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THE SECOND DUAL OF VECTOR-VALUED LIPSCHITZ ALGEBRAS

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Let (X,d) be a locally compact metric space, $0<\alpha\leqslant 1$ and E be a Banach algebra such that the linear span of character space $\Delta(E)$ be norm-dense in E^*. Then $lip^0_\alpha(X,E)^{**}$ is isometrically isomorphic as Banach algebra with $Lip_\alpha(X,E^{**})$. We show that $lip^0_\alpha(X,E)$ is Arens regular and 2−weakly amenable Banach algebra.

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1. Introduction

Let (X,d) be a metric space and $B(X)$ (resp. $C_b(X)$) indicates the Banach space consisting of all bounded complex-valued functions on X, endowed with the norm

$$\|f\|_{sup}=\sup_{x\in X}|f(x)| \qquad (f\in B(X)).$$

Take $\alpha\in\mathbb{R}$ with $\alpha>0$, then $Lip_\alpha X$ is the subspace of $B(X)$, consisting of all bounded complex-valued functions f on X such that

$$p_\alpha(f):=\sup\left\{\frac{|f(x)-f(y)|}{d(x,y)^\alpha}:x,y\in X,\ \ x\neq y\right\}<\infty.$$

It is known that $Lip_\alpha X$ is endowed with the norm $\|.\|_\alpha$ given by

$$\|f\|_\alpha=p_\alpha(f)+\|f\|_{sup};$$

and pointwise product is a unital commutative Banach algebra, called Lipschitz algebra.

Let (X,d) be a metric space with at least two elements and $(E,\|.\|)$ be a Banach space over the scalar field $\mathbb{F}(=\mathbb{R}\ or\ \mathbb{C})$ for a constant $\alpha>0$ and a function $f:X\longrightarrow E$, set

$$p_{\alpha,E}(f):=\sup_{x\neq y}\frac{\|f(x)-f(y)\|}{d(x,y)^\alpha},$$

which is called the Lipschitz constant of f . For any metric space (X,d), any Banach algebra E and any $\alpha>0$, we define the Lipschitz algebra $Lip_\alpha(X,E)$ by

$$Lip_\alpha(X,E):=\{f:X\longrightarrow E: f \text{ is bounded and } p_{\alpha,E}(f)<\infty\},$$

with pointwise multiplication and norm

$$\|f\|_{\alpha,E}:=p_{\alpha,E}(f)+\|f\|_{\infty,E}.$$

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The Lipschitz algebra $lip_\alpha(X,E)$ is a subalgebra of $Lip_\alpha(X,E)$ defined by

$$lip_\alpha(X,E) = \{f : X \longrightarrow E \ : \ \frac{\|f(x)-f(y)\|}{d(x,y)^\alpha} \longrightarrow 0 \text{ as } d(x,y) \longrightarrow 0\}.$$

If X is a locally compact metric space, then $lip^0_\alpha(X,E)$ is a subalgebra of $lip_\alpha(X,E)$ consisting of those functions tend to zero at infinity. The elements of $Lip_\alpha(X,E)$ and $lip_\alpha(X,E)$ are called big and little Lipschitz operators, respectively. Set,

$$|||f|||_{\alpha,E} := \max\{\|f\|_{\infty,E}, p_{\alpha,E}(f)\},$$

for all $f \in Lip_\alpha(X,E)$. The $\|.\|_{\alpha,E}$ and $|||.|||_{\alpha,E}$ are equivalent norms on $Lip_\alpha(X,E)$. Let $C_b(X,E)$ be the set of all bounded continuous functions from X into E. For each $f \in C_b(X,E)$, define the norm

$$\|f\|_{\infty,E} := \sup_{x\in X} \|f(x)\|,$$

and for $f,g \in C_b(X,E)$ and $\lambda \in \mathbb{F}$, define

$$(f+g)(x) = f(x)+g(x),\ (\lambda f)(x) = \lambda f(x),\ (x \in X).$$

It is well known that $(C_b(X,E),\|.\|_{\infty,E})$ becomes a Banach space over $\mathbb{F}$ and $Lip_\alpha(X,E)$ is a linear subspace of $C_b(X,E)$, see [5]. If E is a Banach space (resp; algebra), then $(Lip_\alpha(X,E),\|.\|_{\alpha,E})$, $(lip_\alpha(X,E),\|.\|_{\alpha,E})$ and $(lip^0_\alpha(X,E),\|.\|_{\alpha,E})$ are Banach spaces (resp; algebras) of $C_b(X,E)$, see [4].

It is clear that the Lipschitz algebra $Lip_\alpha(X,E)$ contains the space $Cons(X,E)$, consisting of all constant vector-valued functions on X. The Lipschitz algebras were first considered in [2, 13]. There are valuable works related to some notions of amenability of Lipschitz algebras, [7, 9, 10] discussed amenability of vector-valued Lipschitz algebras. [3, 12] investigated some properties of vector-valued Lipschitz algebras.

Bade, Curtis and Dales [1] studied that if (X,d) is a compact metric space and $0 < \alpha \leqslant 1$, then the second dual space of $lip_\alpha X$ is isometrically isomorphic to $Lip_\alpha X$. The method of their proof is an adaptation of one due to de Leeuw [6] who proved the result, when X is the circle group $\mathbb{T}$. It was shown in [8] that $(lip_\alpha X)^{**}$ is isomorphic to $Lip_\alpha X$ in the case that X is a manifold. In [14], author studied $Lip_\alpha(X,B)$ and $(lip_\alpha(X,B))^{**}$, where X is a compact metric space and B is a Banach space. In general, $Lip_\alpha(X,B)$ is not Banach algebra, unless B is a Banach algebra. Moreover, he claimed that $lip_\alpha(X,B)^{**}$ and $Lip_\alpha(X,B)$ are isometrically isomorphism as Banach algebras. In this paper, we improve these results in a general case.
Moreover, we study that if (X,d) is a locally compact metric space, $0 < \alpha \leqslant 1$ and E is a Banach algebra such that the linear span of character space $\Delta(E)$ is a norm-dense in E^*, then $lip_\alpha(X,E)^{**}$ is isometrically isomorphism as Banach algebras with $Lip_\alpha(X,E^{**})$. Also, we prove that $lip^0_\alpha(X,E)$ is Arens regular and $2-$weakly amenable.

2. **Preliminaries**

Let (X,d) be a metric space and $\alpha > 0$. $Lip_\alpha(X,E)$, $lip_\alpha(X,E)$ and $lip^0_\alpha(X,E)$ are vector spaces, Banach spaces and Banach algebras whenever E is so, respectively. Also, if (X,d) is a metric space and E is a Banach algebra, then $Lip_\alpha(X,E)$ is a commutative (unital) Banach algebra if and only if E is a commutative (unital) Banach algebra. Let E be a $*-$Banach algebra and $f^*(x) = (f(x))^*$ for $x \in X$ and $f \in Lip_\alpha(X,E)$, then $p_\alpha(f^*) = p_\alpha(f)$ and $\|f^*\|_{\infty,E} = \|f\|_{\infty,E}$ so that $Lip_\alpha(X,E)$ is a $*-$Banach algebra.

It is easy to see that $f \in Lip_\alpha(X,E)$ if and only if $\sigma \circ f \in Lip_\alpha X$ for all $\sigma \in E^*$. Also, let (X,d) be a normed space, $\alpha > 1$ and E be Banach algebra. Then $Lip_\alpha(X,E) = cons(X,E)$.

Let (X,d) be a compact metric space and $0 < \alpha \leqslant 1$ and E be a Banach algebra, then $\Delta(C(X,E)) = \{\Delta_{x,\sigma} \ : x \in X,\ \sigma \in \Delta(E)\}$, where

$$\Delta_{x,\sigma}(f) = \sigma(f(x)),\ (f \in Lip_\alpha(X,E),\ x \in X).$$

Define $\varphi : X \times \Delta(E) \to \Delta(C(X,E))$ where $(x,\sigma) \to \Delta_{x,\sigma}$. Then φ is a bijection and $\Delta(C(X,E)) = X \times \Delta(E)$, see [11]. We set

$$\Delta(lip_\alpha(X,E)) = \{\varphi|_{lip_\alpha(X,E)} \ : \ \varphi \in X \times \Delta(E)\} := \{\Delta^l_{x,\sigma} \ : \ x \in X, \ \sigma \in \Delta(E)\},$$

$$\Delta(lip^0_\alpha(X,E)) = \{\varphi|_{lip^0_\alpha(X,E)} \ : \ \varphi \in X \times \Delta(E)\} := \{\Delta^0_{x,\sigma} \ : \ x \in X, \ \sigma \in \Delta(E)\},$$

and

$$\Delta(Lip_\alpha(X,E)) = \{\varphi|_{Lip_\alpha(X,E)} \ : \ \varphi \in X \times \Delta(E)\} := \{\Delta^L_{x,\sigma} \ : \ x \in X, \ \sigma \in \Delta(E)\}.$$

Let A be a commutative Banach algebra. Then the radical of A denoted by $Rad(A)$, is defined by

$$Rad(A) := \cap_{\varphi \in \Delta(A)} \ker \varphi.$$

Clearly, $Rad(A)$ is a closed ideal of A. Also A is called semisimple if

$$Rad(A) = \{0\}.$$

Lemma 2.1. *Let (X,d) be a metric space, E be a commutative Banach algebra and $0 < \alpha \leqslant 1$. Then the following statements are equivalent.*
(i) $C_b(X,E)$ is a semisimple Banach algebra.
(ii) $Lip_\alpha(X,E)$ is a semisimple Banach algebra.
(iii) $lip_\alpha(X,E)$ is a semisimple Banach algebra.
(iv) E is a semisimple Banach algebra.

Proof. $(iv) \Longrightarrow (i)$ Let $x \in X$ and $\theta_x : C_b(X,E) \to E$ is defined by $\theta_x(f) = f(x)$. Then θ_x is linear, continuous and epimorphism. Thus

$$\theta_x(Rad(C_b(X,E))) \subseteq Rad(E) = \{0\}.$$

So

$$Rad(C_b(X,E)) \subseteq \ker(\theta_x) = \{f : \ f(x) = 0\}.$$

Hence $Rad(C_b(X,E)) \subseteq \cap_{x\in X} \ker(\theta_x) = \{0\}$. So $C_b(X,E)$ is semisimple.
$(i) \Longrightarrow (iv)$ Let $\varphi : E \to C_b(X,E)$, define by $\varphi(z) = \varphi_z$, where $\varphi_z(x) = z$ for $x \in X$. Then φ is linear, isometric and homomorphism. Hence

$$\varphi(Rad(E)) \subseteq Rad(C_b(X,E)) = \{0\}.$$

But φ is one-to-one, so $Rad(E) = \{0\}$.

$(ii) \Longrightarrow (iv)$ Let $\varphi : E \to Lip_\alpha(X,E)$ defined by $\varphi(z) = f_z$, where $f_z(x) = z$ for $x \in X$. Therefore $\|f_z\|_{\alpha,E} = \|z\| = \|f_z\|_{\infty,E}$ for each $z \in E$. Hence

$$\varphi(Rad(E)) \subseteq Rad(Lip_\alpha(X,E)) = \{0\}.$$

Then $Rad(E) = \{0\}$.
$(iv) \Longrightarrow (ii)$ Let $\sigma \in \Delta(E)$ and $\varphi_\sigma : Lip_\alpha(X,E) \longrightarrow Lip_\alpha X$ define by $\varphi_\sigma(f) = \sigma \circ f$. Then φ_σ is linear, continuous and epimorphism. Thus

$$\varphi_\sigma(Rad(Lip_\alpha(X,E)) \subseteq Rad(Lip_\alpha X) \subseteq \cap_{x\in X} \delta_x = \{0\},$$

where, $\delta_x(g) = g(x)$ for $g \in Lip_\alpha X$. Hence

$$\begin{aligned} Rad(Lip_\alpha(X,E)) \subseteq \cap_{\sigma\in\Delta(E)} \ker \varphi_\sigma &= \{f : \ \sigma \circ f(x) = 0, \ \sigma \in \Delta(E), \ x \in X\} \\ &= \{f : \ f(x) \in \cap_{\sigma\in\Delta(E)} \ker \sigma, \ x \in X\} \\ &= \{f : \ f(x) \in Rad(E), \ x \in X\} = \{0\}. \end{aligned}$$

$(i) \Longrightarrow (iv)$ Let $\varphi : E \to Lip_\alpha(X,E)$ defined by $\varphi(z) := f_z$, where $f_z(x) = z$ for $x \in X$. Then $\|f_z\|_{\alpha,E} = \|z\| = \|f\|_{\infty,E}$ for each $z \in E$. Thus φ is well-defined.
Also,

$$\varphi(Rad(E)) \subseteq Rad(Lip_\alpha(X,E)) = \{0\},$$

and φ is one-to-one, so $Rad(E) = \{0\}$.
$(iii) \Longleftrightarrow (iv)$ is similar to $(ii) \Longleftrightarrow (iv)$.

□

Recall that (X,d) is called uniformly discrete if there exists $\varepsilon > 0$ such that $d(x,y) \geq \varepsilon$ for all $x,y \in X$ with $x \neq y$.

Lemma 2.2. *Let (X,d) be a uniformly discrete metric space, E be a Banach algebra and $\alpha > 0$. Then $Lip_\alpha(X,E) = B(X,E)$ with equivalent norms.*

Proof. Suppose that (X,d) is uniformly discrete. Thus there exists $\varepsilon > 0$ such that for all $x,y \in X$ with $x \neq y$, we have

$$d(x,y) \geq \varepsilon.$$

Suppose that $f \in B(X,E)$, we have

$$p_\alpha(f) = \sup_{x\neq y} \frac{\|f(x) - f(y)\|}{d(x,y)^\alpha} \leq \frac{1}{\varepsilon^\alpha} \sup_{x\neq y} \|f(x) - f(y)\| \leq \frac{2}{\varepsilon^\alpha}\|f\|_\infty < \infty.$$

It follows that $f \in Lip_\alpha(X,E)$. Moreover

$$\|f\|_\infty \leq \|f\|_\alpha \leq (1 + \frac{2}{\varepsilon^\alpha})\|f\|_\infty,$$

and consequently $B(X,E) = Lip_\alpha(X,E)$ with equivalent norms.

□

3. Second dual of vector-valued Lipschitz algebras

Let (X,d) be a compact metric space and $0 < \alpha \leqslant 1$. Bade, Curtis and Dales [1] showed that $(lip_\alpha X)^{**} \cong Lip_\alpha X$ isometrically isomorphic as Banach algebras.
In this section, we generalized it for locally compact metric space (X,d) and vector-valued Lipschitz algebras with a different proof. In fact, we show that

$$lip^0_\alpha(X,E)^{**} \cong Lip_\alpha(X,E^{**}).$$

Let (X,d) be a locally compact metric space, E be a Banach algebra and $0 < \alpha \leqslant 1$, $x \in X$ and $\sigma \in E^*$. Then $\Delta^0_{x,\sigma} \in lip^0_\alpha(X,E)^*$, where $\Delta^0_{x,\sigma}(f) = \sigma(f(x))$ for all $f \in lip^0_\alpha(X,E)^*$.
Also, if $\sigma \in \Delta(E)$ then $\Delta^0_{x,\sigma} \in \Delta(lip^0_\alpha(X,E))$. We need the following Lemma which its proof follows immediately form [12, Theorem 5.3].

Lemma 3.1. *Let (X,d) be a locally compact metric space, E be a Banach algebra and $0 < \alpha \leqslant 1$. Then the linear space of $\{\Delta^0_{x,\sigma};\ \sigma \in E^*,\ x \in X\}$ is norm-dense in $lip^0_\alpha(X,E)^*$*

We now state the main result of the paper.

Theorem 3.1. *Let (X,d) be a locally compact metric space, $0 < \alpha \leqslant 1$ and E be a Banach algebra such that the linear span of character space $\Delta(E)$ is norm-dense in E^*. Then*

$$lip^0_\alpha(X,E)^{**} \cong (Lip_\alpha(X,E^{**}), |||.|||_{\alpha,E^{**}}),$$

isometrically isomorphic as Banach algebras.

Proof. We the map $\varphi : lip^0_\alpha(X,E)^{**} \longrightarrow Lip_\alpha(X,E^{**})$ is defined by

$$[\varphi(F)(x)](\sigma) := F(\Delta^0_{x,\sigma}),\ (F \in lip^0_\alpha(X,E)^{**},\ \sigma \in E^*,\ x \in X),$$

where,

$$\Delta^0_{x,\sigma}(f) = \sigma \circ f(x),\ (f \in lip^0_\alpha(X,E)).$$

Clearly, $\Delta^0_{x,\sigma} \in lip^0_\alpha(X,E)^*$.

If $F_1 = F_2$, then $F_1(\Delta^0_{x,\sigma}) = F_2(\Delta^0_{x,\sigma})$. Thus

$$[\varphi(F_1)(x)](\sigma) = [\varphi(F_2)(x)](\sigma)\ (\sigma \in E^*).$$

Hence $\varphi(F_1)(x) = \varphi(F_2)(x)$ $(x \in X)$ and $\varphi(F_1) = \varphi(F_2)$. Therefore φ is well defined. It is obvious that φ is linear.

Now since

$$\begin{aligned}\|\Delta^0_{x,\sigma}\| = \sup_{\|g\|_{\alpha,E}\leqslant 1} |\Delta^0_{x,\sigma}(g)| &= \sup_{\|g\|_{\alpha,E}\leqslant 1} |\sigma \circ g(x)| \\ &\leqslant \sup_{\|g\|_{\alpha,E}\leqslant 1} \|\sigma\|\|g(x)\| \\ &\leqslant \|\sigma\| \sup_{\|g\|_{\alpha,E}\leqslant 1} \|g\|_{\infty,E} \leqslant \|\sigma\|,\end{aligned}$$

it follows that

$$|[\varphi(F)(x)](\sigma)| = |F(\Delta^0_{x,\sigma})| \leqslant \|F\|\|\Delta^0_{x,\sigma}\|\ (\sigma \in E^*,\ x \in X),$$

and so $\|\varphi(F)(x)\|_{\infty,E} \leqslant \|F\|$ for $F \in lip^0_\alpha(X,E)^{**}$. Hence $\|\varphi\| \leqslant 1$, and φ is continuous. Also,

$$\begin{aligned}\frac{\|\varphi(F)(x) - \varphi(F)(y)\|_{E^{**}}}{d(x,y)^\alpha} &= \sup_{\|\sigma\|\leqslant 1} \frac{|\varphi(F)(x)(\sigma) - \varphi(F)(y)(\sigma)|}{d(x,y)^\alpha} \\ &= \sup_{\|\sigma\|\leqslant 1} \frac{|F(\Delta^0_{x,\sigma}) - F(\Delta^0_{y,\sigma})}{d(x,y)^\alpha},\end{aligned}$$

and

$$\begin{aligned}\frac{|F(\Delta^0_{x,\sigma}) - F(\Delta^0_{y,\sigma})|}{d(x,y)^\alpha} &\leqslant \frac{\|F\|\|\Delta^0_{x,\sigma}) - \Delta^0_{y,\sigma}\|}{d(x,y)^\alpha} \\ &= \frac{\|F\|}{d(x,y)^\alpha} \sup_{\|g\|_{\alpha,E}\leqslant 1} |\sigma \circ g(x) - \sigma \circ g(y)|.\end{aligned}$$

Hence

$$\begin{aligned}p_{\alpha,E^{**}}(\varphi(F)) &= \sup_{x\neq y} \frac{\|\varphi(F)(x) - \varphi(F)(y)\|_{E^{**}}}{d(x,y)^\alpha} \\ &\leqslant \sup_{x\neq y} \sup_{\|\sigma\|\leqslant 1} \sup_{\|g\|_{\alpha,E}\leqslant 1} \frac{\|F\||\sigma \circ g(x) - \sigma \circ g(y)|}{d(x,y)^\alpha} \\ &= \sup_{x\neq y} \sup_{\|g\|_{\alpha,E}\leqslant 1} \sup_{\|\sigma\|\leqslant 1} \frac{\|F\||\sigma \circ g(x) - \sigma \circ g(y)|}{d(x,y)^\alpha} \\ &= \sup_{x\neq y} \sup_{\|g\|_{\alpha,E}\leqslant 1} \frac{\|F\|\|g(x) - g(y)\|}{d(x,y)^\alpha} \\ &= \sup_{\|g\|_{\alpha,E}\leqslant 1} \sup_{x\neq y} \frac{\|F\|\|g(x) - g(y)\|}{d(x,y)^\alpha} \\ &= \sup_{\|g\|_{\alpha,E}\leqslant 1} \|F\|p_{\alpha,E}(g) \leqslant \|F\|.\end{aligned}$$

Hence $p_{\alpha,E^{**}}(\varphi(F)) \leqslant \|F\|$. Define

$$\|\varphi(F)\|_{\alpha,E^{**}} := \max\{\|\varphi(F)\|_{\infty,E^{**}}, p_{\alpha,E^{**}}(\varphi(F))\}$$

Then $\|\varphi(F)\|_{\alpha,E^{**}} \leqslant \|F\| < \infty$, and so $\varphi(F) \in Lip_\alpha(X,E^{**})$. We show that $\|F\| \leqslant \|\varphi(F)\|_{\infty,E^{**}}$ since the linear space of $\{\Delta^0_{x,\sigma} : \ \|\sigma\| \leqslant 1, \ \sigma \in E^*, \ x \in X\}$ is norm-dense in $lip^0_\alpha(X,E)]^*$ by Lemma 3.1. Thus suppose that $f := \Delta^0_{x,\sigma}$, so

$$|F(f)| = |F(\Delta^0_{x,\sigma})| \leqslant \|\varphi(F)\|_{\infty,E^{**}}.$$

Also, if $f := \sum_{i=1}^n c_i\Delta^0_{x_i,\sigma_i}$ such that $\sum_{i=1}^n |c_i| \leqslant 1$ and $\|\sigma_i\| \leqslant 1$ Then

$$|F(f)| \leqslant \sum_{i=1}^n |c_i||\Delta^0_{x_i,\sigma_i}| \leqslant \|\varphi(F)\|_{\infty,E^{**}}.$$

If $f := norm - \lim_\gamma f_\gamma$, where $f_\gamma \in [lip^0_\alpha(X,E)]^*_1$, then

$$|F(f)| = \lim_\gamma |F(f_\gamma)| \leqslant \|\varphi(F)\|_{\infty,E^{**}}.$$

Therefore

$$\|F\| = \sup_{f\in[lip^0_\alpha(X,E)]^*_1} |F(f)| \leqslant \|\varphi(F)\|_{\infty,E^{**}} \leqslant \|\varphi(F)\|_{\alpha,E^{**}}.$$

Hence $\|\varphi(F)\|_{\alpha,E^{**}} = \|F\|$, and so φ is a isometry.
Let $f \in Lip_\alpha(X,E^{**})$ and $\lambda := \{\Delta^0_{x_1,\sigma_1}, \Delta^0_{x_2,\sigma_2}, ..., \Delta^0_{x_n,\sigma_n}\}$ for which $x_i \in X$ and $\sigma \in E^*$ with $\|\sigma_i\| \leqslant 1$. Set

$$\lambda_1 \leqslant \lambda_2 \Longleftrightarrow \lambda_1 \subseteq \lambda_2.$$

Let $V_\lambda :=< \Delta^0_{x_1,\sigma_1}, \Delta^0_{x_2,\sigma_2}, ..., \Delta^0_{x_n,\sigma_n} >$ and $F_\lambda : V_\lambda \longrightarrow \mathbb{C}$ is defined by

$$F_\lambda(\sum_1^n \lambda_i\Delta^0_{x_i,\sigma_i}) := \sum_{k=1}^n f(x_k)(\sigma_k).$$

Then $\|F_\lambda\| \leqslant \|f\|_{\alpha,E}$ so $F_\lambda \in V^*_\lambda$. Also, F_λ is linear and w^*-continuous, so by Hahn Banach Theorem there exists w^*-continuous extension
$\bar{F_\lambda} : lip^0_\alpha(X,E)^* \longrightarrow \mathbb{C}$ such that $\|\bar{F_\lambda}\| \leqslant \|f\|_{\alpha,E}$ and $\bar{F_\lambda}$ is w^*-continuous. So there exists $f_\lambda \in lip^0_\alpha(X,E)$ such that $\bar{F_\lambda} = \hat{f_\lambda}$. Put $F \in w^* - cl\{\hat{f_\lambda}\}$, then $F \in lip^0_\alpha(X,E)^{**}$ such that $\varphi(F) = f$, so φ is onto.

Now, we show that φ is a homomorphism. Let $F,G \in lip^0_\alpha(X,E)^{**}$, $x \in X$ and $\sigma \in \Delta(E)$. Then

$$F\Box G(\Delta^0_{x,\sigma}) = F(G\Box\Delta^0_{x,\sigma}) = F(G(\Delta^0_{x,\sigma})\Delta^0_{x,\sigma}) = F(\Delta^0_{x,\sigma})G(\Delta^0_{x,\sigma}).$$

Similarly,

$$F\Diamond G(\Delta^0_{x,\sigma}) = F(\Delta^0_{x,\sigma})G(\Delta^0_{x,\sigma}).$$

Hence

$$\begin{aligned}[\varphi(F\Box G)(x)](\sigma) &= F\Box G(\Delta^0_{x,\sigma}) \\ &= F(\Delta^0_{x,\sigma})G(\Delta^0_{x,\sigma}) \\ &= [\varphi(F)(x)](\sigma).[\varphi(G)(x)](\sigma),\end{aligned}$$

for all $x \in X, \ \sigma \in \Delta(E)$. Since the linear span of character space $\Delta(E)$ is norm-dense in E^*, it follows that

$$\varphi(F)(x).\varphi(G)(x)(\sigma) = \varphi(F\Box G)(x)(\sigma), \ ,(x \in X, \ \sigma \in E^*).$$

Then $\varphi(F).\varphi(G) = \varphi(F\Box G)$.

□

Corollary 3.1. *Let (X,d) be a locally compact metric space, $0<\alpha\leqslant 1$ and E be a reflective Banach algebra such that the linear span of character space $\Delta(E)$ is norm-dense in E^*. Then $lip^0_\alpha(X,E)^{**}\cong Lip_\alpha(X,E)$.*

Let (X,d) be a compact metric space. Then $lip^0_\alpha(X,E)=lip_\alpha(X,E)$. By Theorem 3.1, the following corollary is immediate.

Corollary 3.2. *Let (X,d) be a compact metric space, $0<\alpha\leqslant 1$ and E be a reflective Banach algebra such that the linear span of character space $\Delta(E)$ is norm-dense in E^*. Then $lip_\alpha(X,E)^{**}\cong Lip_\alpha(X,E)$.*

The following example show that the condition of locally compact is essential in Theorem 3.1.

Example 3.1. *Let E be a Banach algebra, then $lip^0_\alpha(\mathbb{Q},E)=\{0\}$ and $Cons(X,E)\subseteq Lip_\alpha(\mathbb{Q},E)$. So $\{0\}=(lip^0_\alpha(\mathbb{Q},E))^{**}\neq Lip_\alpha(\mathbb{Q},E^{**})$.*

Example 3.2. *Let (X,d) be a uniformly discrete metric space, $0<\alpha\leqslant 1$ and E be a Banach algebra. Then by Lemma 2.2, we have*

$$Lip_\alpha(X,E^{**})=l_\infty(X,E^{**}),\ lip^0_\alpha(X,E)^{**}=c_0(X,E)^{**}.$$

Therefore

$$c_0(X,E)^{**}=l_\infty(X,E^{**}).$$

Let A be a Banach algebra and $\Box$ (resp; $\Diamond$) be the first (resp; second) Arens product in the second duall A^{**}. Then $(A^{**},\Box)$ and $(A^{**},\Diamond)$ are Banach algebras. Also A is regular if and only if $\Box=\Diamond$. Then algebra A is Arens regular if the algebra $(A^{**},\Diamond)$ is commutative.

Theorem 3.2. *Let (X,d) be a locally compact metric space, E be a Banach algebra such that the linear span of character space $\Delta(E)$ is norm-dense in E^* and $0<\alpha\leqslant 1$. Then $lip^0_\alpha(X,E)$ is Arens regular.*

Proof. Define $\varphi: lip^0_\alpha(X,E)^{**}\longrightarrow Lip_\alpha(X,E^{**})$ where

$$[\varphi(F)(x)](\sigma):=F(\Delta^0_{x,\sigma}),\ (F\in lip^0_\alpha(X,E)^{**},\ \sigma\in E^*,\ x\in X).$$

Then by Theorem 3.1, φ is isometrically isomorphic as Banach algebras. Let, $F,G\in lip^{**}_\alpha(X,E)\cong Lip_\alpha(X,E^{**})$. Therefore

$$F\Box G(\Delta^0_{x,\sigma})=F(\Delta^0_{x,\sigma})G(\Delta^0_{x,\sigma})=F\Diamond G(\Delta^0_{x,\sigma}).$$

Also,

$$F\Box G(\sum_{i=1}^n\lambda_i\Delta^0_{x_i,\sigma_i})=\sum_{i=1}^n\lambda_iF\Box G(\Delta^0_{x_i,\sigma_i})=F\Diamond G((\sum_{i=1}^n\lambda_i\Delta^0_{x_i,\sigma_i}).$$

Since the linear space of $\{\Delta^0_{x,\sigma}:\ \|\sigma\|\leqslant 1,\ \sigma\in E^*,\ x\in X\}$ is norm-dense in $[lip^0_\alpha(X,E)^*]_1$, it follows that $F\Box G(f)=F\Diamond G(f)$ for all $f\in[lip^0_\alpha(X,E)]^*_1$. Hence

$$F\Box G=F\Diamond G,\ (F,G\in lip^0_\alpha(X,E)^{**})$$

Therefore $lip^0_\alpha(X,E)$ is Arens regular. □

If (X,d) is a compact metric space, by Corollary 3.2, the following is immediate.

Corollary 3.3. *Let (X,d) be a compact metric space, E be a Banach algebra such that the linear span of character space $\Delta(E)$ is norm-dense in E^* and $0<\alpha\leqslant 1$. Then $lip_\alpha(X,E)$ is Arens regular.*

If A is a commutative Banach algebra which is Arens regular such that A^{**} is semisimple. Then A is $2-$ weakly amenable Banach algebra, see [7, Corollary 1.11].

Theorem 3.3. *Let (X,d) be a localy metric space, E be a commutative regular and semisimple Banach algebra such that the linear span of character space $\Delta(E)$ is norm-dense in E^* and $0<\alpha\leqslant 1$. Then $lip^0_\alpha(X,E)$ is $2-$weakly amenable.*

Proof. By Theorem 3.1, we have $lip^0_\alpha(X,E)^{**}\cong Lip_\alpha(X,E^{**})$ and by Lemma 2.1, $Lip_\alpha(X,E^{**})$ is semisimple so $lip^0_\alpha(X,E)^{**}$ is semisimple and by Theorem 3.2, $lip^0_\alpha(X,E)$ is Arens regular. Then by [7, Corollary 1.11], $lip^0_\alpha(X,E)$ is a $2-$weakly amenable. □

Corollary 3.4. *Let (X,d) be a compact metric space, E be a commutative regular and semisimple Banach algebra such that the linear span of character space $\Delta(E)$ is norm-dense in E^* and $0<\alpha\leqslant 1$. Then $lip_\alpha(X,E)$ is $2-$weakly amenable.*

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