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A NOTE ON OPERATORS ON NORMED FINITE DIMENSIONAL WEAK HYPERVECTOR SPACES

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In this paper we prove the equivalence of different norms on normal normed finite dimensional weak hypervector spaces, the weak isomorphism between weak hypervector spaces and their correspondent classical vector spaces. Finally, we prove the continuity of operators on normal normed weak hypervector spaces.

Keywords: normed weak hypervector space, finite dimensional weak hypervector space, normal weak hypervector space, operator, weak operator, weak isomorphism

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1. Introduction

The concept of hyperstructure was first introduced by Marty [3] in 1934 and has attracted attention of many authors in last decades and has constructed some other structures such as hyperrings, hypergroups, hypermodules, hyperfields, and hypervector spaces. These constructions has been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. A wealth of applications of this concepts are given in $[1, 2, 4-7, 10, 11]$.

In 1988 the concept of hypervector space was first introduced by Scafati-Tallini. She studied more properties of this new structure in [9]. Note that the hypervector spaces used in this paper are the special case where there is only one hyperoperation, the external one, all the others are ordinary operations. The general hypervector spaces have all operations multivalued also in the hyperfield (see [11]). In [4] we defined the concepts of dimension of weak hypervector spaces and proved some results. Now we want to use that results and prove the equivalence of different norms on normal normed finite dimensional weak hypervector spaces. We define the concept of a weak isomorphism and prove that normal normed weak hypervector spaces and their correspondent classical finite dimensional vector spaces with the same dimension are weak isomorphic. Also we prove another results in this field. Finally, we prove the continuity of operators on normal normed weak hypervector spaces. This paper is arranged as follows. In section 2 we define the preliminary concepts and then in section 3 we state the main results of this note.

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2. Preliminaries

Definition 2.1. [9] *A weak or weakly distributive hypervectorspace over a field F is a quadruple $(X,+,o,F)$ such that $(X,+)$ is an abelian group and $o : F \times X \longrightarrow P_*(X)$ is a multivalued product such that:*

$$(1)\qquad \forall a \in F, \forall x, y \in X,\ [ao(x+y)] \cap [aox + aoy] \neq \emptyset,$$

$$(2)\qquad \forall a, b \in F, \forall x \in X,\ [(a+b)ox] \cap [aox + box] \neq \emptyset,$$

$$(3)\qquad \forall a, b \in F, \forall x \in X,\ ao(box) = (ab)ox,$$

$$(4)\qquad \forall a \in F, \forall x \in X,\ ao(-x) = (-a)ox = -(aox),$$

$$(5)\qquad \forall x \in X,\ x \in 1ox.$$

We call (1) *and* (2) *weak right and left distributive laws, respectively. Note that the set $ao(box)$ in* (3) *is of the form $\cup_{y \in box} aoy$.*

Definition 2.2. [9] *Let $(X, +, o, F)$ be a weak hypervector space over a field F, that is the field of real or complex numbers. We define a pseudonorm in X as a mapping $\|\| : X \longrightarrow R$, of X into the reals such that:*

(i) $\| 0 \| = 0$,
(ii) $\forall x, y \in X,\ \| x + y \| \leq \| x \| + \| y \|$,
(iii) $\forall a \in F,\ \forall x \in X,\ \sup \| aox \| = | a | \| x \|$.

Definition 2.3. *Let X and Y be hypervector spaces over F. A map $T : X \longrightarrow Y$ is called*
(i) linear if and only if
$T(x+y) = T(x) + T(y),\ T(aox) \subseteq aoT(x),\ \forall x, y \in X, a \in F$
(ii) antilinear if and only if
$T(x+y) = T(x) + T(y),\ T(aox) \supseteq aoT(x),\ \forall x, y \in X, a \in F,$
(iii) strong linear if and only if
$T(x+y) = T(x) + T(y),\ T(aox) = aoT(x),\ \forall x, y \in X, a \in F.$

3. Main results

By Lemma 3.1 in [4] we have the following definition.

Definition 3.1. [4] *If $a \in F$ and $x \in X$, then Special point z_{aox} for $0 \neq a$ is the element of aox such that $x \in a^{-1}oz_{aox}$ and for $a = 0$, we define $z_{aox} = 0$.*

Remark 3.1. *Note that if X is a normed weak hypervector space, then $\|z_{aox}\| = |a|\|x\|$.*

As the descriptions in [4], z_{aox} is not unique, necessarily. So the set of all these elements denoted by Z_{aox}. In the mentioned paper we introduced a certain category of weak hypervector spaces. These weak hypervector spaces have been called "normal". In [4], the following lemma stated a criterion for normality of a weak hypervector space.

Lemma 3.1. [4] *Let X be a weak hypervector space over F. X is normal if and only if*

$$z_{a_1ox} + z_{a_2ox} = z_{(a_1+a_2)ox}, \ \forall x \in X, \ \forall a_1, a_2 \in F,$$
$$z_{aox_1} + z_{aox_2} = z_{ao(x_1+x_2)}, \ \forall x_1, x_2 \in X, \ \forall a \in F.$$

Remark 3.2. *Before to state our results we describe some fundamental concepts from* [4]. *Let X be a weak hypervector space over F. A subset $M = \{x_1, ..., x_n\}$ of X is said to be linearly independent if the equation*

$$0 = \sum_{i=1}^{n} z_{\alpha_i ox_i}$$

implies $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$, where $\alpha_1, ..., \alpha_n$ are scalars. An arbitrary subset M is linearly independent if every nonempty finite subset of M is linearly independent. M is a basis of X if M is linearly independent and for any x of X there exists scalars $\alpha_1, ..., \alpha_n$ such that $x = \sum_{i=1}^{n} z_{\alpha_i ox_i}$, where $\{x_1, ..., x_n\}$ is a subspace of M.

Now we are prepare to define the new concept of weak isomorphism between two weak hypervector spaces.

Definition 3.2. *Let X and Y be weak hypervector spaces over F. A map $T : X \longrightarrow Y$ is called weak linear operator if T is additive and satisfies*

$$T(Z_{aox}) \subseteq aoTx, \quad (a \in F, x \in X).$$

Let X and Y be normed spaces. If T is bijective and T and T^{-1} are continuous, then T is called a weak isomorphism and X and Y are said to be weak isomorphic.

Proposition 3.1. [4] *Let X be a normal weak hypervector space over F. Then X with the same defined sum and the following scalar product is a classical vector space*

$$a.x = z_{aox}, \quad (a \in F, x \in X).$$

Proposition 3.2. *Let $(X, +, o)$ be a normal n-dimensional weak hypervector space over F with the basis $\{x_1, ..., x_n\}$. Then $(X, +, .)$ is n-dimensional classical vector space with the same basis such that the operation . is a scalar product with the following definition.*

$$a.x = z_{aox}, \quad (a \in F, x \in X).$$

Furthermore, if $(X, +, o, \| \, . \, \|)$ is a normed weak hypervector space over F, then $(X, +, ., \| \, . \, \|)$ is normed classical vector space.

Proof. Call $(X, +, o)$ and $(X, +, .)$ by X_1 and X_2, respectively. By Proposition 3.1, X_2 is a classical vector space. So it is enough to prove that $\{x_1, ..., x_n\}$ is a basis for X_2. Let $a_1, ..., a_2$ be scalars so that $\sum_{i=1}^{n} a_i.x_i = 0$. This implies $\sum_{i=1}^{n} z_{a_i ox_i} = 0$. Since $\{x_1, ..., x_n\}$ is linearly independent in X_1, we obtain $a_1 = ... = a_n = 0$ that implies the linearly independence of $\{x_1, ..., x_n\}$ in X_2. It is easy to check that $\{x_1, ..., x_n\}$ spans the elements of X_2.

If X is normed weak hypervector space, then for any $a \in F$ and $x \in X$ we have

$$\| a.x \| = \| z_{aox} \| = | a | \| x \| .$$

the remain properties of norm are inherited. □

Remark 3.3. *Denote the classical vector space correspondent to weak hypervector space X by X_c.*

Lemma 3.2. *Let X be a normal normed finite dimensional weak hypervector space. Then the different norms on X are equivalent.*

Proof. Let $\| . \|_1$ and $\| . \|_2$ be two norms on $(X,+,o)$. By Proposition 3.2, X_c with any one of these two norms is a normed finite dimensional classical vector space. We know that the different norms on a normed finite dimensional vector space are equivalent so we can conclude that $\| . \|_1$ and $\| . \|_2$ are equivalent. □

Remark 3.4. *Recall from* [4] *that the coordinate of $x \in X$ with regard to the basis $\{x_1, ..., x_n\}$ is the scalars $(a_1, ..., a_n)$ such that $x = \sum_{i=1}^{n} z_{a_i o x_i}$. It is easy to check that $\| x \| = \sum_{i=1}^{n} \mid a_i \mid$ has the properties of a norm that by preceding lemma is equivalent to other norms on X. Moreover in Lemma 3.10 (b) in* [4] *we have $aoz_{box} = abox$.*

Lemma 3.3. *Let X be a normal normed n-dimensional weak hypervector space over $\mathbb{F}$. Then $\mathbb{F}^n$ and X are weak isomorphic.*

Proof. Define $T : X \longrightarrow \mathbb{F}^n$ by $Tx = (a_1, ..., a_n)$ that therein $(a_1, ..., a_n)$ is the coordinate of x with regard to $\{x_1, ..x_n\}$ as a basis of X. Since X is normal, T is well-defined and additive, by Lemma 3.3. let $a \in F$. By Lemma 3.10 (b) in [4] and Lemma 3.1 we have

$$z_{aox} = z_{ao \sum_{i=1}^{n} z_{a_i o x_i}} = \sum_{i=1}^{n} z_{aoz_{a_i o x_i}} = \sum_{i=1}^{n} z_{aa_i o x_i}$$

that implies

$$Tz_{aox} = (aa_1, ..., aa_n).$$

On the other hand we have

$$aTx = (aa_1, ..., aa_n).$$

Hence we obtain

$$Tz_{aox} = aTx$$

Therefore T is a weak linear operator. Injectivity and surjectivity of T is clear. By Remark 3.4, assume that $\sum_{i=1}^{n} \mid a_i \mid$ is the norm of x. Let this norm be the norm of $(a_1, ..., a_n)$ in $\mathbb{F}^n$. So we have

$$\| x \| = \sum_{i=1}^{n} \mid a_i \mid = \| (a_1, ..., a_n) \| = \| Tx \| .$$

Let $y \in X$ and $(b_1, ..., b_n)$ be the coordinate of y. Since T is additive, we obtain

$$\| Tx - Ty \| = \| x - y \|$$

that imply the continuity of T and T^{-1}. So the proof is completed. □

Corollary 3.1. *We have the following statemens:*

(1) *Any two normal normed n-dimensional weak hypervector space over a field are weak isomorphic.*

(2) *Any normal normed finite dimensional weak hypervector space is a complete space.*

(3) *Any weak subhypervector space of a normal normed finite dimensional weak hypervector space is closed.*

Proof. (1) and (2) are inferred from Lemma 3.11 and then (3) is inferred from (2). □

Theorem 3.1. *Let X be a normal normed finite dimensional weak hypervector space, Y a normed weak hypervector space and $T : X \to Y$ a weak linear operator. Then T is continuous.*

Proof. Let the dimension of X be n, $\{x_1, ..., x_n\}$ be a basis of X and the coordinate of an arbitrary element $x \in X$ be $(a_1, ...a_n)$. By Remark 3.4, $\sum_{i=1}^{n} | a_i |$ can be the norm of x that is equivalent to any other norm on X. So we have

$$\begin{aligned} \| Tx \| = \| T(\sum_{i=1}^{n} z_{a_i o x_i}) \| = \| \sum_{i=1}^{n} T(z_{a_i o x_i}) \| \quad &\leq \quad \sum_{i=1}^{n} \| T(z_{a_i o x_i}) \| \\ &\leq \quad \sum_{i=1}^{n} | a_i | \| T(x_i) \| \end{aligned}$$

the last inequality is because of the weak linearity of T. Setting $k = max\{Tx_1, ..., Tx_n\}$ we obtain

$$\| Tx \| \leq k \sum_{i=1}^{n} | a_i | = k \| x \| .$$

So T is bounded by Definition 3.2 in [6]. It is easy to prove that any additive and bounded operator is continuous. □

Corollary 3.2. *Let X be a normal normed finite dimensional weak hypervector space, Y a normed weak hypervector space and $T : X \to Y$ a linear operator. Then T is continuous.*

Proof. By Definition 2.3, we know $T(aox) \subseteq aoT(x)$ for all $x \in X$ and $a \in F$. Since $T(z_{aox}) \in T(aox)$, T is weak linear. Hence by Theorem 3.1, T is continuous. □

As stated in Introduction, the hypervector spaces used in this paper are the special case where there is only one hyperoperation. So it is natural the following question:

Question. What is the similar argument of this paper for the general hypervector spaces have all operations multivalued?

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REFERENCES

[1] *P. Corsini*, Prolegomena of hypergroup theory, Aviani editore, (1993).

[2] *P. Corsini and V. Leoreanu*, Applications of Hyperstructure theory, Kluwer Academic Publishers, Advances in Mathematics (Dordrecht), (2003).

[3] *F. Marty*, Sur nue generalizeation de la notion de group, 8^{th} congress of the Scandinavic Mathematics, Stockholm, (1934), 45–49.

[4] *A. Taghavi and R. Hosseinzadeh*, A note on dimension of weak hypervector spaces, Italian J. of Pure and Appl. Math, preprint.

[5] *A. Taghavi and R. Hosseinzadeh*, Hahn-Banach Theorem for functionals on hypervector spaces, The Journal of Mathematics and Computer Science, Vol .2 No.4 (2011) 682-690.

[6] *A. Taghavi and R. Hosseinzadeh*, Operators on normed hypervector spaces, Southeast Asian Bulletin of Mathematics, (2011) 35: 367-372.

[7] *A. Taghavi and R. Parvinianzadeh*, Hyperalgebras and Quotient Hyperalgebras, Italian J. of Pure and Appl. Math, Math, No. 26 (2009) 17-24.

[8] *M.Scafati-Tallini*, Characterization of remarkable Hypervector space, Proc. 8^{th} congress on "Algebraic Hyperstructures and Aplications", Samotraki, Greece, (2002), Spanidis Press, Xanthi, (2003), 231-237.

[9] *M.Scafati-Tallini*, Weak Hypervector space and norms in such spaces, Algebraic Hyperstructures and Applications Hadronic Press. (1994), 199–206.

[10] *T. Vougiouklis*, The fundamental relation in hyperrings. The general hyperfield. Algebraic hyperstructures and applications (Xanthi, 1990), World Sci. Publishing, Teaneck, NJ, (1991), 203–211.

[11] *T. Vougiouklis*, Hyperstructures and their representations, Hadronic Press, (1994).