REGULAR EQUIVALENCE RELATIONS ON ORDERED $*$-SEMIHYPERGROUPS

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In this paper, we study regular equivalence relations on ordered $*$-semihypergroups in detail. We introduce the concept of quasi-orders on ordered $*$-semihypergroups. Furthermore, we construct a regular equivalence relation on an ordered $*$-semihypergroup by a quasi-order such that the corresponding quotient structure is also an ordered $*$-semihypergroup. In particular, if the unary operation $*$ is regarded as the identity mapping, then it is a complete solution to the open problem given in [B. Davvaz, P. Corsini and T. Changphas, Relationship between odered semihypergroups and ordered semigroups by using pseuoorders, European J. Combin., 44 (2015), 208–217]. Finally, we establish the relationship between regular equivalence relations and quasi-orders on an ordered $*$-semihypergroup, and as an application of the above results, we obtain a homomorphism theorem of ordered $*$-semihypergroups by quasi-orders.

Keywords: ordered $*$-semihypergroup, regular equivalence, quasi-order, homomorphism.
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1. Introduction

It is well known that an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages and error-correcting codes. Many authors, especially Kehayopulu [12], Kehayopulu and Tsingelis [13, 14], Satyanarayana [22] studied such semigroups with some restrictions. Nordahl and Scheiblich [20] considered a unary operation $*$ on semigroups and introduced the concept of regularity on $*$-semigroups. In [24], Wu imposed the $*$-operation on ordered semigroups under the assumption of order preserving.

On the other hand, hyperstructure theory was introduced in 1934, when Marty [17] defined hypergroups, began to analyze their properties and applied them to groups. In the following decades and nowadays, hyperstructure theory have many applications in several branches of both pure and applied sciences, for example, see [3, 4, 7, 16, 18]. In particular, a semihypergroup is a classic example of algebraic hyperstructure as a generalization

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of a semigroup and a hypergroup. At present, many authors have studied different aspects of semihypergroups, for instance, Bonansinga and Corsini [1], Corsini [2], Davvaz [5], Hasankhani [9], Hila et al. [11], Leoreanu [15], Naz and Shabir [19], Salvo et al. [21], and many others. As a generalization of ordered semigroups, Heidari and Davvaz [10] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, also see [6]. As we know, strongly regular equivalence relation and regular equivalence relation of ordered semihypergroups always play important roles in the study of ordered semihypergroups structure. For more details, the reader is referred to [6, 8]. In the present paper, we shall extend the concept of ordered $*$-semigroups to the hyper version and study the regular equivalence relation of ordered $*$-semihypergroups by using the notion of quasi-orders, which is a generalization of the concept of pseudoorders on an ordered semihypergroup. In addition, since the operation $*$ can be regarded as the identity mapping, the results in this paper also can be considered to be a complete solution to the open problem given by Davvaz et al. in [6].

Let us describe the organization of the present paper. After an introduction, in Section 2 we recall some basic definitions and results of ordered semihypergroups which will be used throughout this paper and introduce the concept of ordered $*$-semihypergroups. In Section 3, we introduce the concepts of quasi-orders on ordered $*$-semihypergroups and discuss the properties of quasi-orders on an ordered $*$-semihypergroup. Furthermore, we construct a regular equivalence relation on an ordered $*$-semihypergroup by a quasi-order such that the corresponding quotient structure is an ordered $*$-semihypergroup. In Section 4, we shall establish the relationship between ordered regular equivalence relations and quasi-orders on an ordered $*$-semihypergroup, and obtain a homomorphism theorem of ordered $*$-semihypergroups by quasi-orders defined in previous section.

2. Preliminaries and some notations

For convenience, let us first give some necessary definitions. A mapping $\circ : S \times S \to \mathcal{P}^\star(S)$ is called a hyperoperation on S, where $\mathcal{P}^\star(S)$ denotes the family of all nonempty subsets of S. The system $(S, \circ)$ is called a *hypergroupoid*. If A and B are two nonempty subsets of S, then we denote $A \circ B = \bigcup\limits_{a\in A, b\in B} a \circ b$. In particular, for any $x \in S$, we write $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. Recall that a *semihypergroup* is a hypergroupoid $(S, \circ)$ such that for every $x, y, z \in S$, $x \circ (y \circ z) = (x \circ y) \circ z$ (see [3]).

Let $(S, \circ)$ be a semihypergroup and A, B be non-empty subsets of S. If ρ is an equivalence relation on S, we put

$$A \overrightarrow{\rho} B \;\Leftrightarrow\; (\forall a \in A)(\exists b \in B)\ a\ \rho\ b,$$

$$A\ \overline{\rho}\ B \Leftrightarrow \begin{cases} (\forall a \in A)(\exists b \in B)\ a\ \rho\ b, \\ (\forall b' \in B)(\exists a' \in A)\ a'\ \rho\ b', \end{cases}$$

and

$$A\ \overline{\overline{\rho}}\ B \Leftrightarrow (\forall a \in A)(\forall b \in B)\ a\ \rho\ b.$$

Recall that a *regular equivalence relation* (see [6, 8]) ρ on a semihypergroup S which means that, if

$$(\forall a, b, x \in S)\ a\ \rho\ b \Rightarrow a \circ x\ \overline{\rho}\ b \circ x \text{ and } x \circ a\ \overline{\rho}\ x \circ b;$$

ρ is said to be *strongly regular* [6, 8] if

$$(\forall a, b, x \in S)\ a\ \rho\ b \Rightarrow a \circ x\ \overline{\overline{\rho}}\ b \circ x \text{ and } x \circ a\ \overline{\overline{\rho}}\ x \circ b.$$

Let $(S, \circ)$ be a semihypergroup and ρ an equivalence relation on S. We denote by $a\rho$ the equivalence ρ-class containing a. From [3], we have the following two theorems.

Theorem 2.1 ([3])**.** *Let $(S, \circ)$ be a semihypergroup and ρ be an equivalence relation on S.*

(1) *If ρ is regular, then S/ρ is a semihypergroup with respect to the following hyperoperation: $x\rho \odot y\rho = \{z\rho \mid z \in x \circ y\}$.*
(2) *If the above hyperoperation is well defined on S/ρ, then ρ is regular.*

Theorem 2.2 ([3])**.** *Let $(S, \circ)$ be a semihypergroup and ρ be an equivalence relation on S.*

(1) *If ρ is strongly regular, then S/ρ is a semigroup with respect to the following operation: $x\rho \odot y\rho = z\rho$, for all $z \in x \circ y$.*
(2) *If the above operation is well defined on S/ρ, then ρ is strongly regular.*

As we know, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation "$\leq$" such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. Furthermore, an order semigroup S with a unary operation $* : S \to S$ is called an *ordered $*$-semigroup* if it satisfies $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for any $x, y \in S$ (see [24]).

We now recall the notion of ordered semihypergroups from [10].

Definition 2.3. An algebraic hyperstructure $(S, \circ, \leq)$ is called an *ordered semihypergroup* (also called *po-semihypergroup* in [10]) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \preceq a \circ y$ and $x \circ a \preceq y \circ a$. Here, if $A, B \in \mathcal{P}^\star(S)$, then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

In the following, we shall generalize the concept of ordered $*$-semigroups to the hyper version. For any $A \in \mathcal{P}^\star(S)$, the notation A^* is defined by $A^* := \{a^* \in S \mid a \in A\}$.

Definition 2.4. An ordered semihypergroup S with a unary operation $* : S \to S$ is called an *ordered $*$-semihypergroup* if it satisfies:

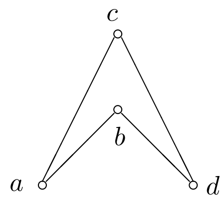
(1) $(\forall x \in S)\ (x^*)^* = x$;
(2) $(\forall x, y \in S)\ (x \circ y)^* = y^* \circ x^*$.

In the above definition, such a unary operation $*$ is called an *involution*. If for any $a, b \in S$ with $a \leq b$, we have $a^* \leq b^*$, then $*$ is called an *order preserving involution*.

Example 2.5. Let $S = \{a, b, c, d\}$ be an ordered semihypergroup (see [23]). The hyperoperation "$\circ$", the order "$\leq$" and the corresponding Hasse diagram are given below. Define the involution $*$ by $a^* = a$, $b^* = c$, $c^* = b$ and $d^* = d$. We can easily check that $(S, \circ, \leq)$ is an ordered $*$-semihypergroup with order preserving involution $*$.

$\circ$	a	b	c	d
a	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$
b	$\{a,d\}$	$\{b\}$	$\{a,d\}$	$\{a,d\}$
c	$\{a,d\}$	$\{a,d\}$	$\{c\}$	$\{a,d\}$
d	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{d\}$

$$\leq := \{(a,a),\ (a,b),\ (a,c),\ (b,b),\ (c,c),\ (d,b),\ (d,c),\ (d,d)\}.$$



Let $(S;\diamond,*;\leq_S)$ and $(T;\circ,\star;\leq_T)$ be two ordered $*$-semihypergroups, $f: S \to T$ a mapping from S into T. f is said to be *isotone* if $x \leq_S y$ implies that $f(x) \leq_T f(y)$ for all $x, y \in S$. f is called *reverse isotone* if $f(x) \leq_T f(y)$ implies that $x \leq_S y$. A *homomorphism* means that f satisfies the following conditions:

(1) f is isotone;
(2) $(\forall x, y \in S)\ f(x) \circ f(y) = \bigcup f(z)$, for any $z \in x \diamond y$;
(3) $(\forall x \in S)\ f(x^*) = f^\star(x)$.

Lemma 2.6. *Let A, B be nonempty subsets of an ordered $*$-semihypergroup S. Then the following statements hold*:

(1) $A \subseteq B$ *implies* $A^* \subseteq B^*$;
(2) $(A \circ B)^* = B^* \circ A^*$;
(3) $(A \cup B)^* = A^* \cup B^*$;
(4) $(A \cap B)^* = A^* \cap B^*$.

Proof. Straightforward. □

Definition 2.7. A regular equivalence relation on ordered $*$-semihypergroup S is an equivalence relation ρ on S such that

(1) $(\forall a, b, x \in S)\ a\rho b \Rightarrow a \circ x\ \overline{\rho}\ b \circ x$ and $x \circ a\ \overline{\rho}\ x \circ b$;
(2) $(\forall a, b \in S)\ a\rho b \Rightarrow a^* \rho b^*$.

Throughout this paper, unless otherwise mentioned, S will denote an ordered $*$-semihypergroup with order preserving involution $*$. The reader is referred to [4, 23] for notation and terminology not defined in this paper.

3. Quasi-orders on ordered $*$-semihypergroups

In this section, we introduce the concept of quasi-orders on an ordered $*$-semihypergroup S and discuss the properties of quasi-orders on S. In particular, we construct a regular equivalence relation on an ordered $*$-semihypergroup by a quasi-order such that the corresponding quotient structure is also an ordered $*$-semihypergroup.

We know that a relation ρ on an ordered semihypergroup $(S, \circ, \leq)$ is called a *pseudoorder* [6] if it satisfies the following conditions: (1) $\leq \subseteq \rho$; (2) $a\rho b$ and $b\rho c$ imply $a\rho c$; (3) $a\rho b$ implies $a \circ c \ \overline{\overline{\rho}}\ b \circ c$ and $c \circ a \ \overline{\overline{\rho}}\ c \circ b$, for all $c \in S$. Furthermore, in the same paper, Davvaz et al. obtained an ordered semigroup from an ordered semihypergroup by means of pseudoorders and posed the following open problem about ordered semihypergroups.

Open Problem 3.1. *Is there a regular equivalence relation ρ on an ordered semihypergroup $(S, \circ, \leq)$ for which S/ρ is an ordered semihypergroup*?

To answer the above open problem, Gu and Tang [8] defined an equivalence relation ρ_I on an ordered semihypergroup S as follows:

$$\rho_I := \{(x, y) \in S\backslash I \times S\backslash I \mid x = y\} \cup (I \times I),$$

where I is a proper hyperideal of S. From the Theorem 4.1 of [8], for an arbitrary ordered semihypergroup S, there exists a regular equivalence relation ρ_I on S such that the corresponding quotient structure S/ρ_I is also an ordered semihypergroup. However, as we know that S doesn't necessarily exist a proper hyperideal in general. In what follows, we investigate the concept of quasi-orders and construct a regular equivalence relation on an ordered ∗-semihypergroup by a quasi-order such that the corresponding quotient structure is an ordered ∗-semihypergroup. As an application of this result, if the unary operation ∗ is regarded as the identity mapping, then it is a complete solution to the Open Problem 3.1.

Let $A, B \in P^\star(S)$, we set

$$A \ \widetilde{\rho}\ B \ \Leftrightarrow \begin{cases} (\forall a \in A)(\exists b \in B) \ a\rho b \text{ and } b\rho a, \\ \\ (\forall b' \in B)(\exists a' \in A) \ a'\rho b' \text{ and } b'\rho a'. \end{cases}$$

Definition 3.2. Let $(S; \circ, *; \leq)$ be an ordered ∗-semihypergroup. A relation ρ on S is said to be a *quasi-order* if it satisfies the following conditions:

(1) $\leq \subseteq \rho$;
(2) $a\rho b$ implies $a^*\rho b^*$;
(3) $a\rho b$ and $b\rho c$ imply $a\rho c$;
(4) $a\rho b$ implies $a \circ c \ \overrightarrow{\rho}\ b \circ c$ and $c \circ a \ \overrightarrow{\rho}\ c \circ b$, for all $c \in S$;
(5) $a\rho b$ and $b\rho a$ imply $a \circ c \ \widetilde{\rho}\ b \circ c$ and $c \circ a \ \widetilde{\rho}\ c \circ b$, for all $c \in S$.

Remark 3.3. For an ordered ∗-semihypergroup S with order preserving involution ∗, there exists a quasi-order relation on S. In fact, it is a routine matter to verify that the order relation “ $\leq$ ” on S is a quasi-order relation.

One can easily observe that every pseudoorder relation on an ordered ∗-semihypergroup S is a quasi-order on S. However, the converse is not true, in general, as shown in the following example.

Example 3.4. Consider the ordered ∗-semihypergroup $(S; \circ, *; \leq)$ given in Example 2.5. Let ρ be a relation on S defined as follows:

$$\rho := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (d, a), (d, b), (d, c)\}.$$

It is easy to check that ρ is a quasi-order on S, but it is not a pseudoorder on S. Since $a \ \rho\ c$ and $a \circ b = \{a, d\} = c \circ b$, but $(a, d) \notin \rho$ which implies that $a \circ b \ \overline{\overline{\rho}}\ c \circ b$ doesn't hold.

Open Problem 3.5. *Is there a regular equivalence relation σ on an ordered $*$-semihypergroup $(S;\circ,*;\leq)$ for which S/σ is an ordered $*$-semihypergroup?*

To reach the target, we construct a regular equivalence relation ρ° on an ordered $*$-semihypergroup S by a quasi-order ρ such that S/ρ° is an ordered $*$-semihypergroup.

Theorem 3.6. *Let $(S;\circ,*;\leq)$ be an ordered $*$-semihypergroup and ρ a quasi-order on S. There is actually a regular equivalence relation ρ° on S such that S/ρ° is an ordered $*$-semihypergroup.*

Proof. We define the relation ρ° on S by

$$\rho^\circ := \{(a,b)\in S\times S \mid a\rho b \text{ and } b\rho a\}.$$

Obviously, it is easy to check that ρ° is an equivalence relation on S. Now, let $a\rho^\circ b$ and $c\in S$. Then $a\rho b$ and $b\rho a$. Since ρ is a quasi-order on S, by Definition 3.2, we have $a\circ c\ \widetilde{\rho}\ b\circ c$ and $c\circ a\ \widetilde{\rho}\ c\circ b$.

It means that, for every $x\in a\circ c$, there exists $y\in b\circ c$ such that $x\rho y$ and $y\rho x$ which imply that $x\rho^\circ y$, and for every $y'\in b\circ c$, there exists $x'\in a\circ c$ such that $x'\rho y'$ and $y'\rho x'$. So $x'\rho^\circ y'$. Thus, we deduce that $a\circ c\ \overline{\rho^\circ}\ b\circ c$. By using a similar argument, we can claim that $c\circ a\ \overline{\rho^\circ}\ c\circ b$. Hence ρ° is a regular equivalence relation on S. Thus, by Theorem 2.1, $(S/\rho^\circ,\odot)$ is a semihypergroup with respect to the following hyperoperation:

$$a\rho^\circ\odot b\rho^\circ = \bigcup_{c\in a\circ b} c\rho^\circ, \text{ for all } a\rho^\circ, b\rho^\circ\in S/\rho^\circ.$$

As follows, we define a relation $\preceq_\rho$ on S/ρ° by:

$$\preceq_\rho := \{(x\rho^\circ, y\rho^\circ)\in S/\rho^\circ\times S/\rho^\circ \mid (x,y)\in\rho\}.$$

Then $(S/\rho^\circ,\preceq_\rho)$ is a poset. In fact, suppose that $x\rho^\circ\in S/\rho^\circ$, where $x\in S$. Clearly, $(x,x)\in\leq\subseteq\rho$ and $x\rho^\circ\preceq_\rho x\rho^\circ$. Let $x\rho^\circ\preceq_\rho y\rho^\circ$ and $y\rho^\circ\preceq_\rho x\rho^\circ$. Then $x\rho y$ and $y\rho x$. Thus $x\rho^\circ y$, that is, $x\rho^\circ = y\rho^\circ$. Now, let $x\rho^\circ\preceq_\rho y\rho^\circ$ and $y\rho^\circ\preceq_\rho z\rho^\circ$, then $x\rho y$ and $y\rho z$. Hence, $x\rho z$, and we obtain that $x\rho^\circ\preceq_\rho z\rho^\circ$.

Furthermore, let $x\rho^\circ, y\rho^\circ\in S/\rho^\circ$ be such that $x\rho^\circ\preceq_\rho y\rho^\circ$. Then $x\rho y$ and for any $z\rho^\circ\in S/\rho^\circ$, since ρ is a quasi-order on S, we can get that $x\circ z\ \overrightarrow{\rho}\ y\circ z$ and $z\circ x\ \overrightarrow{\rho}\ z\circ y$. Thus, for any $a\in x\circ z$ there exists $b\in y\circ z$ such that $a\rho b$. This measns that $a\rho^\circ\preceq_\rho b\rho^\circ$. Hence we have

$$x\rho^\circ\odot z\rho^\circ = \bigcup_{a\in x\circ z} a\rho^\circ \preceq_\rho \bigcup_{b\in y\circ z} b\rho^\circ = y\rho^\circ\odot z\rho^\circ.$$

Similarly, we can also deduce that $z\rho^\circ\odot x\rho^\circ\preceq_\rho z\rho^\circ\odot y\rho^\circ$. Therefore, $(S/\rho^\circ,\odot,\preceq_\rho)$ is an ordered semihypergroup. Meanwhile, the mapping $\varphi: S\to S/\rho^\circ,\ x\mapsto x\rho^\circ$ is isotone. In fact, for any $x,y\in S$, if $x\leq y$, then $(x,y)\in\leq\subseteq\rho$ since ρ is a quasi-order on S, so $(x\rho^\circ, y\rho^\circ)\in\preceq_\rho$, that is $x\rho^\circ\preceq_\rho y\rho^\circ$.

Finally, define a unary operation $\star$ on S/ρ° by:

$$(\forall a\in S)\ (a\rho^\circ)^\star = a^*\rho^\circ.$$

The unary operation $\star$ is well defined. Indeed, let $a\rho^\circ = b\rho^\circ$, i.e., $a\rho^\circ b$. Then $a\rho b$ and $b\rho a$. Note that ρ is a quasi-order on S, we get that $a^*\rho b^*$ and $b^*\rho a^*$. So, $a^*\rho^\circ b^*$, that is, $a^*\rho^\circ = b^*\rho^\circ$. Furthermore, S/ρ° is a ordered $*$-semihypergroup with the unary operation $\star$. Indeed, for any $a\rho^\circ, b\rho^\circ \in S/\rho^\circ$, we have

$$((a\rho^\circ)^\star)^\star = (a^*\rho^\circ)^\star = a\rho^\circ$$

and for all $x \in a \circ b,\ u \in b^* \circ a^*$,

$$(a\rho^\circ \odot b\rho^\circ)^\star = \bigcup_{x\in a\circ b} (x\rho^\circ)^\star = \bigcup_{x\in a\circ b} x^*\rho^\circ = \bigcup_{x^*\in b^*\circ a^*} x^*\rho^\circ = \bigcup_{u\in b^*\circ a^*} u\rho^\circ.$$

On the other hand,

$$b^*\rho^\circ \odot a^*\rho^\circ = \bigcup_{u\in b^*\circ a^*} u\rho^\circ.$$

Also, let $*$ be an order preserving involution on S, and $x\rho^\circ \preceq y\rho^\circ$. Then $x\rho y$, by the definition of quasi-order, we have $x^*\rho y^*$. This means that $x^*\rho^\circ \preceq y^*\rho^\circ$, i.e., $(x\rho^\circ)^\star \preceq (y\rho^\circ)^\star$. Therefore, we conclude that the operation $\star$ is also an order preserving involution on S/ρ°. □

Remark 3.7. From the proof of Theorem 3.6, it is easy to see that the mapping $(\rho^\circ)^\sharp : S \to S/\rho^\circ,\ x \mapsto x\rho^\circ$ is a natural homomorphism from S to S/ρ°.

4. **Homomorphism theorem of ordered ∗-semihypergroups**

In this section, we shall establish the relationships between regular equivalence relations and quasi-orders on an ordered $*$-semihypergroup, and obtain a homomorphism theorem of ordered $*$-semihypergroups by quasi-orders which defined in Section 3.

In order to establish the relationships between regular equivalence relations and quasi-orders on an ordered $*$-semihypergroup, we need the following lemma.

Lemma 4.1. *Let $(S; \circ, *; \le_S)$ be an ordered $*$-semihypergroup and σ a relation on S. Then the following statements are equivalent*:

(1) *σ is a quasi-order on S;*
(2) *There is an ordered $*$-semihypergroup $(T; \diamond, \star; \le_T)$ and a homomorphism $\varphi : S \to T$ such that*

$$ker\varphi := \{(a, b) \in S \times S \mid \varphi(a) \le_T \varphi(b)\} = \sigma,$$

where $ker\varphi$ is called the kernel of φ.

Proof. (1) ⇒ (2). Let σ be a quasi-order on S and the regular relation σ° on S defined by

$$\sigma^\circ := \{(a, b) \in S \times S \mid (a, b) \in \sigma \text{ and } (b, a) \in \sigma\}.$$

By Theorem 3.6, the set $S/\sigma^\circ := \{a\sigma^\circ \mid a \in S\}$ with the hyperoperation $a\sigma^\circ \odot b\sigma^\circ = \bigcup_{c\in a\circ b} c\sigma^\circ$, the unary operation $(a\sigma^\circ)^\dagger = a^*\sigma^\circ$ and the order

$$\le_{S/\sigma^\circ} := \{(x\sigma^\circ, y\sigma^\circ) \in S/\sigma^\circ \times S/\sigma^\circ \mid (x, y) \in \sigma\}$$

is an ordered $*$-semihypergroup. Let $T = (S/\sigma^\circ; \odot, *; \leq_{S/\sigma^\circ})$ and φ be the mapping of S onto S/σ° defined by $\varphi : S \to S/\sigma^\circ$, $a \mapsto a\sigma^\circ$. Then, it is easy to see that φ is a homomorphism from S onto S/σ° and $ker\varphi = \sigma$.

(2) $\Rightarrow$ (1). Let $(S; \circ, *; \leq_S)$ and $(T; \diamond, \star; \leq_T)$ be two an ordered $*$-semi-hypergroups with order preserving involution $*$ and $\star$, respectively. If there exist a homomorphism $\varphi : S \to T$ such that $ker\varphi = \sigma$, then we can claim that σ is a quasi-order on S. In fact, let $(a, b) \in \leq$. By hypothesis, $\varphi(a) \leq_T \varphi(b)$. Thus $(a, b) \in ker\varphi = \sigma$, and we have $\leq \subseteq \sigma$. Meanwhile,

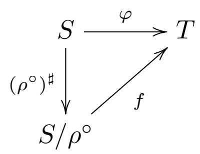
$$a\ \sigma\ b \Rightarrow \varphi(a) \leq_T \varphi(b) \Rightarrow \varphi^\star(a) \leq_T \varphi^\star(b) \Rightarrow \varphi(a^*) \leq_T \varphi(b^*) \Rightarrow a^*\ \sigma\ b^*.$$

Furthermore, let $(a, b) \in \sigma$ and $(b, c) \in \sigma$. Then $\varphi(a) \leq_T \varphi(b) \leq_T \varphi(c)$. Hence $\varphi(a) \leq_T \varphi(c)$, i.e., $(a, c) \in ker\varphi = \sigma$. Also, if $(a, b) \in \sigma$, then $\varphi(a) \leq_T \varphi(b)$. Since $(T; \diamond, \star; \leq_T)$ is an ordered $*$-semihypergroup, we have $\varphi(a) \diamond \varphi(c) \leq_T \varphi(b) \diamond \varphi(c)$ for any $c \in S$. Since φ is a homomorphism from S to T, we get that

$$\bigcup_{x \in a \circ c} \varphi(x) = \varphi(a) \diamond \varphi(c) \preceq \varphi(b) \diamond \varphi(c) = \bigcup_{y \in b \circ c} \varphi(y).$$

Thus, for every $x \in a \circ c$, there exists $y \in b \circ c$ such that $\varphi(x) \leq_T \varphi(y)$, and we have $(x, y) \in ker\varphi = \sigma$, which implies that $a \circ c \overrightarrow{\sigma} b \circ c$. By using a similar argument, we can prove that $c \circ a \overrightarrow{\sigma} c \circ b$. Moreover, let $(a, b) \in \sigma, (b, a) \in \sigma$ and $c \in S$. Then $\varphi(a) \leq_T \varphi(b)$ and $\varphi(b) \leq_T \varphi(a)$. Thus $\varphi(a) = \varphi(b)$, this means that $\varphi(a) \diamond \varphi(c) = \varphi(b) \diamond \varphi(c)$, i.e., $\bigcup_{x \in a \circ c} \varphi(x) = \bigcup_{y \in b \circ c} \varphi(y)$. So, for any $x \in a \circ c$, there exists $y \in b \circ c$ such that $\varphi(x) = \varphi(y)$, and we have $\varphi(x) \leq_T \varphi(y)$ and $\varphi(y) \leq_T \varphi(x)$. Thus $x\sigma y$ and $y\sigma x$. On the other hand, for any $y' \in b \circ c$, there exists $x' \in a \circ c$ such that $\varphi(x') = \varphi(y')$. Hence we have $\varphi(x') \leq_T \varphi(y')$ and $\varphi(y') \leq_T \varphi(x')$, that is, $x'\sigma y'$ and $y'\sigma x'$. Therefore, $a \circ c\ \widetilde{\sigma}\ b \circ c$ as required. Similarly, we can get that $c \circ a\ \widetilde{\sigma}\ c \circ b$. □

Theorem 4.2. *Let $(S; \circ, *; \leq_S)$ and $(T; \diamond, \star; \leq_T)$ be two ordered $*$-semihyper-groups and $\varphi : S \to T$ be a homomorphism. If ρ is a quasi-order on S such that $\rho \subseteq ker\varphi$, then there exists a unique homomorphism $f : S/\rho^\circ \to T$, $a\rho^\circ \mapsto \varphi(a)$ such that the diagram*



commutes, where $\rho^\circ = \rho \cap \rho^{-1}$. Furthermore, $Im(\varphi) = Im(f)$. Conversely, if ρ is a quasi-order on S for which there exists a homomorphism $f : S/\rho^\circ \to T$ such that the above diagram commutes, then $\rho \subseteq ker\varphi$.

Proof. Let ρ be a quasi-order on S such that $\rho \subseteq ker\varphi$, $f : S/\rho^\circ \to T$, $a\rho^\circ \mapsto \varphi(a)$. Then f is well defined. In fact, if $a\rho^\circ = b\rho^\circ$, then $(a, b) \in \rho$. Since $\rho \subseteq ker\varphi$, we have $(\varphi(a), \varphi(b)) \in \leq_T$. Moreover, since $(b, a) \in \rho \subseteq ker\varphi$, we have $(\varphi(b), \varphi(a)) \in \leq_T$. Therefore, $\varphi(a) = \varphi(b)$. Furthermore, f is a homomorphism and $\varphi = f \cdot (\rho^\circ)^\sharp$. In fact, by the proof of Lemma 4.1, $(S/\rho^\circ, \odot, \dagger, \leq_{S/\rho^\circ})$ is an ordered $*$-semihypergroup and the mapping $(\rho^\circ)^\sharp$ is a natural homomorphism. If $a, b \in S$, then

$$a\rho^\circ \leq_{S/\rho^\circ} b\rho^\circ \Rightarrow (a,b) \in \rho \subseteq ker\varphi \Rightarrow \varphi(a) \leq_T \varphi(b),$$

that is, $f(a\rho^\circ) \leq_T f(b\rho^\circ)$ and f is isotone. Meanwhile, since φ is a homomorphism from S to T, we obtain

$$f(a\rho^\circ \odot b\rho^\circ) = \bigcup f(z\rho^\circ) = \bigcup \varphi(z) = \varphi(a) \diamond \varphi(b) = f(a\rho^\circ) \diamond f(b\rho^\circ),$$

for all $z \in a \circ b$. Also, for any $(a\rho^\circ)^\dagger \in S/\rho^\circ$,

$$f((a\rho^\circ)^\dagger) = f(a^*\rho^\circ) = \varphi(a^*) = \varphi^\star(a) = f^\star(a\rho^\circ).$$

Finally, for each $a \in S$, $(f \cdot \rho^\sharp)a = f(\rho^\sharp(a)) = f(a\rho^\circ) = \varphi(a)$. That is, $f \cdot \rho^\sharp = \varphi$. We claim that f is a unique homomorphism from S/ρ° to T. To reach the target, let g is a homomorphism from S/ρ to T such that $\varphi = g \circ \rho^\sharp$. Then

$$f(a\rho^\circ) = \varphi(a) = (g \cdot (\rho^\circ)^\sharp)(a) = g(a\rho^\circ).$$

Consequently, $Im(f) = \{f(a\rho^\circ) \mid a \in S\} = \{\varphi(a) \mid a \in S\} = Im(\varphi)$.

Conversely, let ρ be a quasi-order on S and $f : S/\rho^\circ \to T$ be a homomorphism such that $\varphi = f \cdot (\rho^\circ)^\sharp$. Then, we claim that $\rho \subseteq ker\varphi$. In fact, we have

$$\begin{aligned}(a,b) \in \rho \Rightarrow a\rho^\circ \leq_{S/\rho^\circ} b\rho^\circ &\Rightarrow f(a\rho^\circ) \leq_T f(b\rho^\circ)\\ &\Rightarrow f((\rho^\circ)^\sharp(a)) \leq_T f((\rho^\circ)^\sharp(b))\\ &\Rightarrow (f \cdot \rho^\sharp)(a) \leq_T (f \cdot \rho^\sharp)(b)\\ &\Rightarrow \varphi(a) \leq_T \varphi(b)\\ &\Rightarrow (a,b) \in ker\varphi.\end{aligned}$$

Hence, the proof is completed. □

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