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ITERATIVE ALGORITHMS FOR THE PROXIMAL SPLIT FEASIBILITY PROBLEM

by Yonghong Yao[1], Mihai Postolache[2], Xiaolong Qin[3] and Jen-Chih Yao[4]

The proximal split feasibility problem is considered. An iterative algorithm has been constructed for solving the proximal split feasibility problem. Strong convergence result is given.

Keywords: proximal split feasibility problem, proximal mappings, iterative algorithm, strong convergence.

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1. Introduction

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $\varphi\colon \mathcal{H}_1 \to \mathbb{R} \cup \{+\infty\}$ and $\psi\colon \mathcal{H}_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper and lower semi-continuous convex functions. Let $\mathcal{A}\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint $\mathcal{A}^*$. Let λ be a given positive number. The Moreau envelope of ψ of index λ, also known as the Moreau[9]-Yosida[28] approximate, Yosida approximate or Moreau-Yosida regularization, is defined as

$$\psi_\lambda(x) = \min_{u\in\mathcal{H}_2} \left\{\psi(u) + \frac{1}{2\lambda}\|u - x\|^2\right\},\ x \in \mathcal{H}_2.$$

The proximity operator of ψ is defined by

$$\mathrm{prox}_{\lambda\psi}(x) = \arg\min_{u\in\mathcal{H}_2} \left\{\psi(u) + \frac{1}{2\lambda}\|u - x\|^2\right\},\ x \in \mathcal{H}_2.$$

Recall that the subdifferential $\partial\psi(x^\dagger)$ of ψ at $x^\dagger$ is defined as follows

$$\partial\psi(x^\dagger) = \{x^* \in \mathcal{H}_2 : \psi(x^\ddagger) \geq \psi(x^\dagger) + \langle x^*, x^\ddagger - x^\dagger\rangle,\ \forall x^\ddagger \in \mathcal{H}_2\}.$$

It is easy to see that

$$0 \in \partial\psi(x^\dagger) \Longleftrightarrow x^\dagger = \mathrm{prox}_{\lambda\psi}(x^\dagger).$$

This is to say that the minimizer of any function is the fixed point of its proximity operator. We can apply this equivalent relation to solve optimization problems by using fixed point methods.

Recall that the proximal split feasibility problem is to find a point $x^\dagger \in \mathcal{H}_1$ such that

$$\min_{x^\dagger\in\mathcal{H}_1} \{\varphi(x^\dagger) + \psi_\lambda(\mathcal{A}x^\dagger)\}. \quad (\mathbf{1}.1)$$

In the sequel, we use Γ to denote the solution set of the problem ($\mathbf{1}$.1).

[1] Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China, e-mail: yaoyonghong@aliyun.com

[2] (Corresponding author) China Medical University, Taichung 40402, Taiwan; Gh. Mihoc−C. Iacob Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Bucharest 050711, Romania; University "Politehnica" of Bucharest, Bucharest 060042, Romania, e-mail: emscolar@yahoo.com

[3] Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China, e-mail: qxlxajh@163.com

[4] Center for General Education, China Medical University, Taichung 40402, Taiwan, e-mail: yaojc@mail.cmu.edu.tw

If φ and ψ are the indicator functions of two nonempty closed convex sets $C \subset \mathcal{H}_1$ and $Q \in \mathcal{H}_2$, respectively, then

$$\varphi(x^\dagger) = \delta_C(x^\dagger) = \begin{cases} 0, & \text{if } x^\dagger \in C, \\ +\infty, & \text{otherwise;} \end{cases} \qquad \psi(x^\ddagger) = \delta_Q(x^\ddagger) = \begin{cases} 0, & \text{if } x^\ddagger \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

In this case, the problem (1) reduces to

$$\min_{x^\dagger \in \mathcal{H}_1} \{\delta_C(x^\dagger) + (\delta_Q)_\lambda(\mathcal{A}x^\dagger)\},$$

which is equivalent to

$$\min_{x^\dagger \in C} \Big\{ \frac{1}{2\lambda} \|(I - \mathrm{proj}_Q)(\mathcal{A}x^\dagger)\|^2 \Big\}, \tag{2}$$

where proj_Q is the metric projection from $\mathcal{H}_2$ onto Q.

Solving (2) is exactly to solve the split feasibility problem of finding $x^\dagger$ such that

$$x^\dagger \in C \ \text{ and } \ \mathcal{A}x^\dagger \in Q. \tag{3}$$

The split feasibility problem (3) has received much attention due to its applications in signal processing and image reconstruction [5] with particular progress in intensity modulated therapy [3]. Recently, the split feasibility problem (3) has been studied extensively by many authors (see, for instance, [1, 2, 4, 6, 7, 8, 11, 13, 14], [16]-[25] and [27, 29]).

Note that problem (1) can be converted to the fixed point problem. By the differentiability of the Yosida-approximate ψ_λ, we have

$$\partial(\varphi(x^\dagger) + \psi_\lambda(\mathcal{A}x^\dagger)) = \partial\varphi(x^\dagger) + \mathcal{A}^*\Big(\frac{I - \mathrm{prox}_{\lambda\psi}}{\lambda}\Big)(\mathcal{A}x^\dagger). \tag{4}$$

The optimality condition of (4) is $0 \in \partial\varphi(x^\dagger) + \mathcal{A}^*\Big(\frac{I-\mathrm{prox}_{\lambda\psi}}{\lambda}\Big)(\mathcal{A}x^\dagger)$, which can be rewritten as

$$0 \in \mu\lambda\partial\varphi(x^\dagger) + \mu\mathcal{A}^*(I - \mathrm{prox}_{\lambda\psi})(\mathcal{A}x^\dagger).$$

This relation is equivalent to the following fixed point equation

$$x^\dagger = \mathrm{prox}_{\mu\lambda\varphi}(x^\dagger - \mu\mathcal{A}^*(I - \mathrm{prox}_{\lambda\psi})(\mathcal{A}x^\dagger)). \tag{5}$$

By using the above fixed point equation (5), Moudafi and Thakur [10] presented the following split proximal algorithm to solve problem (1).

Algorithm 1.1

1.	Given an initialization $x_0 \in \mathcal{H}_1$.
2.	Assume that $\{x_n\}$ in $\mathcal{H}_1$ has been constructed. Compute $$\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$$ where $h(x_n) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}x_n\|^2$ and $l(x_n) = \frac{1}{2}\|(I - \mathrm{prox}_{\mu_n\lambda\varphi})x_n\|^2$. If $\theta(x_n) = 0$, then the iterative process stops, otherwise
3.	Compute $$x_{n+1} = \mathrm{prox}_{\mu_n\lambda\varphi}(x_n - \mu_n\mathcal{A}^*(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}x_n)$$ where the step size $\mu_n = \rho_n\frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ in which $0 < \rho_n < 4$.

Subsequently, in [26], Yao *et al.* presented a regularized algorithm. We observe, however, that the stepsize sequence $\{\mu_n\}$, which appeared in Algorithm 1.1, seems to be implicit because of the terms $l(x_n)$ and $\theta(x_n)$. Very recently, Shehu and Iyiola [12] suggested the following split proximal algorithm to solve problem (1).

Algorithm 1.2

1. Given $u \in \mathcal{H}_1$ and $x_1 \in \mathcal{H}_1$, starting points.
2. Set $n = 1$ and compute:
3. $y_n = \alpha_n u + (1-\alpha_n)x_n$
4. $\theta(y_n) = \|\mathcal{A}^*(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}y_n + (I - \mathrm{prox}_{\lambda\varphi})y_n\|$
5. $z_n = y_n - \rho_n \frac{h(y_n)+l(y_n)}{\theta^2(y_n)}(\mathcal{A}^*(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}y_n + (I - \mathrm{prox}_{\lambda\varphi})y_n)$
6. Then compute $x_{n+1} = (1-\beta_n)y_n + \beta_n z_n$
7. If $\mathcal{A}^*(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}y_n = 0 = (I - \mathrm{prox}_{\lambda\varphi})y_n$ and $x_{n+1} = x_n$, the iterative process stops, otherwise
8. Set $n \leftarrow n+1$ and repeat steps 3-6.

Remark 1.1. Note that in Algorithm 1.2, $\mathcal{A}^*(I-\mathrm{prox}_{\lambda\psi})\mathcal{A}y_n = 0 = (I-\mathrm{prox}_{\lambda\varphi})y_n$ implies $\theta(y_n) = 0$. In this case, we can not compute z_n and x_{n+1}.

In the present paper, our main purpose is to suggest a modified proximal split feasibility algorithm for solving the proximal SFP (1). We prove that the generated sequence converges strongly to a solution of the proximal SFP (1) under some appropriate conditions on the iterative parameters.

2. **Preliminaries**

Let $\mathcal{H}$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively and C be a nonempty closed convex subset of $\mathcal{H}$. Recall that a mapping $T\colon C \to C$ is said to be:

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C,$$

(ii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C,$$

where I denotes the identity, which is equivalent to

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$$

for all $x, y \in C$. Also, the mapping $I - T$ is firmly nonexpansive. Throughout, $\mathrm{Fix}(T)$ stands for the set of fixed points of T.

Note that the proximal mapping of ψ is firmly nonexpansive, namely,

$$\langle \mathrm{prox}_{\lambda\psi}(x) - \mathrm{prox}_{\lambda\psi}(y), x - y\rangle \geq \|\mathrm{prox}_{\lambda\psi}(x) - \mathrm{prox}_{\lambda\psi}(y)\|^2$$

for all $x, y \in \mathcal{H}_2$ and it is also the case for complement $I - \mathrm{prox}_{\lambda\psi}$.

For all $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $\mathrm{proj}_C(x)$, such that

$$\|x - \mathrm{proj}_C(x)\| \leq \|x - y\|$$

for all $y \in C$. The mapping proj_C is called the *metric projection* of $\mathcal{H}$ onto C. It is well known that proj_C is a nonexpansive mapping and is characterized by the following property:

$$\langle x - \mathrm{proj}_C(x), y - \mathrm{proj}_C(x)\rangle \leq 0$$

for all $x \in \mathcal{H}$ and $y \in C$.

Now, we introduce two lemmas for our main results in this paper.

Lemma 2.1 ([15])**.** *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \delta_n, n \geq 0,$$

where

(i) $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n \le 0$;
(iii) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Now, we are in a position to introduce a modified proximal split feasibility algorithm for solving problem (1). In the sequel, assume that problem (1) is consistent, i.e., $\Gamma \neq \emptyset$.

Algorithm 3.1

1.	Given fixed point $u \in \mathcal{H}_1$ and initial value $x_1 \in \mathcal{H}_1$.
2.	Assume that the current iteration $x_n \in \mathcal{H}_1$ has been constructed. Compute $\theta(x_n) = \mathcal{A}^*(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}x_n + (I - \mathrm{prox}_{\lambda\varphi})x_n.$ If $\theta(x_n) = 0$, then stop iteration, otherwise proceeds the next step.
3.	Compute the next iteration for $n \ge 1$, $x_{n+1} = (1-\beta_n)(\alpha_n u + (1-\alpha_n)x_n) + \beta_n\left[x_n - \rho_n \frac{h(x_n)+l(x_n)}{\|\theta(x_n)\|^2}\theta(x_n)\right],$ where $h(x_n) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}x_n\|^2$ and $l(x_n) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda\varphi})x_n\|^2$.

Remark 3.1. In Algorithm 3.1, if $\theta(x_n) = 0$, then x_n is a solution of the proximal split feasibility problem (1). As a matter of fact, taking any $\tilde{x} \in \Gamma$, we have $\tilde{x} = \mathrm{prox}_{\lambda\varphi}\tilde{x}$ and $\mathcal{A}\tilde{x} = \mathrm{prox}_{\lambda\psi}\mathcal{A}\tilde{x}$.

Note that $I - \mathrm{prox}_{\lambda\psi}$ and $I - \mathrm{prox}_{\lambda\varphi}$ are firmly-nonexpansive. Hence,

$$\begin{aligned}
0 &= \langle \theta(x_n), x_n - \tilde{x}\rangle \\
&= \langle \mathcal{A}^*(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}x_n + (I - \mathrm{prox}_{\lambda\varphi})x_n, x_n - \tilde{x}\rangle \\
&= \langle \mathcal{A}^*(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}x_n, x_n - \tilde{x}\rangle + \langle (I - \mathrm{prox}_{\lambda\varphi})x_n, x_n - \tilde{x}\rangle \\
&= \langle (I - \mathrm{prox}_{\lambda\psi})\mathcal{A}x_n, \mathcal{A}x_n - \mathcal{A}\tilde{x}\rangle + \langle (I - \mathrm{prox}_{\lambda\varphi})x_n, x_n - \tilde{x}\rangle \\
&\ge \|(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}x_n\|^2 + \|(I - \mathrm{prox}_{\lambda\varphi})x_n\|^2.
\end{aligned} \tag{6}$$

Thus,

$$(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}x_n = 0 \ \text{ and } \ (I - \mathrm{prox}_{\lambda\varphi})x_n = 0.$$

Therefore, $x_n \in \mathrm{Fix}(\mathrm{prox}_{\lambda\varphi})$ and $\mathcal{A}x_n \in \mathrm{Fix}(\mathrm{prox}_{\lambda\psi})$, i.e., $x_n \in \Gamma$.

Theorem 3.1. *Suppose the real sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\rho_n\}$ *satisfy the following conditions:*
(*i*) $\{\alpha_n\} \subset (0,1)$ *with* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = +\infty$*;*
(*ii*) $\{\beta_n\} \subset (0,1)$ *with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$*;*
(*iii*) $\{\rho_n\} \subset (0,+\infty)$ *with* $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$.

Then sequence $\{x_n\}$ *generated by Algorithm* 3.1 *strongly converges to the solution* $z \in \Gamma$*, where* $z = \mathrm{proj}_\Gamma(u)$.

Proof. Set $y_n = \alpha_n u + (1-\alpha_n)x_n$ and $z_n = x_n - \rho_n \frac{h(x_n)+l(x_n)}{\|\theta(x_n)\|^2}\theta(x_n)$ for all $n \ge 1$.

It follows that

$$\begin{aligned}
\|z_n - z\|^2 &= \|x_n - z - \rho_n \frac{h(x_n)+l(x_n)}{\|\theta(x_n)\|^2}\theta(x_n)\| \\
&= \|x_n - z\|^2 - 2\rho_n \frac{h(x_n)+l(x_n)}{\|\theta(x_n)\|^2}\langle \theta(x_n), x_n - z\rangle \\
&\quad + \frac{\rho_n^2(h(x_n)+l(x_n))^2}{\|\theta(x_n)\|^2}.
\end{aligned} \tag{7}$$

By (6), we deduce

$$\begin{aligned} \langle \theta(x_n), x_n - z \rangle &\geq \|(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}x_n\|^2 + \|(I - \mathrm{prox}_{\lambda\varphi})x_n\|^2 \\ &= 2h(x_n) + 2l(x_n). \end{aligned} \tag{8}$$

From (7) and (8), we get

$$\begin{aligned} \|z_n - z\|^2 &\leq \|x_n - z\|^2 - 2\rho_n \frac{h(x_n) + l(x_n)}{\|\theta(x_n)\|^2}(2h(x_n) + 2l(x_n)) \\ &\quad + \frac{\rho_n^2 (h(x_n) + l(x_n))^2}{\|\theta(x_n)\|^2} \\ &= \|x_n - z\|^2 - \rho_n(4 - \rho_n)\frac{(h(x_n) + l(x_n))^2}{\|\theta(x_n)\|^2} \\ &\leq \|x_n - z\|^2. \end{aligned} \tag{9}$$

Note that

$$\begin{aligned} \|y_n - z\| &= \|\alpha_n(u - z) + (1 - \alpha_n)(x_n - z)\| \\ &\leq \alpha_n\|u - z\| + (1 - \alpha_n)\|x_n - z\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1 - \beta_n)(y_n - z) + \beta_n(z_n - z)\| \\ &\leq (1 - \beta_n)\|y_n - z\| + \beta_n\|z_n - z\| \\ &\leq \alpha_n(1 - \beta_n)\|u - z\| + [1 - \alpha_n(1 - \beta_n)]\|x_n - z\| \\ &\leq \max\{\|u - z\|, \|x_n - z\|\}. \end{aligned}$$

An induction induces that

$$\|x_n - z\| \leq \max\{\|u - z\|, \|x_0 - z\|\}.$$

This implies that $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{\mathcal{A}x_n\}$ and $\{z_n\}$ are all bounded.

At the same time, we have

$$\begin{aligned} \|y_n - z\|^2 &= \|\alpha_n(u - z) + (1 - \alpha_n)(x_n - z)\|^2 \\ &= \alpha_n^2\|u - z\|^2 + (1 - \alpha_n)^2\|x_n - z\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n)\langle u - z, x_n - z \rangle, \end{aligned} \tag{10}$$

and

$$\|x_{n+1} - z\|^2 \leq (1 - \beta_n)\|y_n - z\|^2 + \beta_n\|z_n - z\|^2. \tag{11}$$

In light of (9)-(11), we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1 - \beta_n)[\alpha_n^2\|u - z\|^2 + 2\alpha_n(1 - \alpha_n)\langle u - z, x_n - z \rangle \\ &\quad + (1 - \alpha_n)^2\|x_n - z\|^2] + \beta_n\|x_n - z\|^2 \\ &\quad - \beta_n\rho_n(4 - \rho_n)\frac{(h(x_n) + l(x_n))^2}{\|\theta(x_n)\|^2} \\ &\leq (1 - \beta_n)\alpha_n\Bigg[\alpha_n\|u - z\|^2 + 2(1 - \alpha_n)\langle u - z, x_n - z \rangle \\ &\quad - \beta_n\rho_n(4 - \rho_n)\frac{(h(x_n) + l(x_n))^2}{\alpha_n(1 - \beta_n)\|\theta(x_n)\|^2}\Bigg] \\ &\quad + [1 - \alpha_n(1 - \beta_n)]\|x_n - z\|^2. \end{aligned} \tag{12}$$

Set $\delta_n = \|x_n - z\|^2$ and

$$\begin{aligned}\sigma_n = {}& \alpha_n\|u-z\|^2 + 2(1-\alpha_n)\langle u-z, x_n - z\rangle \\ & - \beta_n\rho_n(4-\rho_n)\frac{(h(x_n)+l(x_n))^2}{\alpha_n(1-\beta_n)\|\theta(x_n)\|^2}.\end{aligned} \tag{13}$$

for all $n \geq 1$.

By virtue of (12) and (13), we obtain

$$\delta_{n+1} \leq [1-(1-\beta_n)\alpha_n]\delta_n + (1-\beta_n)\alpha_n\sigma_n, n \geq 1. \tag{14}$$

Next, we show that $\limsup_{n\to\infty}\sigma_n$ is finite. From (13), we get

$$\sigma_n \leq \alpha_n\|u-z\|^2 + 2(1-\alpha_n)\langle u-z, x_n-z\rangle \leq \|u-z\|^2 + 2\|u-z\|\|x_n - z\|.$$

Since $\{x_n\}$ is bounded, it follows that $\limsup_{n\to\infty}\sigma_n < +\infty$.

Next we prove $\limsup_{n\to\infty}\sigma_n \geq -1$ by contradiction. If we assume on the contrary $\limsup_{n\to\infty}\sigma_n < -1$, then there exists m_0 such that $\sigma_n \leq -1$ for all $n \geq m_0$. It then follows from (14) that

$$\delta_{n+1} \leq \delta_n - (1-\beta_n)\alpha_n$$

for all $n \geq m_0$.

By induction, we have

$$\delta_{n+1} \leq \delta_{m_0} - \sum_{i=m_0}^{n}(1-\beta_i)\alpha_i. \tag{15}$$

By taking lim sup as $n \to \infty$ in (15), we have

$$\limsup_{n\to\infty}\delta_n \leq \delta_{m_0} - \lim_{n\to\infty}\sum_{i=m_0}^{n}(1-\beta_i)\alpha_i = -\infty,$$

which induces a contradiction. So,

$$-1 \leq \limsup_{n\to\infty}\sigma_n < +\infty.$$

Hence, $\limsup_{n\to\infty}\sigma_n$ exists. Thus, we can take a subsequence $\{n_k\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}\sigma_n &= \lim_{k\to\infty}\sigma_{n_k} \\ &= \lim_{k\to\infty}\Big[\alpha_{n_k}\|u-z\|^2 + 2(1-\alpha_{n_k})\langle u-z, x_{n_k}-z\rangle \\ &\quad - \beta_{n_k}\rho_{n_k}(4-\rho_{n_k})\frac{(h(x_{n_k})+l(x_{n_k}))^2}{\alpha_{n_k}(1-\beta_{n_k})\|\theta(x_{n_k})\|^2}\Big].\end{aligned} \tag{16}$$

Since x_{n_k} is a bounded real sequence, without loss of generality, we may assume $\lim_{k\to\infty}x_{n_k} = z^\dagger$. Consequently, from (16), the following limit also exists

$$\lim_{k\to\infty}\beta_{n_k}\rho_{n_k}(4-\rho_{n_k})\frac{(h(x_{n_k})+l(x_{n_k}))^2}{\alpha_{n_k}(1-\beta_{n_k})\|\theta(x_{n_k})\|^2}. \tag{17}$$

Note that $\lim_{k\to\infty}\alpha_{n_k} = 0$. It follows from (17) that

$$\lim_{k\to\infty}\frac{(h(x_{n_k})+l(x_{n_k}))^2}{\|\theta(x_{n_k})\|^2} = 0.$$

Noting that $\theta(x_{n_k})$ is bounded, we deduce immediately that

$$\lim_{k\to\infty}(h(x_{n_k})+l(x_{n_k})) = 0.$$

Therefore,

$$\lim_{k\to\infty}h(x_{n_k}) = \lim_{k\to\infty}l(x_{n_k}) = 0.$$

By the lower semicontinuity of h, we have
$$0 \le h(z^\dagger) \le \liminf_{i\to\infty} h(x_{n_i}) = 0.$$
Thus, we obtain
$$h(z^\dagger) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda\psi})\mathcal{A}z^\dagger\|^2 = 0,$$
that is, $Az^\dagger \in \mathrm{Fix}(\mathrm{prox}_{\lambda\psi})$.

Similarly, from the lower semi-continuity of l, we derive
$$0 \le l(z^\dagger) \le \liminf_{i\to\infty} l(x_{n_i}) = 0.$$
Therefore,
$$l(z^\dagger) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda\varphi})z^\dagger\|^2 = 0,$$
that is, $z^\dagger \in \mathrm{Fix}(\mathrm{prox}_{\lambda\varphi})$. Hence $z^\dagger \in \Gamma$.

So,
$$\begin{aligned}\limsup_{n\to\infty}\langle u - z, x_n - z\rangle &= \lim_{i\to\infty}\langle u - z, x_{n_i} - z\rangle \\ &= \langle u - z, z^\dagger - z\rangle \\ &\le 0.\end{aligned}$$
It follows from (16) that $\limsup_{n\to\infty}\sigma_n \le 0$.

Finally, applying Lemma 2.1 to (14) to get that $x_n \to z$. This completes the proof. □

We can apply our algorithm and theorem to the split feasibility problem (3).

Algorithm 3.2

1.	Given fixed point $u \in \mathcal{H}_1$ and initial value $x_1 \in \mathcal{H}_1$
2.	Assume that the current iteration $x_n \in \mathcal{H}_1$ has been constructed. Compute $\theta(x_n) = \mathcal{A}^*(I - \mathrm{proj}_Q)\mathcal{A}x_n + (I - \mathrm{proj}_C)x_n.$ If $\theta(x_n) = 0$, then stop iteration, otherwise proceeds the next step.
3.	Compute the next iteration $x_{n+1} = (1-\beta_n)(\alpha_n u + (1-\alpha_n)x_n) + \beta_n\left[x_n - \rho_n\frac{h(x_n)+l(x_n)}{\|\theta(x_n)\|^2}\theta(x_n)\right], n \ge 1,$ where $h(x_n) = \frac{1}{2}\|(I - \mathrm{proj}_Q)\mathcal{A}x_n\|^2$ and $l(x_n) = \frac{1}{2}\|(I - \mathrm{proj}_C)x_n\|^2$.

Corollary 3.1. *Assume that $C \cap A^{-1}(Q) \ne \emptyset$. Suppose the real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\rho_n\}$ satisfy the following conditions:*

(i) $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = +\infty$;

(ii) $\{\beta_n\} \subset (0,1)$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{\rho_n\} \subset (0,+\infty)$ with $\liminf_{n\to\infty}\rho_n(4-\rho_n) > 0$.

Then sequence $\{x_n\}$ generated by Algorithm 3.2 strongly converges to the solution $z \in C \cap A^{-1}(Q)$, where $z = \mathrm{proj}_{C\cap A^{-1}(Q)}(u)$.

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