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SOME RESULTS ON FRAMES AND G-FRAMES

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Abstract: It is shown that there is a fruitful relation between g-frames and g-frame sequences, from the viewpoint of closed range operators on Hilbert spaces. Also a new method for characterization of g-frames is introduced.

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1. Introduction

Frames were first introduced in the context of non-harmonic Fourier series [7]. Outside the signal processing, frames did not seem to generate much interest until the ground breaking work of [6]. Since then the theory of frames began to be more widely studied. During the last 20 years the theory of frames has been growing rapidly, several new applications have been developed. For example, besides traditional application as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effects of losses in pocket-based communication systems and hence to improve the robustness of data transmission [4], and to design high-rate constellation with full diversity in multiple-antenna code design [8]. The intrusted reader can find details on frames in the introductory book [5]. In [1, 2, 3] some applications have been developed.

G-frames have been introduced by W. Sun in 2006. They are generalized frames and include ordinary frames and many recent generalizations of them, e.g., bounded quasi-projectors and frames of subspaces ([10]). Afterward, A. Najati was able to complete g-frames in [9] and he proved some new theorems.

In Section 2, we investigate frames and the effect of closed range operators on them will be checked. In Section 3, we review g-frames and obtain some results. Finally, in Section 4, we express a new and useful method for characterization of g-frames and we show that the effect of closed range operator on g-frame gives a g-frame sequences.

Throughout this paper H and K are Hilbert spaces and $\mathcal{B}(K,H)$ is the collection of all bounded linear operators of K into H. If $K = H$, then $\mathcal{B}(H,H)$ will be denoted by $\mathcal{B}(H)$.

If an operator u has closed range, then there exists a right-inverse operator $u^\dagger$ (pseudo-inverse of u) in the following sense (see [5]).

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Lemma 1. *Let $u \in \mathcal{B}(K,H)$ be a bounded operator with closed range $\mathcal{R}_u$. Then there exists a bounded operator $u^\dagger \in \mathcal{B}(H,K)$ for which*

$$uu^\dagger x = x, \quad x \in \mathcal{R}_u.$$

Lemma 2. *Let $u \in \mathcal{B}(K,H)$ and u^* be the adjoint of u. Then the following assertions hold:*

(1) *$\mathcal{R}_u$ is closed in H if and only if $\mathcal{R}_{u^*}$ is closed in K.*
(2) *$(u^*)^\dagger = (u^\dagger)^*$.*
(3) *The orthogonal projection of H onto $\mathcal{R}_u$ is given by $uu^\dagger$.*
(4) *The orthogonal projection of K onto $\mathcal{R}_{u^\dagger}$ is given by $u^\dagger u$.*
(5) *$\mathcal{N}_{u^\dagger} = \mathcal{R}_u^\perp$ and $\mathcal{R}_{u^\dagger} = \mathcal{N}_u^\perp$.*

Proof. See [5]. □

2. A necessary condition for frames sequence

Definition 1. *Let $\{f_k\}_{k=1}^\infty$ be a sequence of members of H. We say that $\{f_k\}_{k=1}^\infty$ is a frame for H if there exist $0 < A \leq B < \infty$ such that*

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad f \in H. \tag{1}$$

The constants A and B are called frame bounds. We say that $\{f_k\}_{k=1}^\infty$ is a Bessel sequence with bound B, if just the right-hand side inequality of (1) holds. We say that $\{f_k\}_{k=1}^\infty$ is a frame sequence, if it is a frame for $\overline{\text{span}}\{f_k\}_{k=1}^\infty$.

Let $\{f_k\}_{k=1}^\infty$ be a Bessel sequence for H. The *pre-frame operator* for H will be denoted by $T_{\{f_k\}}$, and is defined by:

$$T_{\{f_k\}} : \ell^2(\mathbb{N}) \to H,$$
$$T_{\{f_k\}}\{c_k\}_{k=1}^\infty = \sum_{k=1}^{\infty} c_k f_k.$$

For the proof of Theorems 1 and 2, refer to [5].

Theorem 1. *A sequence $\{f_k\}_{k=1}^\infty$ in H is a frame for H if and only if*

$$T_{\{f_k\}} : \{c_k\}_{k=1}^\infty \mapsto \sum_{k=1}^{\infty} c_k f_k$$

defines a well-defined and surjective operator from $\ell^2(\mathbb{N})$ into H.

Theorem 2. *A sequence $\{f_k\}_{k=1}^\infty$ in H is a frame sequence for H if and only if*

$$T_{\{f_k\}} : \{c_k\}_{k=1}^\infty \mapsto \sum_{k=1}^{\infty} c_k f_k$$

defines a well-defined operator from $\ell^2(\mathbb{N})$ into H with closed range.

Theorem 3. *For $u \in \mathcal{B}(H,K)$, if $\{f_k\}_{k=1}^\infty$ is a Bessel sequence for H with bound B, then the following assertions are satisfied:*

(1) *the sequence* $\{uf_k\}_{k=1}^{\infty}$ *is a Bessel sequence for* K *with bound* $||u||^2B$*, and*

$$T_{\{uf_k\}} = uT_{\{f_k\}}. \tag{2}$$

(2) *Let* $\{f_k\}_{k=1}^{\infty}$ *be a frame for* H*. Then* $\{uf_k\}_{k=1}^{\infty}$ *is a frame for* K *if and only if* u *is surjective.*

Proof. (1) For all $f \in K$,

$$\sum_k |\langle f, uf_k\rangle|^2 = \sum_k |\langle u^*(f), f_k\rangle|^2 \leq ||u||^2B||f||^2.$$

Therefore $\{uf_k\}_{k=1}^{\infty}$ is a Bessel sequence for K with bound $||u||^2B$. Since for all $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$,

$$uT_{\{f_k\}}\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k u(f_k) = T_{\{uf_k\}}\{c_k\}_{k=1}^{\infty},$$

so $uT_{\{f_k\}} = T_{\{uf_k\}}$.

(2) Suppose that $\{uf_k\}_{k=1}^{\infty}$ is a frame for K. So, $T_{\{uf_k\}}$ is surjective, and since $\{f_k\}_{k=1}^{\infty}$ is a frame for H, then $T_{\{f_k\}}$ is surjective. Thus, according to (2), u should be surjective. Conversely, now suppose that u is surjective. Since $T_{\{f_k\}}$ is surjective, then $T_{\{uf_k\}}$ is surjective too. Whence, $\{uf_k\}_{k=1}^{\infty}$ is a frame for K. □

Theorem 4. *If* $\{f_k\}_{k=1}^{\infty}$ *is a frame for* H *and* $u \in \mathcal{B}(H)$ *with closed range, then* $\{uf_k\}_{k=1}^{\infty}$ *is a frame sequence.*

Proof. See [5]. □

Now, we prove the converse of the assertion stated in Theorem 4.

Theorem 5. *Let* $\{f_k\}_{k=1}^{\infty}$ *be a frame for* H*, and* $u \in \mathcal{B}(H)$*. If* $\{uf_k\}_{k=1}^{\infty}$ *is a frame sequence for* H*, then* u *has closed range.*

Proof. Let $\{uf_k\}_{k=1}^{\infty}$ be a frame sequence for H. According to Theorem 2, $T_{\{uf_k\}}$ is a well-defiend operator from $\ell^2(\mathbb{N})$ into H with closed range. By Theorem 3,

$$\mathcal{R}_{T_{\{uf_k\}}} = \mathcal{R}_{uT_{\{f_k\}}} \subseteq \mathcal{R}_u.$$

Let $y \in \mathcal{R}_u$. Then there is $x \in H$ such that $u(x) = y$. Since by Theorem 1, $T_{\{f_k\}}$ is surjective, so there exists $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ such that $T_{\{f_k\}}\{c_k\}_{k=1}^{\infty} = x$. We have $y = uT_{\{f_k\}}\{c_k\}_{k=1}^{\infty}$, and this means that $y \in \mathcal{R}_{T_{\{uf_k\}}}$. Therefore $\mathcal{R}_u = \mathcal{R}_{T_{\{uf_k\}}}$, so $\mathcal{R}_u$ is closed. □

3. Characterization of g-frames

Throughout this paper $\{H_j\}_{j\in\mathbb{J}}$ will be a sequence of Hilbert spaces, where $\mathbb{J}$ is a subset of $\mathbb{Z}$.

The notation $(\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$ will indicate the space

$$\big\{\{f_j\}_{j\in\mathbb{J}} \mid f_j \in H_j \ , \ \sum_{j\in\mathbb{J}} \|f_j\|^2 < \infty\big\},$$

which is a Hilbert space with pointwise operations, and inner product defined by

$$\langle\{f_j\},\{g_j\}\rangle := \sum_{j\in\mathbb{J}}\langle f_j, g_j\rangle, \quad \{f_j\},\{g_j\}\in(\sum_{j\in\mathbb{J}}\oplus H_j)_{\ell^2}.$$

Definition 2. *A family $\{\Lambda_j \in \mathcal{B}(H,H_j)\}_{j\in\mathbb{J}}$ is called a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, if there exist $0 < A \le B < \infty$ such that*

$$A\|f\|^2 \le \sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2 \le B\|f\|^2, \quad f\in H. \tag{3}$$

The constans A and B are called the lower and upper g-frame bounds for $\{\Lambda_j \in \mathcal{B}(H,H_j)\}_{j\in\mathbb{J}}$, respectively. We say simply that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H (and delete the expression "with respect to $\{H_j\}_{j\in\mathbb{J}}$"), whenever the space sequence $\{H_j\}_{j\in\mathbb{J}}$ is clear.

We say that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence with bound B, if just the right-hand side inequality of (3) holds; and $\{\Lambda_j\}_{j\in\mathbb{J}}$ is said a g-frame sequence, if it is a g-frame for $\overline{\text{span}}\{\Lambda_j^*(H_j)\}_{j\in\mathbb{J}}$.

We say that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is g-complete, if

$$\{f \mid \Lambda_j f = 0\ ,\ j\in\mathbb{J}\} = \{0\}. \tag{4}$$

For the proof of the following Theorems, we refer to [9] and [10].

Theorem 6. *If $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H, then*

$$\overline{span}\{\Lambda_j^*(H_j)\}_{j\in\mathbb{J}} = H.$$

Theorem 7. *$\{\Lambda_j\}_{j\in\mathbb{J}}$ is g-complete if and only if $\overline{span}\{\Lambda_j^*(H_j)\}_{j\in\mathbb{J}} = H$.*

Theorem 8. *$\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence for H with bound B if and only if for any finite subset $\mathbb{J}_1\subset\mathbb{J}$,*

$$\|\sum_{j\in\mathbb{J}_1}\Lambda_j^* g_j\|^2 \le B\sum_{j\in\mathbb{J}_1}\|g_j\|^2, \quad g_j\in H_j.$$

For the proof of Theorems 9 and 10, we refer to [9].

Theorem 9. *A sequence $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence for H with bound B if and only if*

$$T_{\{\Lambda_j\}} : \{f_j\} \mapsto \sum_j \Lambda_j^*(f_j)$$

is a well-defind and bounded operator from $(\sum_{j\in\mathbb{J}}\oplus H_j)_{\ell^2}$ to H with $\|T_{\{\Lambda_j\}}\| \le \sqrt{B}$.

Theorem 10. *A sequence $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H if and only if*

$$T_{\{\Lambda_j\}} : \{f_j\} \mapsto \sum_j \Lambda_j^*(f_j)$$

is a well-defind and bounded operator from $(\sum_{j\in\mathbb{J}}\oplus H_j)_{\ell^2}$ onto H.

Definition 3. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-Bessel sequence for H. The pre-frame operator is denoted by $T_{\{\Lambda_j\}}$, which is defined by*

$$T_{\{\Lambda_j\}} : (\sum_{j\in\mathbb{J}}\oplus H_j)_{\ell^2} \to H,$$

$$T_{\{\Lambda_j\}}(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}}\Lambda_j^*(f_j).$$

The adjoint operator of $T_{\{\Lambda_j\}}$ is called the *analysis operator,* and is defined by

$$T^*_{\{\Lambda_j\}} : H \to (\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2},$$
$$T^*_{\{\Lambda_j\}}(f) = \{\Lambda_j f\}_{j\in\mathbb{J}}.$$

Finally, we can define the g-frame operator by $S_{\{\Lambda_j\}} = T_{\{\Lambda_j\}}T^*_{\{\Lambda_j\}}$, so

$$S_{\{\Lambda_j\}}(f) = \sum_{j\in\mathbb{J}} \Lambda^*_j \Lambda_j f, \quad f \in H.$$

Theorem 11. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a sequence of bounded operators from H to H_j. Then the following assertions are satisfied:*

(1) *$\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence for H if and only if $\sum_{j\in\mathbb{J}} \Lambda^*_j(f_j)$ converges for all $\{f_j\}_{j\in\mathbb{J}} \in (\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$.*

(2) *$\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame sequence if and only if*

$$T_{\{\Lambda_j\}} : \{f_j\} \mapsto \sum_j \Lambda^*_j(f_j)$$

is a well-defind operator from $(\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$ into H with closed range.

Proof. (1) Suppose that $\sum_{j\in\mathbb{J}} \Lambda^*_j(f_j)$ converges for all $\{f_j\}_{j\in\mathbb{J}} \in (\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$. We define

$$T : (\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2} \to H,$$
$$T(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} \Lambda^*_j(f_j).$$

Then T is well-defind. Let for each $n \in \mathbb{N}$,

$$T_n : (\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2} \to H,$$
$$T_n(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j=1}^{n} \Lambda^*_j(f_j).$$

Let $B_n := \left(\sum_{j=1}^n \|\Lambda^*_j\|^2\right)^{\frac{1}{2}}$. Since $\|T_n(\{f_j\}_{j=1}^\infty)\| \leq B_n \|\{f_j\}_{j=1}^\infty\|$, $\{T_n\}$ is a sequence of bounded linear operators, which converges pointwise to T. Hence by the Banach-Steinhaus theorem, T is a bounded operator with

$$||T|| \leq \liminf ||T_n||.$$

So by Theorem 9, $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence for H.

The converse is evident.

(2) By Theorem 10, it is enough to prove that if $T_{\{\Lambda_j\}}$ has closed range, then $\overline{\text{span}}\{\Lambda^*_j(H_j)\}_{j\in\mathbb{J}} = \mathcal{R}_{T_{\{\Lambda_j\}}}$. Let $f \in \overline{\text{span}}\{\Lambda^*_j(H_j)\}_{j\in\mathbb{J}}$. Then

$$f = \lim_{n\to\infty} g_n, \quad g_n \in span\{\Lambda^*_j(H_j)\}_{j\in\mathbb{J}} \subseteq \mathcal{R}_{T_{\{\Lambda_j\}}} = \overline{\mathcal{R}}_{T_{\{\Lambda_j\}}}.$$

Thus $\overline{\text{span}}\{\Lambda^*_j(H_j)\}_{j\in\mathbb{J}} \subseteq \overline{\mathcal{R}}_{T_{\{\Lambda_j\}}} = \mathcal{R}_{T_{\{\Lambda_j\}}}$. On the other hand, if $f \in \mathcal{R}_{T_{\{\Lambda_j\}}}$, then $f \in \text{span}\{\Lambda^*_j(H_j)\}_{j\in\mathbb{J}} \subseteq \overline{\text{span}}\{\Lambda^*_j(H_j)\}_{j\in\mathbb{J}}$. The proof is completed. □

4. **Main Results**

In this section we shall establish some methods for characterization of g-frames.

Theorem 12. *A sequence $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame sequence for H with respect to $\{H_j\}$ if and only if*

$$f \mapsto \{\Lambda_j f\}_{j\in\mathbb{J}} \tag{5}$$

defines a map from H onto a closed subspace of $(\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$.

Proof. First, assume that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame sequence. Then by Theorem 11, $T_{\{\Lambda_j\}}$ is well-defind and $\mathcal{R}_{T_{\{\Lambda_j\}}}$ is closed. Therefore $T^*_{\{\Lambda_j\}}$ is well-defined and has closed range.

For the opposite implication, by hypothesis

$$\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 < \infty, \quad f \in H.$$

Let

$$B := \sup\{\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 : f \in H, ||f|| = 1\}.$$

Let $g_j \in H_j$, and $\mathbb{J}_1 \subset \mathbb{J}$ be finite. We have

$$\begin{aligned} \Big\| \sum_{j\in\mathbb{J}_1} \Lambda_j^* g_j \Big\|^2 &= \big(\sup_{\|f\|=1} |\langle \sum_{j\in\mathbb{J}_1} \Lambda_j^* g_j, f\rangle| \big)^2 \\ &\le \big(\sup_{\|f\|=1} \sum_{j\in\mathbb{J}_1} |\langle g_j, \Lambda_j f\rangle| \big)^2 \\ &\le \big(\sum_{j\in\mathbb{J}_1} \|g_j\|^2 \big) \big(\sup_{\|f\|=1} \sum_{j\in\mathbb{J}_1} \|\Lambda_j f\|^2 \big) \\ &\le B\big(\sum_{j\in\mathbb{J}_1} \|g_j\|^2 \big). \end{aligned}$$

Thus by Theorem 8, $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence for H. Therefore $T_{\{\Lambda_j\}}$ is well-defined and bounded. Furthermore, if the range of the map in (5) is closed, the same is true for $T_{\{\Lambda_j\}}$. So by Theorem 11, item 2, $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame sequence. □

Theorem 13. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame sequence for H with respect to $\{H_j\}_{j\in\mathbb{J}}$. Then $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ if and only if the map*

$$f \mapsto \{\Lambda_j f\}_{j\in\mathbb{J}} \tag{6}$$

from H onto a closed subspace of $(\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$ is injective.

Proof. Let for all $j \in \mathbb{J}$, $\Lambda_j(f) = 0$. Then the value of the map (6) at f is zero, and since it is injective, $f = 0$. By (4), this means that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is g-complete. Since $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame sequence for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, by Theorem7, $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$.

The converse is evident. □

In [9] Najati has shown that if $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ and $u \in \mathcal{B}(H, K)$ has closed range, then $\{\Lambda_j u^*\}_{j\in\mathbb{J}}$ is a g-frame sequence for H with respect to $\{H_j\}_{j\in\mathbb{J}}$.

Now we show that its converse is also true.

Theorem 14. *Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for H, $u \in \mathcal{B}(H)$ and let $\{\Lambda_j u^*\}_{j\in\mathbb{J}}$ be a g-frame sequence for H. Then u has closed range.*

Proof. By Theorem 11, item 2, $T_{\{\Lambda_j u^*\}}$ is a well-defiend operator from $(\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$ to H with closed range. Since

$$\sum_{j\in\mathbb{J}} \|\Lambda_j u^* f\|^2 \leq B\|u^* f\|^2 \leq B\|u\|^2\|f\|^2,$$

$\{\Lambda_j u^*\}_{j\in\mathbb{J}}$ is a g-Bessel sequence for H. If $\{f_j\}_{j\in\mathbb{J}} \in (\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$, then

$$\begin{aligned} uT_{\{\Lambda_j\}}\{f_j\}_{j\in\mathbb{J}} &= \sum_{j\in\mathbb{J}} u\Lambda_j^*(f_j) \\ &= \sum_{j\in\mathbb{J}} (\Lambda_j u^*)^*(f_j) \\ &= T_{\{\Lambda_j u^*\}}\{f_j\}_{j\in\mathbb{J}}. \end{aligned}$$

Therefore $uT_{\{\Lambda_j\}} = T_{\{\Lambda_j u^*\}}$. Thus $uT_{\{\Lambda_j\}}$ has closed range too. We have $\mathcal{R}_u = \mathcal{R}_{uT_{\{\Lambda_j\}}}$. Indeed, it is clear that $\mathcal{R}_{uT_{\{\Lambda_j\}}} \subseteq \mathcal{R}_u$. Let $y \in \mathcal{R}_u$, then there exists $x \in H$ such that $u(x) = y$. Since by Theorem 10, $T_{\{\Lambda_j\}}$ is surjective, there is $\{f_j\}_{j\in\mathbb{J}} \in (\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$ such that $T_{\{\Lambda_j\}}(\{f_j\}_{j\in\mathbb{J}}) = x$. Thus $y = uT_{\{\Lambda_j\}}(\{f_j\}_{j\in\mathbb{J}})$, and the proof is completed. □

Theorem 15. *A sequence $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with bounds A and B if and only if the following two conditions are satisfied:*

(1) *$\{\Lambda_j\}_{j\in\mathbb{J}}$ is g-complete.*

(2) *The operator*

$$T_{\{\Lambda_j\}} : \{f_j\} \mapsto \sum_j \Lambda_j^*(f_j)$$

is well-defined from $(\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$ into H, and for each $\{f_j\}_{j\in\mathbb{J}} \in \mathcal{N}^{\perp}_{T_{\{\Lambda_j\}}}$,

$$A\sum_{j\in\mathbb{J}} \|f_j\|^2 \leq \|T_{\{\Lambda_j\}}\{f_j\}\|^2 \leq B\sum_{j\in\mathbb{J}} \|f_j\|^2. \tag{7}$$

Proof. First, suppose that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame. By Theorems 6 and 7, item 1 is satisfied.

By Theorem 9, $T_{\{\Lambda_j\}}$ is a well-defind operator from $(\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell^2}$ to H, and $\|T_{\{\Lambda_j\}}\|^2 \leq B$. Now, we have

$$\|T_{\{\Lambda_j\}}\{f_j\}_j\|^2 \leq \|T_{\{\Lambda_j\}}\|^2\|\{f_j\}_j\|^2 \leq B\sum_{j\in\mathbb{J}} \|f_j\|^2,$$

and the right-hand inequality in item 2 is proved.

By Theorem 7, $T_{\{\Lambda_j\}}$ is surjective. So $\mathcal{R}_{T_{\{\Lambda_j\}}}$ is closed. Therefore $\mathcal{R}_{T^*_{\{\Lambda_j\}}}$ is closed . Thus

$$\mathcal{N}^{\perp}_{T_{\{\Lambda_j\}}} = \overline{\mathcal{R}_{T^*_{\{\Lambda_j\}}}} = \mathcal{R}_{T^*_{\{\Lambda_j\}}}.$$

Now if $\{f_j\}_{j\in\mathbb{J}} \in \mathcal{N}^{\perp}_{T_{\{\Lambda_j\}}}$, then $\{f_j\}_{j\in\mathbb{J}} \in \mathcal{R}_{T^*_{\{\Lambda_j\}}}$ and hence

$$\{f_j\}_{j\in\mathbb{J}} = T^*_{\{\Lambda_j\}}(g) = \{\Lambda_j g\}_{j\in\mathbb{J}}$$

for some $g \in H$. Therefore

$$\begin{aligned}(\sum_{j\in\mathbb{J}}\|f_j\|^2)^2 &= (\sum_{j\in\mathbb{J}}\|\Lambda_j g\|^2)^2 = |\langle S_{\{\Lambda_j\}}(g), g\rangle|^2\\ &\le \|S_{\{\Lambda_j\}}g\|^2\|g\|^2\\ &\le \|S_{\{\Lambda_j\}}g\|^2\big(\frac{1}{A}\sum_{j\in\mathbb{J}}\|\Lambda_j g\|^2\big).\end{aligned}$$

This implies

$$A\sum_{j\in\mathbb{J}}\|f_j\|^2 \le \|S_{\{\Lambda_j\}}g\|^2 = \|T_{\{\Lambda_j\}}\{\Lambda_j g\}\|^2 = \|T_{\{\Lambda_j\}}\{f_j\}\|^2,$$

and statement 2 is proved.

Conversly, assume that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is g-complete and inequlities (7) is satisfied. Consider $\{t_j\}_{j\in\mathbb{J}} = \{f_j\}_{j\in\mathbb{J}} + \{g_j\}_{j\in\mathbb{J}}$, where $\{f_j\}_{j\in\mathbb{J}} \in \mathcal{N}_{T_{\{\Lambda_j\}}}$ and $\{g_j\}_{j\in\mathbb{J}} \in \mathcal{N}^{\perp}_{T_{\{\Lambda_j\}}}$. We have

$$\begin{aligned}\|T_{\{\Lambda_j\}}\{t_j\}_{j\in\mathbb{J}})\|^2 &= \|T_{\{\Lambda_j\}}\{g_j\}_{j\in\mathbb{J}}\|^2\\ &\le B\sum_{j\in\mathbb{J}}\|g_j\|^2\\ &\le B\|\{f_j\} + \{g_j\}\|^2\\ &= B\|\{t_j\}_{j\in\mathbb{J}}\|^2.\end{aligned}$$

Therefore $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence.

Now let $\{y_n\}$ be a sequence of members of $\mathcal{R}_{T_{\{\Lambda_j\}}}$ such that $y_n \to y$ for some $y \in H$. So there is a $\{x_n\} \in \mathcal{N}^{\perp}_{T_{\{\Lambda_j\}}}$ such that $T_{\{\Lambda_j\}}(x_n) = y_n$. By (7), we have

$$\begin{aligned}A\|\{x_n - x_m\}\|^2 &\le \|T_{\{\Lambda_j\}}\{x_n - x_m\})\|^2\\ &= \|T_{\{\Lambda_j\}}\{x_n\} - T_{\{\Lambda_j\}}\{x_m\}\|^2\\ &= \|y_n - y_m\|^2.\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in $(\sum_{j\in\mathbb{J}}\oplus H_j)_{\ell^2}$. Therefore $\{x_n\}$ converges to some $x \in (\sum_{j\in\mathbb{J}}\oplus H_j)_{\ell^2}$ and the continuity of $T_{\{\Lambda_j\}}$ implies $y = T_{\{\Lambda_j\}}(x) \in \mathcal{R}_{T_{\{\Lambda_j\}}}$, showing that $\mathcal{R}_{T_{\{\Lambda_j\}}}$ is closed. Since $span\{\Lambda_j^*(H_j)\} \subseteq \mathcal{R}_{T_{\{\Lambda_j\}}}$, by assumption in item 1, we have $\mathcal{R}_{T_{\{\Lambda_j\}}} = H$.

Let $T^{\dagger}_{\{\Lambda_j\}}$ denotes the pseudo-inverse of $T_{\{\Lambda_j\}}$. By Lemma 2, item 3, $T_{\{\Lambda_j\}}T^{\dagger}_{\{\Lambda_j\}}$ is the orthogonal projection onto $\mathcal{R}_{T_{\{\Lambda_j\}}} = H$. Thus for any $\{f_j\}_{j\in\mathbb{J}} \in (\sum_j\oplus H_j)_{\ell^2}$,

$$A\|T^{\dagger}_{\{\Lambda_j\}}T_{\{\Lambda_j\}}\{f_j\}_j\|^2 \le \|T_{\{\Lambda_j\}}T^{\dagger}_{\{\Lambda_j\}}T_{\{\Lambda_j\}}\{f_j\}_j\|^2 = \|T_{\{\Lambda_j\}}\{f_j\}_j\|^2.$$

By Lemma 2, item 5, we have $\mathcal{N}_{T^{\dagger}_{\{\Lambda_j\}}} = \mathcal{R}^{\perp}_{T_{\{\Lambda_j\}}}$, and therefore

$$\|T^{\dagger}_{\{\Lambda_j\}}\|^2 \le \frac{1}{A}.$$

Also by Lemma 2, item 2, we have

$$\|(T^*_{\{\Lambda_j\}})^{\dagger}\|^2 \le \frac{1}{A}.$$

But $(T^*_{\{\Lambda_j\}})^\dagger T^*_{\{\Lambda_j\}}$ is the ortogonal projection onto

$$\mathcal{R}_{(T^*_{\{\Lambda_j\}})^\dagger} = \mathcal{R}_{(T^\dagger_{\{\Lambda_j\}})^*} = \mathcal{N}^\perp_{T^\dagger_{\{\Lambda_j\}}} = \mathcal{R}_{T_{\{\Lambda_j\}}} = H.$$

So, for all $f \in H$,

$$\begin{aligned} \|f\|^2 &= \|(T^*_{\{\Lambda_j\}})^\dagger T^*_{\{\Lambda_j\}} f\|^2 \\ &\leq \frac{1}{A}\|T^*_{\{\Lambda_j\}} f\|^2 \\ &= \frac{1}{A}\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2. \end{aligned}$$

This implies that $\{\Lambda_j\}_{j\in\mathbb{J}}$ satisfies the lower g-frame condition. □

5. **Conclusions**

Some new results on the g-frames on the Hilbert spaces were presented. Studying g-frames showed that one can take the existence of the Bessel sequence to be equivalent with the convergence of the series of the synthesis operator. Also, the existence of a g-frame sequence for H is equivalent with the existence of a surjective operator from H onto the representation space of the g-frame. This operator is injective in case we consider g-frame in place of g-frame sequence. The relationship between a closed range operator on H with a g-frame has been fully characterized. As an important consequence, we have also specified the relation between g-completeness of a g-frame and the well-definedness of synthesis operator in a special case.

The behavior of a g-frame after deletion of some of its members, and also approximating every member of H by the remaining members of the g-frame, is an open field to explore.

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