduced and studied some notions about approximate amenability and approximate contractibility of Fréchet algebras.

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In the present work, we introduce and study the notion of weak amenability of Fréchet algebras. In fact we investigate Theorems (2.8.63), (2.8.64) and (2.8.66) from [3], for the case where $\mathcal{A}$ is a Fréchet algebra. It should be noted that all of the proofs in this paper, have been obtained by some slight modifications from the Banach algebra case. We first present some definitions and known results related to Fréchet algebras, which will be required throughout the paper. See [7], [11] and [16], for more information.

A locally convex topological vector space E is a topological vector space in which the origin has a local base of absolutely convex absorbent sets. Recall that $S \subseteq E$ is called bounded if for every zero neighborhood U, there exists scalar λ such that $S \subseteq \lambda U$; it is called balanced if for each $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $\alpha S \subseteq S$. Moreover S is called absorbing if for each $x \in E$, there is the scalar λ such that $x \in \lambda S$.

A collection $\mathcal{U}$ of zero neighborhoods in E is called a fundamental system of zero neighborhoods, if for every zero neighborhood U there exists a $V \in \mathcal{U}$ and an $\varepsilon > 0$ with $\varepsilon V \subset U$. Throughout the paper, all locally convex spaces are assumed to be Hausdorff.

A barrel set in a topological vector space E is a set which is convex, balanced, absorbing and closed. Moreover, E is called barrelled, if it is a Hausdorff topological vector space for which every barrel set in the space is a neighborhood for the zero element.

A family $(p_\alpha)_{\alpha\in A}$ of continuous seminorms on E is called a fundamental system of seminorms, if the sets

$$U_\alpha = \{x \in E : p_\alpha(x) < 1\} \qquad (\alpha \in A)$$

form a fundamental system of zero neighborhoods. See [16, page 251], for more information. By [16, Lemmas 22.4,22.5], every locally convex space E has a fundamental system of seminorms $(p_\alpha)_{\alpha\in A}$; equivalently a family of the seminorms satisfying the following properties:

(i) For every $x \in E$ with $x \neq 0$, there exists an $\alpha \in A$ with $p_\alpha(x) > 0$;
(ii) For all $\alpha, \beta \in A$, there exists $\gamma \in A$ and $C > 0$ such that

$$\max(p_\alpha(x), p_\beta(x)) \leq C p_\gamma(x) \qquad (x \in E).$$

In the following we recall [16, Proposition 22.6] which is very useful in our discussions.

Proposition 1.1. *Let E and F be locally convex spaces with the fundamental system of seminorms $(p_\alpha)_{\alpha\in A}$ in E and $(q_\beta)_{\beta\in B}$ in F. Then for every linear mapping $T : E \longrightarrow F$, the following assertions are equivalent;*

(i) *T is continuous.*
(ii) *T is continuous at 0.*
(iii) *For each $\beta \in B$ there exists an $\alpha \in A$ and $C > 0$, such that*

$$q_\beta(T(x)) \leq C p_\alpha(x),$$

for all $x \in E$.

It should be noted that by [11, page 24], if E, F and G are locally convex spaces and $(p_\mu)_{\mu\in\Lambda_1}$, $(q_\lambda)_{\lambda\in\Lambda_2}$ and $(r_\nu)_{\nu\in\Lambda_3}$ are fundamental systems of seminorms on E, F and G, respectively, and $R : E \times F \longrightarrow G$ is a bilinear map, then R is jointly continuous if and only if for any $\nu_0 \in \Lambda_3$ there exist $\mu_0 \in \Lambda_1$ and $\lambda_0 \in \Lambda_2$ such that R is jointly continuous when we consider only (E, p_{μ_0}), (F, q_{λ_0}) and (G, r_{ν_0}). Moreover by [18, chapter.III.5.1], separate continuity and joint continuity coincide in the class of Fréchet and in particular, Banach spaces.

A topological algebra $\mathcal{A}$ is an algebra, which is a topological vector space, such that the multiplication

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A},$$

defined by $(a, b) \longrightarrow ab$ is a separately continuous mapping; see [7, Definition (3.1.5)]. A Fréchet algebra is a complete topological algebra, whose topology is given by the countable family of increasing submultiplicative seminorms; see [7] and [12] for more information. A quasinormable Fréchet algebra is a Fréchet algebra such that for each zero neighborhood U there is a zero neighborhood V so that for every $\varepsilon > 0$ there exists a bounded set B in $\mathcal{A}$ with $V \subseteq B + \varepsilon U$.

Let $\mathcal{A}$ be a Fréchet algebra. A Fréchet space X is called a Fréchet $\mathcal{A}-$bimodule, if X is an algebraic $\mathcal{A}-$bimodule and the actions on both sides are continuous. We know from [15] that if X is a Fréchet space, then the dual space X^* of X will be endowed with the strong topology, which means the topology of uniform convergence on bounded subsets of X. Note that this coincides with the usual norm topology of X, if X is a normed space. Moreover X^* is a locally convex space. Note that given a Fréchet $\mathcal{A}-$bimodule X, X^* is a locally convex $\mathcal{A}-$bimodule with the continuous actions in the usual way. In fact these actions are separately continuous. By [19], the action of $\mathcal{A}$ on X^* often fails to be jointly continuous.

2. **Main results**

We first recall some of the basic facts about derivations in the algebra setting. Let $\mathcal{A}$ be an algebra, and let X be an $\mathcal{A}-$bimodule. A linear map $D : \mathcal{A} \to X$ is called a derivation if

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in \mathcal{A}).$$

For each $x \in X$, we define a map $ad_x : \mathcal{A} \to X$ by

$$ad_x(a) = a.x - x.a \qquad (a \in \mathcal{A}),$$

which is in fact a derivation. Derivations of this form are called inner derivations.

We commence with the following definitions, which are in fact inspired by the same definitions related to Banach algebras.

Definition 2.1. The Fréchet algebra $\mathcal{A}$ is called weakly amenable if every continuous derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is inner.

Definition 2.2. Let $\mathcal{A}$ be a Fréchet algebra and $\varphi \in \sigma(\mathcal{A})$, the set consisting of all non-zero continuous characters on $\mathcal{A}$. A point derivation d at φ is a linear functional satisfying

$$d(xy) = d(x)\varphi(y) + \varphi(x)d(y), \quad (x, y \in \mathcal{A});$$

i.e. d is a derivation into the $\mathcal{A}-$bimodule $\mathbb{C}$, where the module actions is defined by

$$x.\lambda = \lambda.x = \lambda\varphi(x), \ \ x \in \mathcal{A}, \ \ \lambda \in \mathbb{C}.$$

As the first result of the present work, we show that [3, Theorem 2.8.63] is also valid for the case where $\mathcal{A}$ is a Fréchet algebra. Our proof is a slight modification of the proof of this result. First recall from [3] that if $\mathcal{A}$ is an algebra and E is an $\mathcal{A}-$bimodule, then E is called commutative if $a.x = x.a$, for all $a \in \mathcal{A}$ and $x \in E$. Moreover

$$A.E = \{a.x : \ \ a \in \mathcal{A}, x \in E\}.$$

Also $\mathcal{A}E$ is the linear span of the set $\mathcal{A}.E$.

Theorem 2.3. *Let $\mathcal{A}$ be a weakly amenable Fréchet algebra. Then the following assertions hold;*

(i) *$\mathcal{A}$ is essential; i.e. $\overline{\mathcal{A}^2} = \mathcal{A}$.*
(ii) *There are no non-zero, continuous point derivations on $\mathcal{A}$.*
(iii) *In the case where $\mathcal{A}$ is commutative, each continuous derivation from $\mathcal{A}$ into E is zero, for each commutative complete barrelled locally convex $\mathcal{A}-$bimodule E.*

Proof. (i). Let $(p_n)_{n\in\mathbb{N}}$ be the fundamental system of seminorms on $\mathcal{A}$. Assume towards a contradiction that $\overline{\mathcal{A}^2} \neq \mathcal{A}$. Take $a_0 \in \mathcal{A} \setminus \overline{\mathcal{A}^2}$. By the Hahn Banach Theorem, choose $\lambda_0 \in \mathcal{A}^*$ with $\lambda_0|_{\mathcal{A}^2} = 0$ and $< a_0, \lambda_0 >= 1$, and define $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ by $D(a) =< a, \lambda_0 > \lambda_0$. We show that D is a continuous derivation on $\mathcal{A}$. Suppose that $a \in \mathcal{A}$ and $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$ such that $\lim_{n\to\infty} a_n = a$, in the topology of $\mathcal{A}$. Since $\lambda_0 \in \mathcal{A}^*$, there exist $n_0 \in \mathbb{N}$ and $k > 0$ such that

$$| < x, \lambda_0 > | \leq kp_{n_0}(x),$$

for all $x \in \mathcal{A}$. Moreover by [16, Remark 23.2], $\sup_{x\in B} p_n(x) < \infty$ for each n, where $B \subseteq \mathcal{A}$ is a bounded set. Therefore there exists $M_{n_0} > 0$ such that

$$\sup_{x\in B} p_{no}(x) \leq M_{n_0}.$$

Hence for each $x \in B$, we have

$$\begin{aligned} | < x, D(a_n - a) > | &= | < x, < a_n - a, \lambda_0 > \lambda_0 > | \\ &= | < a_n - a, \lambda_0 > || < x, \lambda_0 > | \\ &\leq kp_{n_0}(x)| < a_n - a, \lambda_0 > |. \end{aligned}$$

Thus

$$\begin{aligned}\sup_{x\in B}|<x,D(a_n-a)>| &= \sup_{x\in B}\{|<a_n-a,\lambda_0>||<x,\lambda_0>|\}\\ &\le kM_{n_0}|<a_n-a,\lambda_0>|.\end{aligned}$$

The right hand side of the above inequality tends to zero, and therefore D is continuous. Also one can prove that D is a derivation, exactly similar to the Banach case. By the hypothesis, $\mathcal{A}$ is weakly amenable. Thus D is inner and so there exists $\lambda\in\mathcal{A}^*$ such that $D(a)=a.\lambda-\lambda.a$ and so $D(a_0)=a_0.\lambda-\lambda.a_0$. Consequently

$$\begin{aligned}<a_0,D(a_0)> &= <a_0,a_0.\lambda-\lambda.a_0>\\ &= <a_0,a_0.\lambda>-<a_0,\lambda.a_0>\\ &= <{a_0}^2,\lambda>-<a_0^2,\lambda>\\ &= 0.\end{aligned}$$

On the other hand

$$<a_0,D(a_0)>=<a_0,<a_0,\lambda_0>\lambda_0>=<a_0,\lambda_0><a_0,\lambda_0>=1$$

which is a contradiction. It follows that $\overline{\mathcal{A}^2}=\mathcal{A}$.

(ii). Let d be a continuous point derivation on $\mathcal{A}$ at a character $\varphi\in\sigma(\mathcal{A})$. Thus d is a continuous linear functional on $\mathcal{A}$ such that

$$d(ab)=\varphi(a)d(b)+d(a)\varphi(b),$$

for $a,b\in\mathcal{A}$. Then the map

$$D:\mathcal{A}\longrightarrow\mathcal{A}^*$$

defined as $D(a)=d(a)\varphi$ is a continuous derivation. Indeed,

$$D(ab)=d(ab)\varphi=\varphi(a)d(b)\varphi+\varphi(b)d(a)\varphi$$

and also $a.D(b)=a.d(b)\varphi$. Moreover

$$<x,a.d(b)\varphi>=<xa,d(b)\varphi>=d(b)<xa,\varphi>=d(b)<x,\varphi(a)\varphi>,$$

for all $x\in A$. It follows that

$$a.D(b)=a.d(b)\varphi=d(b)\varphi(a)\varphi.$$

Similarly $D(a).b=d(a)\varphi(b)\varphi$ and so

$$D(ab)=a.D(b)+D(a).b.$$

Now we show that D is continuous. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$ such that $\lim_{n\to\infty}a_n=a$, in the topology of $\mathcal{A}$. Then

$$|<x,D(a_n-a)>|=|<x,d(a_n-a)\varphi>|=|d(a_n-a)||\varphi(x)|.$$

Since φ is continuous, there exist $n_0\in\mathbb{N}$ and $k>0$ such that $|\varphi(x)|\le kp_{n_0}(x)$, for each $x\in\mathcal{A}$. Also

$$\sup_{x\in B}p_{n_0}(x)\le M_{n_0},$$

for some positive constant M_{n_0}, where B is a bounded subset of $\mathcal{A}$. Consequently

$$\sup_{x\in B}|\varphi(x)| \leq k \sup_{x\in B} p_{n_0}(x) \leq kM_{n_0}.$$

Therefore

$$\begin{aligned}\sup_{x\in B}|< x, D(a_n - a) >| &= \sup_{x\in B}|< x, d(a_n - a)\varphi >| \\ &= |d(a_n - a)| \sup_{x\in B}|\varphi(x)| \\ &\leq kM_{n_0}|d(a_n - a)|.\end{aligned}$$

Since d is continuous, the right hand side of the above inequality tends to zero, and so D is continuous. By the weak amenability of $\mathcal{A}$, there exists $\lambda \in \mathcal{A}^*$ such that $D(a) = a.\lambda - \lambda.a$, for each $a \in \mathcal{A}$ and so

$$\begin{aligned}d(a)\varphi(a) &= < a, d(a)\varphi >=< a, D(a) > \\ &= < a, a.\lambda > - < a, \lambda.a > \\ &= < a^2, \lambda > - < a^2, \lambda > \\ &= 0.\end{aligned}$$

Consequently for each $a \in \mathcal{A}$, $d(a)\varphi(a) = 0$. Thus $d(x) = 0$, for all $x \notin ker\varphi$. It follows that $d = 0$.

(iii). Assume that there exists a non-zero continuous derivation D from $\mathcal{A}$ to the complete barrelled commutative locally convex $\mathcal{A}-$bimodule E. By part (i), $\overline{\mathcal{A}^2} = \mathcal{A}$ and so there exists $a_0 \in \mathcal{A}$ with $D({a_0}^2) \neq 0$. In fact if for each $a \in \mathcal{A}$, $D(a^2) = 0$ then continuity of D together with part (i) imply that $D = 0$, which contradicts the hypothesis. Moreover

$$D({a_0}^2) = a_0.D(a_0) + D(a_0).a_0 = 2a_0.D(a_0).$$

Thus $a_0.D(a_0) \neq 0$ and so by the Hahn Banach Theorem, there exists $\lambda \in E^*$ with $< a_0.D(a_0), \lambda >= 1$. Let

$$R_\lambda : E \longrightarrow \mathcal{A}^*; \quad < a, R_\lambda(x) >=< a.x, \lambda >, \quad (x \in E, a \in \mathcal{A}).$$

We show that R_λ is a continuous $\mathcal{A}-$bimodule homomorphism. For all $a, b \in \mathcal{A}$ and $x \in E$ we have

$$\begin{aligned}< a, R_\lambda(b.x) > &= < a.(b.x), \lambda >=< (ab).x, \lambda > \\ &= < ab, R_\lambda(x) >=< a, b.R_\lambda(x) >\end{aligned}$$

and so $R_\lambda(b.x) = b.R_\lambda(x) = R_\lambda(x).b$. Suppose that $(p_\mu)_{\mu\in\Lambda}$ is the fundamental system of seminorms on E, and $\lim_\alpha x_\alpha = x$ in E. Since λ is continuous on E, there exist $\mu_0 \in \Lambda$ and $k > 0$ with

$$|< (a.(x_\alpha - x)), \lambda >| \leq kp_{\mu_0}(a.(x_\alpha - x)),$$

for all $a \in \mathcal{A}$ and α. Moreover, since E is a barrelled $\mathcal{A}-$bimodule, for every bounded subset B of $\mathcal{A}$ and every $\varepsilon > 0$, there is a zero neighborhood U in E such that

$$B.U \subseteq p_{\mu_0}^{-1}([0, \varepsilon));$$

see [18, 5.2, page 89], for more details. Also there exists α_0 such that $x_\alpha - x \in U$, for each $\alpha \geq \alpha_0$. It follows that for all $a \in B$ and $\alpha \geq \alpha_0$

$$p_{\mu_0}(a.x_\alpha - a.x) < \varepsilon.$$

Consequently

$$\sup_{a\in B} | < a.x_\alpha - a.x, \lambda > | \leq k \sup_{a\in B} p_{\mu_0}(a.(x_\alpha - x)) \leq k\varepsilon,$$

for each $\alpha \geq \alpha_0$. This implies the continuity of R_λ. It is clear that $R_\lambda \circ D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is continuous. We show that $R_\lambda \circ D$ is a derivation. For $a, b, x \in \mathcal{A}$ we have

$$\begin{aligned} < x, (R_\lambda \circ D)(ab) > &= < x, R_\lambda(D(ab)) >=< x.D(ab), \lambda > \\ &= < xa.D(b), \lambda > + < x.D(a).b, \lambda > \\ &= < xa.D(b), \lambda > + < bx.D(a), \lambda > . \end{aligned}$$

On the other hand

$$\begin{aligned} < x, (R_\lambda \circ D)(a).b + a.(R_\lambda \circ D)(b) > &= < bx, (R_\lambda \circ D)(a) > \\ &\quad + < xa, (R_\lambda \circ D)(b) > \\ &= < bx.D(a), \lambda > \\ &\quad + < xa.D(b), \lambda >, \end{aligned}$$

which implies that $R_\lambda \circ D$ is a derivation. Since $\mathcal{A}$ is weakly amenable $R_\lambda \circ D$ is inner and so there exists $\varphi \in \mathcal{A}^*$ with $(R_\lambda \circ D)(a) = a.\varphi - \varphi.a$, for all $a \in \mathcal{A}$. Since $\mathcal{A}$ is commutative, it follows that $R_\lambda \circ D = 0$. But

$$< a_0, (R_\lambda \circ D)(a_0) >=< a_0.D(a_0), \lambda >= 1,$$

which is a contradiction. This completes the proof. □

The following result provides us a sufficient condition under which weak amenability of $\mathcal{A}$ is obtained. Recall that an element p of $\mathcal{A}$ is called an idempotent if $p^2 = p$.

Proposition 2.4. *Let $\mathcal{A}$ be a commutative Fréchet algebra. Suppose that $\mathcal{A}$ is spanned by its idempotents. Then $\mathcal{A}$ is weakly amenable.*

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a continuous derivation and p be an idempotent element of $\mathcal{A}$. Then $p.D(p).p = 0$. Indeed $D(p) = D(p^2) = p.D(p) + D(p).p$, and hence

$$p.D(p) = p^2.D(p) + p.D(p).p,$$

which implies that $p.D(p).p = 0$. On the other hand $\mathcal{A}$ is commutative, hence $p.D(p) = D(p).p$. It follows that $p.D(p) = D(p).p = 0$ and consequently

$$D(p) = D(p^2) = p.D(p) + D(p).p = 0.$$

It follows that $D = 0$. Thus D is inner and so $\mathcal{A}$ is weakly amenable. □

Remark 2.5. Let $\mathcal{A}$ be a Fréchet algebra, whose topology is defined by the fundamental system of seminorms $(p_n)_{n\in\mathbb{N}}$. Let I be a proper closed ideal of $\mathcal{A}$.

(i) By [7, (3.2.10)], $\dfrac{\mathcal{A}}{I}$ endowed with the quotient topology is a Fréchet space and this topology is defined by the seminorms

$$q_n(a+I) = \inf\{p_n(a+b) : \quad b \in I\}.$$

Moreover the multiplication

$$(a+I, b+I) \mapsto ab+I, \qquad\qquad (a,b \in \mathcal{A})$$

on $\dfrac{\mathcal{A}}{I}$ is continuous, i.e. $\dfrac{\mathcal{A}}{I}$ is a topological algebra and each q_n is submultiplicative. It follows that $\dfrac{\mathcal{A}}{I}$ is a Fréchet algebra. Moreover by [16, Lemma 22.10], the quotient map $\pi : \mathcal{A} \to \frac{\mathcal{A}}{I}$ is continuous and open.

(ii) $(\frac{\mathcal{A}}{I})^*$ is a complete locally convex $\frac{\mathcal{A}}{I}-$bimodule. Also we have $(\frac{\mathcal{A}}{I})^* = I^\circ$, as two vector spaces. But this equality is an isomorphism, whenever I is quasinormable; see [16, Proposition 26.18] and [16, Remark 26.5].

Theorem 2.6. *Let $\mathcal{A}$ be a Fréchet algebra and I be a quasinormable closed ideal of $\mathcal{A}$. If I and $\frac{\mathcal{A}}{I}$ are weakly amenable, then $\mathcal{A}$ is also weakly amenable.*

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a continuous derivation and $\iota : I \longrightarrow \mathcal{A}$ be the natural embedding. Then $\iota^* \circ D \circ \iota : I \longrightarrow I^*$ is a derivation, obviously. By [16, Proposition (23.30)], $\iota^* \circ D \circ \iota$ is continuous. By the hypothesis, I is weakly amenable. Thus there exists $\lambda_1 \in I^*$ with $(\iota^* \circ D \circ \iota)(a) = a.\lambda_1 - \lambda_1.a$. One can extend λ_1 to a continuous linear functional $\overline{\lambda_1}$ on $\mathcal{A}$. By replacing D by $D - ad_{\overline{\lambda_1}}$, we may suppose that $(\iota^* \circ D)|_I = 0$. For all $a, b \in I$ and $c \in \mathcal{A}$ we have

$$\begin{aligned} < c, D(ab) > &= < c, a.D(b) > + < c, D(a).b > \\ &= < ca, D(b) > + < bc, D(a) > \\ &= < ca, (\iota^* \circ D)(b) > + < bc, (\iota^* \circ D)(a) > \\ &= 0 \end{aligned}$$

and so $D|_{I^2} = 0$. By Theorem 2.3 part (i), $\overline{I^2} = I$ and consequently $D|_I = 0$. Set

$$F = \overline{I\mathcal{A} + \mathcal{A}I}.$$

So, we have $F = \overline{I^2} = I$. Then F is a closed ideal of $\mathcal{A}$ and so it is a closed Fréchet $\mathcal{A}-$bimodule and also a closed $\mathcal{A}-$submodule of $\mathcal{A}$. For $a \in \mathcal{A}$ and $b \in I$ we have $ab \in I$ and so $a.D(b) = 0$ and $D(ab) = 0$; and so $D(a).b = 0$. Take $x \in \mathcal{A}$. Then

$$< bx, D(a) >=< x, D(a).b >= 0$$

and so $D(a)|_{I.\mathcal{A}} = 0$. Similarly $D(a)|_{\mathcal{A}.I} = 0$ and so $D(a)|_F = 0$ which implies that $D(\mathcal{A}) \subseteq F^\circ$. Since $\mathcal{A}$ and I are Fréchet algebras, $\frac{\mathcal{A}}{I}$ is a Fréchet $\frac{\mathcal{A}}{I}-$bimodule and hence $(\frac{\mathcal{A}}{I})^*$ is a locally convex $\frac{\mathcal{A}}{I}-$bimodule. Now consider the map

$$D_I : \frac{\mathcal{A}}{I} \longrightarrow (\frac{\mathcal{A}}{I})^* = F^\circ,$$

defined by $D_I(a+I) = D(a)$, which is well defined by the explanations given in Remark 2.5. So one can easily show that D_I is a derivation. We show that D_I is continuous. Let $a+I \in \frac{\mathcal{A}}{I}$ and $(a_n+I)_{n\in\mathbb{N}}$ be a sequence in $\frac{\mathcal{A}}{I}$ such that $a_n+I \to a+I$, in the topology of $\frac{\mathcal{A}}{I}$. Without loss of generality one can assume that $a_n \to a$, in the topology of $\mathcal{A}$. In fact since I is quasinormable, by [16, Proposition 26.18], for each bounded subset B of $\frac{\mathcal{A}}{I}$ there exists a bounded subset C of $\mathcal{A}$ such that $\pi(C) = B$. Moreover $D(\mathcal{A})|_I = 0$. Thus

$$\begin{aligned} sup_{(d+I)\in B}|<d+I, D_I(a_n-a+I)>| &= sup_{(d+I)\in\pi(C)}|<d+I, D_I(a_n-a+I)>| \\ &= sup_{c\in C}|<c+I, D_I(a_n-a+I))>| \\ &= sup_{c\in C}|<c, D(a_n-a)>| \\ &= sup_{c\in C}|<c, D(a_n-a)>|. \end{aligned}$$

Since D is continuous, $D(a_n - a) \to 0$ in the topology of $\mathcal{A}^*$. Consequently

$$D_I(a_n+I) \to D_I(a+I),$$

in the topology of $(\frac{\mathcal{A}}{I})^*$. Thus D_I is continuous. By the hypothesis, $\frac{\mathcal{A}}{I}$ is weakly amenable. Thus there exists $\lambda_2 \in F^\circ$ such that

$$D(x) = x.\lambda_2 - \lambda_2.x.$$

It follows that $\mathcal{A}$ is weakly amenable. □

Remark 2.7. Let $\mathcal{A}$ and $\mathcal{B}$ be Fréchet algebras, and $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous homomorphism. We show that $\mathcal{B}$ is a Fréchet $\mathcal{A}-$bimodule by the maps defined as

$$\mathcal{A}\times\mathcal{B} \longrightarrow \mathcal{B}, \quad (a,b) \longrightarrow a.b, \quad a.b = \theta(a)b$$

$$\mathcal{B}\times\mathcal{A} \longrightarrow \mathcal{B}, \quad (b,a) \longrightarrow b.a, \quad b.a = b\theta(a).$$

It is clear that $\mathcal{B}$ is an algebraic $\mathcal{A}-$bimodule. Let $(p_n)_{n\in\mathbb{N}}$ and $(q_m)_{m\in\mathbb{N}}$ be the fundamental system of seminorms on $\mathcal{A}$ and $\mathcal{B}$, respectively. The map

$$\mathcal{A}\times\mathcal{B} \longrightarrow \mathcal{B}, \quad (a,b) \longrightarrow a.b = \theta(a)b$$

is continuous. Indeed, since q_m is submultiplicative

$$q_m(b_1b_2) \le q_m(b_1)q_m(b_2),$$

for every $b_1, b_2 \in \mathcal{B}$. On the other hand θ is a continuous homomorphism. Then for every $m \in \mathbb{N}$ there exist $n_0 \in \mathbb{N}$ and $k_0 > 0$ such that $q_m(\theta(a)) \le k_0 p_{n_0}(a)$, for each $a \in \mathcal{A}$. Thus for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$

$$q_m(\theta(a)b) \le q_m(\theta(a))q_m(b) \le k_0 p_{n_0}(a)q_m(b),$$

for every $a \in \mathcal{A}, b \in \mathcal{B}$. So the left action of the algebra $\mathcal{A}$ on $\mathcal{B}$ is continuous. Similarly

$$\mathcal{B}\times\mathcal{A} \longrightarrow \mathcal{B}, \quad (b,a) \longrightarrow b.a = b\theta(a)$$

is continuous and so $\mathcal{B}$ is a Fréchet $\mathcal{A}-$bimodule. Thus $\mathcal{B}^*$ is a complete locally convex $\mathcal{A}-$bimodule by the maps defined as

$$\mathcal{A}\times\mathcal{B}^* \longrightarrow B^*, \quad (a,\lambda) \longrightarrow a.\lambda, \quad <b, a.\lambda> = <b.a, \lambda>$$

and

$$\mathcal{B}^* \times \mathcal{A} \longrightarrow \mathcal{B}^*, \quad (\lambda, a) \longrightarrow \lambda.a, \quad < b, \lambda.a >=< a.b, \lambda >,$$

where $a \in \mathcal{A}, b \in \mathcal{B}$ and $\lambda \in \mathcal{B}^*$. Also we have

$$< b, a.\lambda >=< b.a, \lambda >=< b\theta(a), \lambda >=< b, \theta(a).\lambda >,$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\lambda \in \mathcal{B}^*$. Therefore $a.\lambda = \theta(a).\lambda$. Similarly $\lambda.a = \lambda.\theta(a)$.

Proposition 2.8. *Let $\mathcal{A}$ be a Fréchet algebra and $\mathcal{B}$ be a Banach algebra. Suppose that $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous homomorphism with $\overline{\theta(\mathcal{A})} = \mathcal{B}$. If $\mathcal{A}$ is commutative and weakly amenable, then $\mathcal{B}$ is weakly amenable.*

Proof. Let $D : \mathcal{B} \longrightarrow \mathcal{B}^*$ be a continuous derivation. It is not hard to obtain that $D \circ \theta : \mathcal{A} \longrightarrow \mathcal{B}^*$ is a continuous derivation. Moreover, for all $a, b \in \mathcal{A}$ and $\lambda \in \mathcal{B}^*$,

$$\begin{aligned} < \theta(b), a.\lambda > &= < \theta(b)\theta(a), \lambda >=< \theta(ba), \lambda > \\ &= < \theta(a)\theta(b), \lambda >=< \theta(b), \lambda.a > . \end{aligned}$$

Now continuity of θ together with the fact that $\overline{\theta(\mathcal{A})} = \mathcal{B}$, yield that

$$< c, \lambda.a >=< c, a.\lambda >,$$

for all $c \in \mathcal{B}$, $a \in \mathcal{A}$ and $\lambda \in \mathcal{B}^*$. Consequently $a.\lambda = \lambda.a$ and so $\mathcal{B}^*$ is a commutative Banach $\mathcal{A}-$bimodule. Thus $\mathcal{B}^*$ is barreled, and so by Theorem 2.3 part (iii), we have $D \circ \theta = 0$. Since $\overline{\theta(\mathcal{A})} = \mathcal{B}$, it follows that $D = 0$. Hence D is an inner derivation, which implies that $\mathcal{B}$ is weakly amenable. □

In the sequel, we provide weak amenability of $\frac{\mathcal{A}}{I}$ by the weak amenability of $\mathcal{A}$, under some extra conditions. First we introduce the concept of trace extension property of closed ideals in the Fréchet algebras, which comes from this definition for Banach algebras. Recall that a linear functional τ on $\mathcal{A}$ is called a trace if $\tau(ab) = \tau(ba)$, for all $a, b \in \mathcal{A}$.

Definition 2.9. Let I be a closed ideal in a Fréchet algebra $\mathcal{A}$. Then I has the trace extension property if, for each $\lambda \in I^*$ with $a.\lambda = \lambda.a$ $(a \in \mathcal{A})$, there exists a continuous trace, τ on $\mathcal{A}$ such that $\tau|_I = \lambda$.

Proposition 2.10. *Let $\mathcal{A}$ be a Fréchet algebra and I be a quasinormable closed ideal of $\mathcal{A}$.*

(i) *Suppose that $\frac{\mathcal{A}}{I}$ is weakly amenable. Then I has the trace extension property.*
(ii) *Suppose that $\mathcal{A}$ is weakly amenable and that I has the trace extension property. Then $\frac{\mathcal{A}}{I}$ is weakly amenable.*

Proof. (i). Take $\lambda \in I^*$, with $a.\lambda = \lambda.a$ $(a \in \mathcal{A})$, and choose $\Lambda \in \mathcal{A}^*$ with $\Lambda|_I = \lambda$. Define

$$D : \frac{\mathcal{A}}{I} \longrightarrow I^\circ = (\frac{\mathcal{A}}{I})^*, \quad D(a + I) = a.\Lambda - \Lambda.a.$$

Remark 2.5 implies that D is well defined. Also D is clearly a derivation. Some arguments similar to the proof of the continuity of derivation D_I in Theorem 2.6

with the fact that the module actions of $\mathcal{A}$ on $\mathcal{A}^*$ is continuous, provide that D is continuous. Since $\frac{\mathcal{A}}{I}$ is weakly amenable, there exists $\mu \in (\frac{\mathcal{A}}{I})^* = I^\circ$ such that

$$D(a+I) = (a+I).\mu - \mu.(a+I) = a.\mu - \mu.a, \quad (a \in \mathcal{A}).$$

Set

$$\tau = \Lambda - \mu \in \mathcal{A}^*.$$

Then $a.\tau = \tau.a$ for each $a \in \mathcal{A}$ and so τ is a continuous trace on $\mathcal{A}$ with $\tau|_I = \lambda$. In fact

$$\tau|_I = \Lambda|_I - \mu|_I = \lambda - 0 = \lambda.$$

Consequently I has the trace extension property.

(ii). Let

$$D : \frac{\mathcal{A}}{I} \longrightarrow (\frac{\mathcal{A}}{I})^* = I^\circ$$

be a continuous derivation and set $\tilde{D} = \pi^* \circ D \circ \pi$. Note that again we have used Remark 2.5. Then $\tilde{D} : \mathcal{A} \longrightarrow \mathcal{A}^*$ is a continuous derivation and so there exists $\lambda \in \mathcal{A}^*$ such that $\tilde{D}(a) = a.\lambda - \lambda.a, (a \in \mathcal{A})$. Thus $a.\lambda|_I - \lambda|_I.a = 0$, by $\tilde{D}(a)|_I = 0$. Since I has the trace extension property, there exists $\tau \in \mathcal{A}^*$ such that $a.\tau = \tau.a,\ (a \in \mathcal{A})$ and $\tau|_I = \lambda|_I$. Consequently $\lambda - \tau \in I^\circ = (\frac{\mathcal{A}}{I})^*$. Moreover it is not hard to see that

$$D(a+I) = a.(\lambda - \tau) - (\lambda - \tau).a \quad (a \in A),$$

and so D is inner. Therefore $\frac{\mathcal{A}}{I}$ is weakly amenable. □

The following result is immediately obtained from Proposition 2.10.

Corollary 2.11. *Let $\mathcal{A}$ be a commutative Fréchet algebra and I be a quasinormable closed ideal of $\mathcal{A}$. If $\mathcal{A}$ is weakly amenable, then $\frac{\mathcal{A}}{I}$ is also weakly amenable.*

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