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UNIQUENESS OF INVERSE PROBLEMS FOR DIFFERENTIAL PENCILS WITH THE SPECTRAL DISCONTINUITY CONDITION

Yasser KHALILI[1], Abdolhadi DABBAGHIAN[2], Mohsen Khaleghi MOGHADAM[3]

In this paper we investigate the inverse problem for a differential pencil with the non-smooth solutions on the finite interval $[0,1]$. *We establish properties of the spectral characteristics and by taking the Weyl function, we prove the uniqueness theorem for this inverse problem.*

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1. Introduction

The boundary value problem L for the differential pencil is written as

$$y''(x) + (\rho^2 - 2\rho p(x) - q(x))y(x) = 0, \qquad x \in [0,1], \tag{1}$$

$$U(y) := y'(0) - hy(0) = 0, \tag{2}$$

$$V(y) := y'(1) + Hy(1) = 0, \tag{3}$$

with the discontinuity conditions

$$\begin{cases} y(a+0,\rho) = \alpha y(a-0,\rho), \\ y'(a+0,\rho) = \alpha^{-1} y'(a-0,\rho) + (\beta\rho + \gamma) y(a-0,\rho), \end{cases} \tag{4}$$

in an interior point $x = a$. The functions $p(x)$ and $q(x)$ are real-valued and $p(x) \in W_2^1[0,1]$, $q(x) \in L^2[0,1]$. Also ρ is a spectral parameter, the coefficients α, β, γ, h and H are real numbers and $\alpha \neq \pm 1$.

Various problems of natural sciences can be modeled by Sturm-Liouville equations and parameters of these problems can be recovered by inverse problem theories. For example, an inverse spectral technique is used to reconstruct some

[1,]Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, 578 Sari, Iran, e-mail: y.khalili@sanru.ac.ir (Corresponding author)
[2]Department of Mathematics, Neka Branch, Islamic Azad University, Neka, Iran
[3] Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, 578 Sari, Iran

components of the residual stress tensor in the arterial wall (see, [10,11]). Discontinuous boundary value problems which the main discontinuity is caused by reflection of the shear waves at the base of the crust appear in geophysical models for oscillations of the earth (see, [3,16]). Inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics (see, [15,19]). After reducing the corresponding mathematical model, we obtain the boundary value problem L with discontinuities in an interior point where the potential function $Q(x) = 2\rho p(x) + q(x)$ must be constructed from the given spectral information. Spectral information can be also used to reconstruct the conductivity profiles of a one dimensional discontinuous medium.

Boundary value problems with discontinuous conditions often appear in mathematics, physics, geophysics, mechanics and other branches of natural sciences. The important results for inverse problems without discontinuity conditions have been studied in [8,14,17]. The behavior of the spectrum and study of the inverse problem for discontinuous boundary value problems are more complicated (see, [6,12,13]). Inverse problems for Sturm-Liouville operators and quadratic pencils of Sturm-Liouville operators with the jump condition have been studied in many works (see, [5,9,18,20]). For example, Yurko has studied non-self-adjoint differential pencils with quasi-periodic boundary conditions and spectral jump conditions in [23]. Yang has considered the inverse problems for differential pencils with spectral boundary conditions and jump conditions (see, [21]). Spectral problems for discontinuous operators with the conditions like

$$\begin{cases} y(a+0,\rho) = \alpha_1 y(a-0,\rho), \\ y'(a+0,\rho) = \alpha_2 y'(a-0,\rho) + \alpha_3 y(a-0,\rho), \end{cases}$$

were also studied in [1,2,13,22]. We would like to study the inverse problem for differential pencils with Robin boundary conditions and spectral jump conditions that is a new work in this field. In this work we study the inverse problem for discontinuous diffusion operators using the Weyl function. We apply the method of the spectral mappings which is an impressive technique for studying a various class of the inverse problems.

To give the Weyl function, the fundamental system of solutions (FSS) plays an important role. By taking this function and applying the spectral mappings method, we prove the uniqueness solution of the inverse problem. In Sec. 2, we establish the asymptotic form of the solutions and eigenvalues. In Sec. 3, we prove the uniqueness theorem. Finally, Sec. 4 contains some conclusion.

2. The behavior of the spectrum

Let the functions $C(x,\rho)$, $S(x,\rho)$, $\varphi(x,\rho)$ and $\psi(x,\rho)$ be the solutions of Eq. (1) under the initial conditions $C(0,\rho) = S'(0,\rho) = \varphi(0,\rho) = \psi(1,\rho) = 1$, $C'(0,\rho) = S(0,\rho) = 0$, $\varphi'(0,\rho) = h$, $\psi'(1,\rho) = -H$ and the jump condition (4).

Let $C_0(x,\rho)$ and $S_0(x,\rho)$ be the smooth solutions of Eq. (1) on the interval $[0,1]$ under the initial conditions $C_0(0,\rho) = S_0'(0,\rho) = 1$ and $C_0'(0,\rho) = S_0(0,\rho) = 0$. For each fixed $x \in [0,1]$, the functions $C_0(x,\rho)$, $S_0(x,\rho)$, $C(x,\rho)$, $S(x,\rho)$, $\varphi(x,\rho)$ and $\psi(x,\rho)$ together with their derivatives with respect to x are entire in ρ.

We can write for $x < a$ and $x > a$,

$$\begin{cases} C(x,\rho) = A_{11}(\rho)C_0(x,\rho) + A_{12}(\rho)S_0(x,\rho), \\ S(x,\rho) = A_{21}(\rho)C_0(x,\rho) + A_{22}(\rho)S_0(x,\rho). \end{cases} \tag{5}$$

By using the initial conditions at $x = 0$, it is trivial that for $x < a$,

$$A_{11}(\rho) = A_{22}(\rho) = 1, \quad A_{12}(\rho) = A_{21}(\rho) = 0,$$

and therefore

$$\begin{cases} C(x,\rho) = C_0(x,\rho), \\ S(x,\rho) = S_0(x,\rho). \end{cases}$$

By taking these solutions and the jump condition (4), we have for $x > a$,

$$\begin{cases} A_{11}(\rho) = \alpha C_0(a,\rho)S_0'(a,\rho) - \alpha^{-1}S_0(a,\rho)C_0'(a,\rho) - (\beta\rho + \gamma)C_0(a,\rho)S_0(a,\rho), \\ A_{12}(\rho) = (\alpha^{-1} - \alpha)C_0(a,\rho)C_0'(a,\rho) + (\beta\rho + \gamma){C_0}^2(a,\rho), \\ A_{21}(\rho) = (\alpha - \alpha^{-1})S_0(a,\rho)S_0'(a,\rho) - (\beta\rho + \gamma){S_0}^2(a,\rho), \\ A_{22}(\rho) = \alpha^{-1}C_0(a,\rho)S_0'(a,\rho) - \alpha S_0(a,\rho)C_0'(a,\rho) + (\beta\rho + \gamma)C_0(a,\rho)S_0(a,\rho). \end{cases} \tag{6}$$

Denote the characteristic function for the boundary value problem L by

$$\Delta(\rho) := \langle \psi(x,\rho), \varphi(x,\rho) \rangle,$$

where $\langle y,z \rangle := yz' - y'z$ is the Wronskian of the functions $y(x)$ and $z(x)$. By virtue of Liouville's formula for the Wronskian, we have

$$\Delta(\rho) = -V(\varphi) = U(\psi) \tag{7}$$

(see [9]). This function is entire in ρ.

From [4,9], we know that for $|\rho| \to \infty$, uniformly in $x < a$,

$$\varphi(x,\rho) = cos(\rho x - \mathcal{P}(x)) + O\big(\rho^{-1}exp(|\tau|x)\big), \tag{8}$$

$$\varphi'(x,\rho) = -\rho sin(\rho x - \mathcal{P}(x)) + O\big(exp(|\tau|x)\big), \qquad (9)$$

where $\mathcal{P}(x) = \int_0^x p(t)dt$, $\lambda = \rho^2, \tau = Im\,\rho$.

Theorem 2.1. We derive the following asymptotical expression for $|\rho| \to \infty$,

$$\Delta(\rho) = \rho\big(-b_+ sin\big(\rho - \mathcal{P}(1)\big) + b_- sin(\rho(2a-1) - 2\mathcal{P}(a) + \mathcal{P}(1))$$
$$-\frac{\beta}{2}\big(cos\big(\rho - \mathcal{P}(1)\big) + 2cos(\rho(2a-1) - 2\mathcal{P}(a) + \mathcal{P}(1))\big)\big)$$
$$+O\big(exp(|\tau|)\big), \qquad (10)$$

where $b_+ = \frac{\alpha+\alpha^{-1}}{2}, b_- = \frac{\alpha-\alpha^{-1}}{2}$.

Proof: From [4,9], we know that for $|\rho| \to \infty$,

$$C_0(x,\rho) = cos(\rho x - \mathcal{P}(x)) + \frac{Q(x)\,sin(\rho x - \mathcal{P}(x))}{\rho}$$
$$+o\big(\rho^{-1}exp(|\tau|x)\big),$$

$$C_0'(x,\rho) = -\rho\,sin(\rho x - \mathcal{P}(x)) + Q(x)\,cos(\rho x - \mathcal{P}(x))$$
$$+o\big(exp(|\tau|x)\big),$$

where $Q(x) = \frac{1}{2}\int_0^x q(t)\,dt$. Also

$$S_0(x,\rho) = \frac{sin(\rho x - \mathcal{P}(x))}{\rho} - \frac{Q(x)cos(\rho x - \mathcal{P}(x))}{\rho^2}$$
$$+o\big(\rho^{-2}exp(|\tau|x)\big),$$

$$S_0'(x,\rho) = cos(\rho x - \mathcal{P}(x)) + \frac{Q(x)sin(\rho x - \mathcal{P}(x))}{\rho}$$
$$+o\big(\rho^{-1}exp(|\tau|x)\big).$$

By applying these functions and (5), (6), we obtain for $x > a$,

$$C(x,\rho) = b_+ cos\big(\rho x - \mathcal{P}(x)\big) + b_-\,cos(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x))$$
$$+\frac{\beta}{2}\big(sin\big(\rho x - \mathcal{P}(x)\big) - 2sin(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x))\big)$$
$$+O\big(\rho^{-1}exp(|\tau|x)\big),$$

$$S(x,\rho) = \frac{1}{\rho}\big(b_+\,\sin\big(\rho x - \mathcal{P}(x)\big) + b_-\,sin(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x))$$
$$-\frac{\beta}{2}\big(\cos\big(\rho x - \mathcal{P}(x)\big) - cos(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x))\big)\big)$$
$$+O\big(\rho^{-2}exp(|\tau|x)\big).$$

Since the functions $C(x,\rho)$ and $S(x,\rho)$ are the fundamental system of solutions of (1), we can write

$$\varphi(x,\rho) = H_1(\rho)C(x,\rho) + H_2(\rho)\, S(x,\rho).$$

Therefore

$$\varphi'(x,\rho) = H_1(\rho)C'(x,\rho) + H_2(\rho)\, S'(x,\rho).$$

Considering the initial conditions for the solutions $\varphi(x,\rho)$, $C(x,\rho)$ and $S(x,\rho)$, we will have

$$\begin{cases} \varphi(0,\rho) = H_1(\rho)C(0,\rho) + H_2(\rho)\, S(0,\rho), \\ \varphi'(0,\rho) = H_1(\rho)C'(0,\rho) + H_2(\rho)\, S'(0,\rho), \end{cases} \Rightarrow \begin{cases} 1 = H_1(\rho)\times 1 + H_2(\rho)\ \times 0, \\ h = H_1(\rho)\times 0 + H_2(\rho)\ \times 1. \end{cases}$$

So $H_1(\rho) = 1$ and $H_2(\rho) = h$ and consequently $\varphi(x,\rho) = C(x,\rho) + h\, S(x,\rho)$. Thus

$$\begin{aligned} \varphi(x,\rho) = {} & b_+ \cos\big(\rho x - \mathcal{P}(x)\big) + b_- \cos(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x)) \\ & + \tfrac{\beta}{2}\big(\sin\big(\rho x - \mathcal{P}(x)\big) - 2\sin(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x))\big) \\ & + \tfrac{h}{\rho}\Big(b_+ \sin\big(\rho x - \mathcal{P}(x)\big) + b_- \sin(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x)) \\ & - \tfrac{\beta}{2}\big(\cos\big(\rho x - \mathcal{P}(x)\big) - \cos(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x))\big)\Big) \\ & + O\big(\rho^{-2} exp(|\tau| x)\big), \qquad x > a. \end{aligned} \tag{11}$$

So

$$\begin{aligned} \varphi'(x,\rho) = {} & \rho\Big(-b_+ \sin\big(\rho x - \mathcal{P}(x)\big) + b_- \sin(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x)) \\ & + \tfrac{\beta}{2}\Big(\cos\big(\rho x - \mathcal{P}(x)\big) + 2\cos(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x))\Big)\Big) \\ & + h\Big(b_+ \cos\big(\rho x - \mathcal{P}(x)\big) - b_- \cos\big(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x)\big) \\ & + \tfrac{\beta}{2}\big(\sin\big(\rho x - \mathcal{P}(x)\big) + \sin(\rho(2a-x) - 2\mathcal{P}(a) + \mathcal{P}(x))\big)\Big) \\ & + O\big(\rho^{-1} exp(|\tau| x)\big), \qquad x > a. \end{aligned} \tag{12}$$

Using (3) and (7) into account, These yield (10). □

Lemma 2.1. ([24]). Let $\{a_i\}_{i=1}^{p}$ be the set of real numbers satisfying the inequalities $a_0 > a_1 > \cdots > a_{p-1} > 0$ and $\{b_i\}_{i=1}^{p}$ be the set of complex numbers. If $b_p \neq 0$, then the roots of the equation

$$e^{a_0\lambda} + b_1 e^{a_1\lambda} + \cdots + b_{p-1} e^{a_{p-1}\lambda} + b_p = 0,$$

have the form

$$\lambda_n = \frac{2\pi n i}{a_0} + h(n), \qquad n = 0, \pm 1, \ldots,$$

where $h(n)$ is a bounded sequence.

Theorem 2.2. For sufficiently large n, the function $\Delta(\rho)$ has simple zeros of the form

$$\rho_n = n\pi + \mathcal{P}(1) + \sigma_1 + O(n^{-1}), \tag{13}$$

where $\sigma_1 = \frac{1}{2i} ln\left(\frac{i-\kappa_1}{i+\kappa_1}\right)$.

Proof: By the well-known standard method, we can write the zeros of the function $\Delta(\rho)$ by the following form (see [7])

$$\rho_n = \rho_n^0 + O(n^{-1}), \quad |n| \to \infty,$$

where ρ_n^0 are the zeros of the function

$$\Delta^0(\rho) = \rho\big(-b_+ sin(\rho - \mathcal{P}(1)) + b_- sin(\rho(2a-1) - 2\mathcal{P}(a) + \mathcal{P}(1))$$
$$+ \frac{\beta}{2}\big(cos(\rho - \mathcal{P}(1)) + 2cos(\rho(2a-1) - 2\mathcal{P}(a) + \mathcal{P}(1))\big)\big).$$

Considering $\kappa_1 = \frac{\beta}{2b_+}$ and $\kappa_2 = \frac{\beta}{b_-}$, we have

$$\sqrt{1+\kappa_1{}^2}\, b_+\left(\frac{1}{\sqrt{1+\kappa_1{}^2}} sin(\rho - \mathcal{P}(1)) - \frac{\kappa_1}{\sqrt{1+\kappa_1{}^2}} cos(\rho - \mathcal{P}(1))\right)$$
$$= \sqrt{1+\kappa_2{}^2}\, b_-\left(\frac{1}{\sqrt{1+\kappa_2{}^2}} sin(\rho(2a-1) - 2\mathcal{P}(a) + \mathcal{P}(1))\right.$$
$$\left.+ \frac{\kappa_2}{\sqrt{1+\kappa_2{}^2}} cos(\rho(2a-1) - 2\mathcal{P}(a) + \mathcal{P}(1))\right).$$

So

$$\sqrt{1+\kappa_1{}^2}\, b_+\big(cos\,\sigma_1 sin(\rho - \mathcal{P}(1)) - sin\,\sigma_1\, cos(\rho - \mathcal{P}(1))\big)$$
$$= \sqrt{1+\kappa_2{}^2}\, b_-\big(cos\,\sigma_2\, sin(\rho(2a-1) - 2\mathcal{P}(a) + \mathcal{P}(1))$$
$$+ sin\,\sigma_2\, cos(\rho(2a-1) - 2\mathcal{P}(a) + \mathcal{P}(1))\big),$$

where $\sigma_2 = \frac{1}{2i} ln\left(\frac{i-\kappa_2}{i+\kappa_2}\right)$. Now assuming $K = \frac{\sqrt{1+\kappa_1{}^2} b_+}{\sqrt{1+\kappa_2{}^2} b_-}$, we can write

$$K\, sin(\rho - \mathcal{P}(1) - \sigma_1) = sin(\rho(2a-1) - 2\mathcal{P}(a) + \mathcal{P}(1) + \sigma_2).$$

Since the trigonometric function $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, by using Lemma 2.1, we have for sufficiently large n,

$$\rho_n^0 = n\pi + \mathcal{P}(1) + \sigma_1.$$

By applying the Rouche's theorem, we arrive at the eigenvalues (13). □

Corollary 2.1. For $|\rho| \to \infty$, It follows from (8), (9), (11) and (12) that

$$\left|\varphi^{(m)}(x,\rho)\right| \le C|\rho|^m exp(|\tau|x), \qquad x \in [0,1]. \tag{14}$$

Put $\phi(x,\rho) = \frac{\psi(x,\rho)}{\Delta(\rho)}$. Since $\psi(x,\rho)$ is the solution of Eq. (1) under the boundary condition $V(\psi) = 0$, it is trivial that $\phi(x,\rho)$ is the solution of Eq. (1) and $V(\phi) = 0$. Also from $\Delta(\rho) = U(\psi)$, we can result clearly that $U(\phi) = 1$. The functions $\phi(x,\rho)$ and $M(\rho) := \phi(0,\rho)$ are called the Weyl solution and Weyl function for the boundary value problem L, respectively. By virtue of Liouville's formula for the Wronskian, we have

$$\langle\varphi(x,\rho), S(x,\rho)\rangle = \varphi(0,\rho)S'(0,\rho) - \varphi'(0,\rho)S(0,\rho) = 1\times 1 - 0\times h = 1,$$

and therefore we can write

$$\phi(x,\rho) = h_1(\rho)S(x,\rho) + h_2(\rho)\varphi(x,\rho).$$

Now considering the initial conditions at $x = 0$, we have

$$\begin{cases}\phi(0,\rho) = h_1(\rho)S(0,\rho) + h_2(\rho)\varphi(0,\rho),\\ \phi'(0,\rho) = h_1(\rho)S'(0,\rho) + h_2(\rho)\varphi'(0,\rho).\end{cases}$$

Since $\psi'(0,\rho) = \Delta(\rho) + h\psi(0,\rho)$.we can write $\Delta(\rho) = \psi'(0,\rho) - h\psi(0,\rho)$, So

$$\phi'(0,\rho) = \frac{\psi'(0,\rho)}{\Delta(\rho)} = \frac{\Delta(\rho)+h\psi(0,\rho)}{\Delta(\rho)} = 1 + hM(\rho).$$

By regarding to the above system and substituting the initial conditions, we give

$$\begin{cases}M(\rho) = h_1(\rho)\times 0 + h_2(\rho)\times 1, & \Rightarrow \quad h_2(\rho) = M(\rho),\\ 1 + hM(\rho) = h_1(\rho)\times 1 + M(\rho)\times h, & \Rightarrow \quad h_1(\rho) = 1.\end{cases}$$

Thus

$$\phi(x,\rho) = S(x,\rho) + M(\rho)\varphi(x,\rho). \tag{15}$$

Furthermore by taking the initial conditions for the functions , $\phi(x,\rho)$ and $\varphi(x,\rho)$ we have

$$\langle\varphi(x,\rho),\phi(x,\rho)\rangle=\varphi(0,\rho)\phi'(0,\rho)-\varphi'(0,\rho)\phi(0,\rho)=1\times(1+hM(\rho))-hM(\rho)=1. \quad (16)$$

Theorem 2.3. We have the following asymptotic representations for $|\rho|\to\infty$,

$$\psi(x,\rho)=\cos(\rho(1-x)-\mathcal{P}(1)+\mathcal{P}(x))+O\left(\rho^{-1}exp(|\tau|(1-x))\right),\ x\in(a,1], \quad (17)$$

$$\begin{aligned}\psi(x,\rho)=\ &b_+\cos(\rho(1-x)-\mathcal{P}(1)+\mathcal{P}(x))\\&-b_-\cos(\rho(1-2a-x)-\mathcal{P}(1)-2\mathcal{P}(a)+\mathcal{P}(x))\\&+\frac{\beta}{2}(\sin(\rho(1-x)-\mathcal{P}(1)+\mathcal{P}(x))\\&+\sin(\rho(1-2a-x)-\mathcal{P}(1)-2\mathcal{P}(a)+\mathcal{P}(x)))\\&+O\left(\rho^{-1}exp(|\tau|(1-x))\right), \qquad x\in[0,a]. \end{aligned} \quad (18)$$

Proof: Taking the FSS $C_0(x,\rho),S_0(x,\rho)$ and the standard method, we have for $x\in(a,1]$,

$$\psi(x,\rho)=B_{11}(\rho)C_0(x,\rho)+B_{12}(\rho)S_0(x,\rho). \quad (19)$$

Applying the conditions of the functions $C_0(x,\rho),S_0(x,\rho)$ and $\psi(x,\rho)$ at $x=1$, we have

$$\begin{cases}\psi(1,\rho)=B_{11}(\rho)C_0(1,\rho)+B_{12}(\rho)S_0(1,\rho),\\ \psi'(1,\rho)=B_{11}(\rho)C_0'(1,\rho)+B_{12}(\rho)S_0'(1,\rho),\end{cases}\Rightarrow$$
$$\begin{cases}1=B_{11}(\rho)C_0(1,\rho)+B_{12}(\rho)S_0(1,\rho),\\ -H=B_{11}(\rho)C_0'(1,\rho)+B_{12}(\rho)S_0'(1,\rho).\end{cases}$$

Now by using Cramer's rule, we can give

$$B_{11}(\rho)=\cos(\rho-\mathcal{P}(1))+(Q(1)+H)\frac{\sin(\rho-\mathcal{P}(1))}{\rho}+O\left(\frac{1}{\rho^2}\right),$$
$$B_{12}(\rho)=\rho\sin(\rho-\mathcal{P}(1))-(Q(1)+H)\cos(\rho-\mathcal{P}(1))+O\left(\frac{1}{\rho}\right).$$

These coefficients together with the FSS $C_0(x,\rho),S_0(x,\rho)$ yield (17). Analogously by using again the FSS $C_0(x,\rho),S_0(x,\rho)$, the solution (17) and the jump condition (4), we get (18). □

From (17) and (18), we have

$$\begin{aligned}\psi'(x,\rho)=\ &\rho\sin(\rho(1-x)-\mathcal{P}(1)+\mathcal{P}(x))\\&+O\left(exp(|\tau|(1-x))\right), \qquad x\in(a,1]. \end{aligned}\quad (20)$$

$$\psi'(x,\rho) = \rho\big(b_+ \sin(\rho(1-x) - \mathcal{P}(1) + \mathcal{P}(x))$$
$$-b_- \sin(\rho(1-2a-x) - \mathcal{P}(1) - 2\mathcal{P}(a) + \mathcal{P}(x))$$
$$-\frac{\beta}{2}(\cos(\rho(1-x) - \mathcal{P}(1) + \mathcal{P}(x))$$
$$+\cos\left(\rho(1-2a-x) - \mathcal{P}(1) - 2\mathcal{P}(a) + \mathcal{P}(x)\right)\big)\big)$$
$$+O\left(exp(|\tau|(1-x))\right), \quad x \in [0,a). \tag{21}$$

Inverse Problem 2.1. Suppose that α, β, γ and a are known a priori. Given the Weyl function $M(\rho)$, construct the functions $p(x), q(x)$ and the coefficients h, H.

3. Uniqueness theorem

Now we prove the uniqueness theorem for the solution of the inverse problem. We consider together with $L = L(p(x), q(x), h, H)$ a boundary value problem $\tilde{L} = L(\tilde{p}(x), \tilde{q}(x), \tilde{h}, \tilde{H})$ of the same form (1)-(4) but with different coefficients. If a certain symbol denotes an object related to L, then the same symbol with tilde will denote the analogs object related to $\tilde{L}$.

Theorem 3.1. If $M(\rho) = \tilde{M}(\rho)$, then $L = \tilde{L}$. Thus, the specification of the Weyl function uniquely determines the boundary value problem L.

Proof: We assume that α, β, γ and a are known a priori. At first, we define the matrix $P(x,\rho) = [P_{jk}(x,\rho)]_{j,k=1,2}$, by the formula

$$P(x,\rho)\begin{bmatrix} \tilde{\varphi}(x,\rho) & \tilde{\phi}(x,\rho) \\ \tilde{\varphi}'(x,\rho) & \tilde{\phi}'(x,\rho) \end{bmatrix} = \begin{bmatrix} \varphi(x,\rho) & \phi(x,\rho) \\ \varphi'(x,\rho) & \phi'(x,\rho) \end{bmatrix}. \tag{22}$$

By virtue of (16), this yields

$$\begin{cases} P_{j1}(x,\rho) = \varphi^{(j-1)}(x,\rho)\tilde{\phi}'(x,\rho) - \phi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho), \\ P_{j2}(x,\rho) = \phi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{\phi}(x,\rho). \end{cases} \tag{23}$$

Also we have

$$\begin{cases} \varphi(x,\rho) = P_{11}(x,\rho)\tilde{\varphi}(x,\rho) + P_{12}(x,\rho)\tilde{\varphi}'(x,\rho), \\ \phi(x,\rho) = P_{11}(x,\rho)\tilde{\phi}(x,\rho) + P_{12}(x,\rho)\tilde{\phi}'(x,\rho). \end{cases} \tag{24}$$

Using (15) and (23), we obtain

$$\begin{cases} P_{j1}(x,\rho) = \varphi^{(j-1)}(x,\rho)\tilde{S}'(x,\rho) - S^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho) \\ \qquad +\widehat{M}(\rho)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho), \\ P_{j2}(x,\rho) = S^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{S}(x,\rho) \\ \qquad -\widehat{M}(\rho)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho), \end{cases} \tag{25}$$

where $\widehat{M}(\rho) = \tilde{M}(\rho) - M(\rho)$. Since $\tilde{M}(\rho) = M(\rho)$, we deduce that $\widehat{M}(\rho) = 0$, and consequently, for each fixed x in $[0,1]$, the functions $P_{jk}(x,\rho), k = 1,2$, are entire in ρ.

Fix $\delta > 0$. Denote $G_\delta = \{\rho \in \mathbb{C}: |\rho - \rho_n| \geq \delta\}$. It follows from (10), (15), (17), (18), (20) and (21) that

$$|\psi^{(m)}(x,\rho)| \leq C|\rho|^m exp\left(|\tau|(1-x)\right), \qquad x \in [0,1], \tag{26}$$

$$|\Delta(\rho)| \geq C|\rho|\, exp\left(|\tau|\right), \qquad \rho \in G_\delta, \tag{27}$$

$$|\phi^{(m)}(x,\rho)| \leq C|\rho|^{m-1} exp\left(|\tau|(1-x)\right), \qquad x \in [0,1], \qquad \rho \in G_\delta. \tag{28}$$

For each fixed $x \in [0,1]$, one gets from (14), (23) and (28) that

$$|P_{11}(x,\rho)| \leq C_1, \quad |P_{12}(x,\rho)| \leq C_2|\rho|^{-1}, \qquad \rho \in G_\delta.$$

Therefore $P_{11}(x,\rho) = P_1(x)$ and $P_{12}(x,\rho) = 0$ for $x \in [0,1]$. Together with (24), this yields

$$\varphi(x,\rho) = P_1(x)\tilde{\varphi}(x,\rho), \qquad \phi(x,\rho) = P_1(x)\tilde{\phi}(x,\rho). \tag{29}$$

Since for sufficiently large ρ, we have

$$sin\, \rho x \cong \frac{1}{2} i e^{-i\rho x}, \qquad cos\, \rho x \cong \frac{1}{2} e^{-i\rho x},$$

by taking (8), (10), (11) , (15) , (17) and (18), we can obtain as $|\rho| \to \infty$, arg $\rho \in \left(0, \frac{\pi}{2}\right)$,

$$\frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = exp\left(i\left(\mathcal{P}(x) - \tilde{\mathcal{P}}(x)\right)\right), \qquad \frac{\phi(x,\rho)}{\tilde{\phi}(x,\rho)} = exp\left(-i\left(\mathcal{P}(x) - \tilde{\mathcal{P}}(x)\right)\right). \tag{30}$$

From (29) and these relations, we result that $\mathcal{P}(x) = \tilde{\mathcal{P}}(x)$ and consequently $p(x) = \tilde{p}(x)$. Also we can get $P_1(x) = 1$ and therefore $\varphi(x,\rho) = \tilde{\varphi}(x,\rho)$ and $\phi(x,\rho) = \tilde{\phi}(x,\rho)$ for all x, ρ. So $q(x) = \tilde{q}(x)$, $h = \tilde{h}$ and $H = \tilde{H}$. The proof is completed. □

4. Conclusion

Through review of the papers, we have noted a lack in the inverse problem field for differential pencils. Indeed inverse problems for pencils of operators with Robin boundary conditions and spectral jump conditions have not been studied yet. We use the method of the spectral mappings and by taking the Weyl function we prove the uniqueness solution of the inverse problem.

R E F E R E N C E S

[1] *R. Kh. Amirov*, On Sturm-Liouville operators with discontinuity conditions inside an interval, J. Math. Anal. Appl., 317(2006), 163-176.

[2*] R. Kh. Amirov, A. Nabiev*, Inverse problems for the quadratic pencil of the Sturm-Liouville equations with impulse, Abstr. Appl. Anal., Art. ID 361989(2013), 10pp.

[3] *R. S. Anderssen*, The effect of discontinuities in density and shear velocity on the asymptotic overtone structure of torsional eigenfrequencies of the earth, Geophys. J. R. astr. Soc., 50(1997), 303-309.

[4] *S. A. Buterin, C. T. Shieh*, Inverse nodal problems for differential pencils, Applied Mathematics Letters, 22(8)(2009), 1240-1247.

[5] *Y. Cakmak, S. Isık*, Recovery of the impulsive diffusion operator with discontinuous coefficient, Appl. Math. Inf. Sci., 8(5)(2014), 2267-2275.

[6] *R. Carlsone*, An inverse spectral problem for Sturm-Lioville operator with discontinues coefficients, Proceedings of the American Mathematical Society, 120(2) (1994), 475-484.

[7] *J. B. Conway*, Functions of One Complex Variable, Springer-Verlag, New York, 1995.

[8] *G. Freiling, V. A. Yurko*, Inverse problems for Sturm–Liouville equations with boundary conditions polynomially dependent on the spectral parameter, Inverse Problems, 26(2010), 055003.

[9] *G. Freiling, V. A. Yurko*, Inverse Sturm-Liouville Problems and their Applications, NOVA Science Publisher, New York, 2001.

[10] *K. Gou, Z. Chen*, Inverse Sturm-Liouville problems and their biomedical engineering applications, JSM Math Stat., 2(1)(2015), 1008.

[11*] K. Gou, S. Joshi, J. R. Walton*, Recovery of material parameters of soft hyperelastic tissue by an inverse spectral technique, International J. Engineering Science, 56(2012), 1-16.

[12] *Y. Güldü, M. Arslanta*, Direct and inverse problems for Sturm-Liouville operator which has discontinuity conditions and coulomb potential, Chinese J. Mathematics, (2014), 804383.

[13] *O. H. Hald*, Discontinues inverse eigenvalue problems, Communications on Pure and Applied Mathematics, 37(5)(1984), 539-577.

[14] *H. Koyunbakan*, Inverse problem for a quadratic pencil of Sturm–Liouville operator, J. Mathematical Analysis and Applications, 378(2011), 549-554.

[15] *R. J. Krueger*, Inverse problems for non absorbing media with discontinuous material properties, J. Math. Phys., 23(1982), 396-404.

[16] *F. R. Lapwood, T. Usami*, Free Oscillations of the Earth, Cambridge University Press, Cambridge, 1981.

[17] A. Neamaty, S. Mosazadeh, On the canonical solution and dual equations of Sturm-Liouville problem with singularity and turning point, Iranian J. Science & Technology, Transaction A, 34(A1)(2010), 71-88.

[18] *A. Neamaty, Y. Khalili*, A uniqueness theorem for the inverse BVP with the jump condition and turning point, Scientific Bulletin, 77(3)(2015), 195-206.
[19] *D. G. Shepelsky*, The inverse problem of reconstruction of the medium's conductivity in a class of discontinuous and increasing functions, Advances in Soviet Math., 19(1994), 209-231.
[20] *C. T. Shieh, V. A. Yurko*, Inverse nodal and inverse spectral problems for discontinuous boundary value problems, J. Math. Anal. Appl., 347(2008), 266-272.
[21] *C-F. Yang*, Uniqueness theorems for differential pencils with eigenparameter boundary conditions and transmission conditions, J. Diff. Equations, 255(9) (2013), 2615-2635.
[22] *V. A. Yurko*, Integral transforms connected with discontinuous boundary value problems, Integral Transforms Spec. Funct., 10(2000), 141-164.
[23] *V. A. Yurko*, Inverse problem for quasi-periodic differential pencils with jump conditions inside the interval, Complex Analysis and Operator Theory, 10(6)(2016), 1203-1212.
[24] *V. F. Zhdanovich*, Formulae for the zeros of Dirichlet polynomials and quasi-polynomials, Dokl. Acad. Nauk SSSR, 135(8) (1960), 1046-1049.