FIXED POINT THEOREMS IN b-MULTIPLICATIVE METRIC SPACES

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In this paper, we introduce the new notion of b-multiplicative metric space. We prove fixed point theorems for single and multivalued mappings on b-multiplicative metric spaces, endowed with a graph. We construct examples to illustrate our notions and results. As illustrative application, we give an existence theorem for the solution of a class of Fredholm multiplicative integral equations.

Keywords: Multiplicative metric space; b-multiplicative metric space; graph; fixed point.

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1. Introduction

Grossman and Katz [11] introduced a new kind of Calculus called multiplicative (or non-Newtonian) calculus by interchanging the roles of subtraction and addition with the role of division and multiplication, respectively. By using the ideas of Grossman and Katz [11], Bashirov *et al.* [4] defined the notion of multiplicative metric.

Özavsar and Cevikel [19] investigated the multiplicative metric spaces along with their topological properties and proved some fixed point theorems for contraction mappings of multiplicative metric spaces. Effective contribution in this direction is due to: Abbas *et al.* [1] for a study of common fixed points of generalized rational type cocyclic mappings; He *et al.* [13] for introducing common fixed point results for weak commutative mappings; Yamaod and Sintunavarat [24] for their contribution to fixed points for generalized contraction mappings with cyclic (α, β)-admissible mapping; Gu and Cho [12] for intersting results concerning common fixed point for four maps satisfying ϕ-contractive condition; Mongkolkeha and Sintunavarat [17] for their study on best proximity points for multiplicative proximal contraction mapping; Rome and Sarwar [21] for a characterization of multiplicative metric completeness.

Czerwik [5] introduced the notion of b-metric space which is a generalization of a metric space. There are some fixed point theorems in b-metric spaces. Huang *et al.* [14] introduced fixed point results for rational Geraghty contractive mappings; Ozturk and Turkoglu [20] studied fixed points for generalized alpha-psi-contractions; Shatanawi *et al.* [22] established a study of contraction conditions using comparison functions.

In their elegant survey, Došenović *et al.* [7] show that the fixed point results for various multiplicative contractions are in fact equivalent with the corresponding fixed point results in (standard) metric spaces. We address reader to other valuable sources: Abodayeh *et al.* [2], Agarwal *et al.* [3], Došenović and S. Radenović [6], Shukla [23].

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Even if these remarks are true, we realized this article to introduce the notion of b-multiplicative metric space and mention some of its topological properties. Then we prove fixed point theorems for mappings on b-multiplicative metric spaces endowed with a graph, for single and multivalued mappings. We illustrate our results with the help of examples. As novel application, we give an existence theorem for the solution of a class of Fredholm multiplicative integral equations. Last, but not least, let us note that these fixed point theorems are new in the setting of b-metric space endowed with a graph, as far as we know.

2. Main results

We begin this section by introducing b-multiplicative metric spaces.

Definition 2.1. Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $m\colon X \times X \to [1, \infty)$ is called a b-multiplicative metric if the following conditions hold:
(m_1) $m(x, y) > 1$ for all $x, y \in X$ with $x \neq y$ and $m(x, y) = 1$ if and only if $x = y$;
(m_2) $m(x, y) = m(y, x)$ for all $x, y \in X$;
(m_3) $m(x, z) \leq m(x, y)^s \cdot m(y, z)^s$ for all $x, y, z \in X$.
The triplet (X, m, s) is called a b-multiplicative metric space.

Example 2.1. Let $X = [0, \infty)$. Define a mapping

$$m_a\colon X \times X \to [1, \infty), \quad m_a(x, y) = a^{(x-y)^2},$$

where $a > 1$ is any fixed real number. Then for each a, m_a is b-multiplicative metric on X with $s = 2$. Note that m_a is not a multiplicative metric on X.

Example 2.2. If $p \in (0, 1)$, then $l^p(\mathbb{R}) = \Big\{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\Big\}$ endowed with the functional

$$m_e\colon l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to [1, \infty), \quad m_e(\{x_n\}, \{y_n\}) = e^{\left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}}$$

for each $\{x_n\}, \{y_n\} \in l^p(\mathbb{R})$, is a b-multiplicative metric space with $s = 2^{\frac{1}{p}-1}$. Note that m_e is neither a metric nor a b-metric on X.

Following concepts are defined in the same way as defined for multiplicative metric spaces by Özavsar and Cevikel [19].

Let (X, m, s) is a b-multiplicative metric space. Then the multiplicative open and closed ball of radius $\varepsilon > 1$ having center at x is of the form:

$$B_\varepsilon(x) = \{y \in X \ : \ m(x, y) < \varepsilon\} \text{ and } \overline{B}_\varepsilon(x) = \{y \in X \ : \ m(x, y) \leq \varepsilon\}$$

respectively. A point $x \in X$ is said to be multiplicative limit point of $S \subset X$, if and only if $(B_\varepsilon(x) \backslash \{x\}) \cap S \neq \emptyset$ for every $\varepsilon > 1$. A set $S \subset X$ is multiplicative closed in (X, m) if S contains all of its multiplicative limit points. A set S is multiplicative bounded if there exist $x \in X$ and $M > 1$ such that $S \subseteq B_M(x)$. A sequence $\{x_n\}$ is a multiplicative convergent to $x \in X$ denoted by $x_n \to_b x$, if for every multiplicative open ball $B_\varepsilon(x)$, there exists a natural number N_0 such that $n \geq N_0 \Rightarrow x_n \in B_\varepsilon(x)$. That is, for each $\varepsilon > 1$, there exists some $N_0 \in \mathbb{N}$ such that $m(x_n, x) < \varepsilon$ for each $n \geq N_0$. Similarly, a sequence $\{x_n\}$ is multiplicative Cauchy, if for each $\varepsilon > 1$, there exists $N_0 \in N$ such that $m(x_m, x_n) < \varepsilon$ for each $m, n \geq N_0$. A mapping $f\colon X \to X$ is said to be a multiplicative continuous at the point $x \in X$ if and only if $f(x_n) \to_b f(x)$ for every sequence $\{x_n\}$ with $x_n \to_b x$. A multiplicative metric space is complete if every multiplicative Cauchy sequence in it is multiplicative convergent to some $x \in X$.

Lemma 2.1. Let (X, m, s) is a b-multiplicative metric space. If a sequence $\{x_n\}$ is a multiplicative convergent, then the multiplicative limit point is unique.

Proof. Suppose that $x_n \to_b x$ and $x_n \to_b y$. Then for each $\varepsilon > 1$, we have $N_0 \in \mathbb{N}$ such that $m(x_n, x) < \varepsilon^{\frac{1}{2s}}$ and $m(x_n, y) < \varepsilon^{\frac{1}{2s}}$. By using the multiplicative triangular inequality, we get

$$m(x, y) \leq m(x, x_n)^s \cdot m(x_n, y)^s < \varepsilon.$$

Since $\varepsilon > 1$ is arbitrary. Thus, we have $m(x, y) = 1$, that is $x = y$. □

Let (X, m, s) be a b-multiplicative metric space. Subsequently, $G = (V, E)$ is a directed graph such that the set of its vertices V coincides with X (i.e., $V = X$) and the set of its edges E is such that $E \supseteq \triangle$, where $\triangle = \{(x, x) : x \in X\}$. Also assume that G has no parallel edges. Jachymski [15] introduced the notion of G-continuity on metric space. We are going to extend this notion to b-multiplicative metric space. A mapping $f\colon X \to X$ is said to be G-continuous if for each sequence $\{x_n\}$ in X such that $x_n \to_b x$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N} \cup \{0\}$, we have $fx_n \to_b fx$. Jachymski [15] also introduced the notion of edge preserving mappings as: A mapping $f\colon X \to X$ is said to be an edge preserving mapping if for each $x, y \in X$ with $(x, y) \in E$, we have $(fx, fy) \in E$.

Now, we move towards our first result.

Theorem 2.1. Let (X, m, s) be a complete b-multiplicative metric space endowed the graph G and let $f\colon X \to X$ be an edge preserving mapping such that for each $(x, y) \in E$, we have

$$m(fx, fy) \leq \max\{m(x, y), m(x, fx), m(y, fy), m(x, fy)^{\frac{1}{2s}} \cdot m(y, fx)\}^{\kappa} \tag{1}$$

where $\kappa \in [0, \frac{1}{s})$. Assume that the following conditions hold:

(i) there exists $x_0 \in X$ such that $(x_0, fx_0) \in E$;

(ii) a. f is G-continuous;

or

b. for each sequence $\{x_n\} \subseteq X$ such that $(x_n, x_{n+1}) \in E$ and $x_n \to_b x$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then f has a fixed point.

Proof. By hypothesis, we have $x_0 \in X$ such that $(x_0, x_1) \in E$, where $x_1 = fx_0$. From (1), we have

$$\begin{aligned} m(x_1, x_2) &= m(f(x_0), f(x_1)) \\ &\leq \max\{m(x_0, x_1), m(x_0, fx_0), m(x_1, fx_1), m(x_0, fx_1)^{\frac{1}{2s}} \cdot m(x_1, fx_0)\}^{\kappa} \\ &= \max\{m(x_0, x_1), m(x_0, x_1), m(x_1, x_2), m(x_0, x_2)^{\frac{1}{2s}} \cdot m(x_1, x_1)\}^{\kappa} \\ &= \max\{m(x_0, x_1), m(x_1, x_2)\}^{\kappa} \\ &= m(x_0, x_1)^{\kappa}, \end{aligned}$$

otherwise we have a contradiction, that is, $m(x_1, x_2) \leq m(x_1, x_2)^{\kappa}$. As f is an edge preserving, we have $(x_1, x_2) \in E$.

Again, from (1), we obtain

$$\begin{aligned} m(x_2, x_3) &= m(f(x_1), f(x_2)) \\ &\leq \max\{m(x_1, x_2), m(x_1, fx_1), m(x_2, fx_2), m(x_1, fx_2)^{\frac{1}{2s}} \cdot m(x_2, fx_1)\}^{\kappa} \\ &= \max\{m(x_1, x_2), m(x_1, x_2), m(x_2, x_3), m(x_1, x_3)^{\frac{1}{2s}} \cdot m(x_2, x_2)\}^{\kappa} \\ &= \max\{m(x_1, x_2), m(x_2, x_3)\}^{\kappa} \\ &= m(x_1, x_2)^{\kappa} \\ &\leq m(x_0, x_1)^{\kappa^2}. \end{aligned}$$

Continuing in the same way, we construct a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n$, $(x_n, x_{n+1}) \in E$ and $m(x_n, x_{n+1}) \leq m(x_0, x_1)^{\kappa^n}$ for each $n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$, then by the multiplicative triangular inequality, we get

$$\begin{aligned} m(x_n, x_{n+m}) &\leq m(x_n, x_{n+1})^{s^n} \cdot m(x_{n+1}, x_{n+2})^{s^{n+1}} \cdots m(x_{n+m-1}, x_{n+m})^{s^{n+m-1}} \\ &\leq m(x_0, x_1)^{\kappa^n s^n} \cdot m(x_0, x_1)^{\kappa^{n+1} s^{n+1}} \cdots m(x_0, x_1)^{\kappa^{n+m-1} s^{n+m-1}} \\ &\leq m(x_0, x_1)^{(\kappa s)^n + (\kappa s)^{n+1} + \cdots + (\kappa s)^{n+m-1}} \\ &\leq m(x_0, x_1)^{\frac{(\kappa s)^n}{1-(\kappa s)}}. \end{aligned}$$

Letting $n \to \infty$, in above inequality, we get $m(x_n, x_{n+m}) \to_b 1$. Hence the sequence $\{x_n\}$ is multiplicative Cauchy sequence. By the completeness of X, there exists $x^* \in X$ such that $x_n \to_b x^*$. If f is G-continuous, then $x_{n+1} = fx_n \to_b fx^*$. Since the multiplicative limit is unique, thus $x^* = fx^*$. Suppose that $(ii - b)$ holds, then we have $(x_n, x^*) \in E$. From (1), and multiplicative triangular inequality, we have

$$\begin{aligned} & m(x^*, fx^*) \\ & \leq m(x^*, x_{n+1})^s \cdot m(x_{n+1}, fx^*)^s \\ & = m(x^*, x_{n+1})^s \cdot m(fx_n, fx^*)^s \\ & \leq m(x^*, x_{n+1})^s \cdot \max\{m(x_n, x^*), m(x_n, fx_n), m(x^*, fx^*), m(x_n, fx^*)^{\frac{1}{2s}} \cdot m(x^*, fx_n)\}^{\kappa s} \\ & = m(x^*, x_{n+1})^s \cdot \max\{m(x_n, x^*), m(x_n, x_{n+1}), m(x^*, fx^*), m(x_n, fx^*)^{\frac{1}{2s}} \cdot m(x^*, x_{n+1})\}^{\kappa s} \\ & \leq m(x^*, x_{n+1})^s \cdot \max\{m(x_n, x^*), m(x_n, x_{n+1}), m(x^*, fx^*), \\ & \quad m(x_n, x^*)^{\frac{1}{2}} \cdot m(x^*, fx^*)^{\frac{1}{2}} \cdot m(x^*, x_{n+1})\}^{\kappa s}. \end{aligned}$$

Suppose that $m(x^*, fx^*) > 1$. Letting $n \to \infty$, in above inequality, we get

$$m(x^*, fx^*) \leq \max\{1, m(x^*, fx^*), m(x^*, fx^*)^{\frac{1}{2}}\}^{\kappa s} = m(x^*, fx^*)^{\kappa s}.$$

This is a contradiction to our assumption, since $\kappa s < 1$. Thus, $m(x^*, fx^*) = 1$. That is, $x^* = fx^*$. □

Example 2.3. Let $X = [0, \infty)$ be endowed with a b-multiplicative metric $m(x, y) = 3^{(x-y)^2}$, with $s = 2$. Define

$$f \colon X \to X, \quad fx = \begin{cases} 0 \text{ if } x < 1 \\ \frac{x+7}{3} \text{ otherwise.} \end{cases}$$

Consider a graph $G = (V, E)$ as $V = X$ and $E = \{(x, y) : x, y \geq 1\} \cup \{(x, x) : x \in X\}$. For each $(x, y) \in E$, we have

$$m(fx, fy) = 3^{(\frac{x}{3} - \frac{y}{3})^2} = (m(x, y))^{\frac{1}{9}}.$$

Thus, (1) holds. Furthermore, it is easy to see that all other conditions of Theorem 2.1 hold. Therefore, f has a fixed point.

Example 2.4. Let $X = \mathbb{R}^2$ endowed with a multiplicative metric defined by the formula $m(x, y) = |x_1 - y_1| + |x_2 - y_2|$ for each $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Consider the mapping

$$f \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = \left(\frac{5x}{7} - \frac{3y}{7} + 1, \frac{8y}{9} - \frac{4x}{9} + 1\right), \text{ for each } (x, y) \in X.$$

Define the graph $G = (V, E)$ such that $V = \mathbb{R}^2$ and $E = \{(x, y) : x, y \in X\}$. Thus, it is easy to see that (1) and all other conditions of Theorem 2.1 hold. Thus, f has a fixed point, that is $x = fx$, where $x = (2, 1)$.

We denote by $CL(X)$, the class of all nonempty multiplicative closed subsets of X. Note that a point $x_0 \in X$ is a fixed point of mapping $F \colon X \to CL(X)$ if we have $x_0 \in Fx_0$.

Theorem 2.2. Let (X, m, s) be a complete b-multiplicative metric space endowed the graph $G = (V, E)$ and let $F\colon X \to CL(X)$ be a mapping such that for each $(x, y) \in E$ and $u \in Fx$, there exists $v \in Fy$ satisfying

$$m(u, v) \leq m(x, y)^{\kappa} \cdot m(y, u)^{\omega} \tag{2}$$

where $\kappa \in [0, \frac{1}{s})$ and $\omega \geq 0$. Assume that the following conditions hold:

(i) there exists $x_0 \in X$ and $x_1 \in Fx_0$ such that $(x_0, x_1) \in E$;

(ii) for each $u \in Fx$ and $v \in Fy$ such that $m(u, v) < m(x, y)$ we have $(u, v) \in E$, whenever $(x, y) \in E$;

(iii) for each sequence $\{x_n\} \subseteq X$ such that $(x_n, x_{n+1}) \in E$ and $x_n \to_b x$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then F has a fixed point.

Proof. By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Fx_0$ such that $(x_0, x_1) \in E$. From (2), for $(x_0, x_1) \in E$ and $x_1 \in Fx_0$, there exists $x_2 \in Fx_1$ such that

$$m(x_1, x_2) \leq m(x_0, x_1)^{\kappa} \cdot m(x_1, x_1)^{\omega} = m(x_0, x_1)^{\kappa}. \tag{3}$$

By hypothesis (ii) and (3), we have $(x_1, x_2) \in E$.

Again from (2), for $(x_1, x_2) \in E$ and $x_2 \in Fx_1$, there exists $x_3 \in Fx_2$ such that

$$m(x_2, x_3) \leq m(x_1, x_2)^{\kappa} \cdot m(x_2, x_2)^{\omega} = m(x_1, x_2)^{\kappa} \leq m(x_0, x_1)^{\kappa^2}.$$

Continuing in the same way, we construct a sequence $\{x_n\}$ in X such that $x_n \in Fx_{n-1}$, $(x_n, x_{n+1}) \in E$ and $m(x_n, x_{n+1}) \leq m(x_0, x_1)^{\kappa^n}$ for each $n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$, then by the multiplicative triangle inequality, we have

$$\begin{aligned} m(x_n, x_{n+m}) &\leq m(x_n, x_{n+1})^{s^n} \cdot m(x_{n+1}, x_{n+2})^{s^{n+1}} \cdots m(x_{n+m-1}, x_{n+m})^{s^{n+m-1}} \\ &\leq m(x_0, x_1)^{\kappa^n s^n} \cdot m(x_0, x_1)^{\kappa^{n+1} s^{n+1}} \cdots m(x_0, x_1)^{\kappa^{n+m-1} s^{n+m-1}} \\ &\leq m(x_0, x_1)^{(\kappa s)^n + (\kappa s)^{n+1} + \cdots + (\kappa s)^{n+m-1}} \\ &\leq m(x_0, x_1)^{\frac{(\kappa s)^n}{1-(\kappa s)}}. \end{aligned}$$

Letting $n \to \infty$ in above inequality, we get $m(x_n, x_{n+m}) \to_b 1$. Hence the sequence $\{x_n\}$ is a multiplicative Cauchy sequence. By the completeness of X, there exists $x^* \in X$ such that $x_n \to_b x^*$. By hypothesis (iii), we have $(x_n, x^*) \in E$. From (2), for each $(x_n, x^*) \in E$ and $x_{n+1} \in Fx_n$, there exists $v^* \in Fx^*$ such that

$$m(x_{n+1}, v^*) \leq m(x_n, x^*)^{\kappa} \cdot m(x^*, x_{n+1})^{\omega}.$$

From the multiplicative triangle inequality, we have

$$\begin{aligned} m(x^*, v^*) &\leq m(x^*, x_{n+1})^s \cdot m(x_{n+1}, v^*)^s \\ &\leq m(x^*, x_{n+1})^s \cdot m(x_n, x^*)^{\kappa s} \cdot m(x^*, x_{n+1})^{\omega s}. \end{aligned}$$

Letting $n \to \infty$ in above inequality, we get $m(x^*, v^*) = 1$. That is, $x^* = v^*$. Thus, we have $x^* \in Fx^*$. □

Example 2.5. Let $X = [0, \infty)$ be endowed with a b-multiplicative metric $m(x, y) = e^{(x-y)^2}$, with $s = 2$. Define

$$F\colon X \to CL(X), \quad Fx = \begin{cases} [0, \frac{x}{2}], & \text{if } x < 2 \\ \{2\}, & \text{if } x = 2 \\ [0, \ln(x+1)], & \text{otherwise.} \end{cases}$$

Consider a graph $G = (V, E)$ as $V = X$ and $E = \{(x, y) : 0 \leq x, y \leq 2\} \cup \{(x, x) : x \in X\}$. To see (2) holds with $k = \frac{1}{4}$ and $\omega = 1$, we need to take the following cases:

(i) For each $(x,y) \in E$ with $x, y < 2$, and $u \in Fx$, there exists $v \in Fy$ such that

$$m(u,v) \le e^{(\frac{x}{2}-\frac{y}{2})^2} \le m(x,y)^{\frac{1}{4}} \cdot m(y,u);$$

(ii) For each $(x,y) \in E$ with $x < 2$, $y = 2$ and $u \in Fx$, there exists $v \in Fy = \{2\}$ such that

$$m(u,v) \le m(x,y)^{\frac{1}{4}} \cdot m(y,u) = m(x,y)^{\frac{1}{4}} \cdot m(v,u);$$

Thus, (2) holds. Furthermore, it is easy to see that all other conditions of Theorem 2.2 hold. Therefore, F has a fixed point.

As special case of Theorem 2.1 we have the following corollary:

Corollary 2.1. Let (X, m, s) be a complete b-multiplicative metric space endowed the graph G and let $f : X \to X$ be an edge preserving mapping such that for each $(x,y) \in E$, one of the following inequality hold:
(i) $m(fx, fy) \le m(x,y)^{\kappa}$;
(ii) $m(fx, fy) \le m(x, fx)^{\kappa}$;
(iii) $m(fx, fy) \le m(y, fy)^{\kappa}$;
(iv) $m(fx, fy) \le \{m(x, fy)^{\frac{1}{2s}} \cdot m(y, fx)\}^{\kappa}$,
where $\kappa \in [0, \frac{1}{s})$. Assume that the following conditions hold:
(i) there exists $x_0 \in X$ such that $(x_0, fx_0) \in E$;
(ii) a. f is G-continuous;
or
b. for each sequence $\{x_n\} \subseteq X$ such that $(x_n, x_{n+1}) \in E$ and $x_n \to_b x$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.
Then f has a fixed point.

Consider the graph $G = (V, E)$ as $V = X$ and $E = \{(x,y) : x \preceq y\}$, then by Theorem 2.1 we have the following result:

Corollary 2.2. Let $(X, m, s, \preceq)$ be a complete b-multiplicative ordered metric space and let $f : X \to X$ be an ordered preserving mapping such that for each $x \preceq y$, we have

$$m(fx, fy) \le \max\{m(x,y), m(x,fx), m(y,fy), m(x,fy)^{\frac{1}{2s}} \cdot m(y,fx)\}^{\kappa}$$

where $\kappa \in [0, \frac{1}{s})$. Assume that the following conditions hold:
(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;
(ii) a. f is continuous;
or
b. for each sequence $\{x_n\} \subseteq X$ such that $x_n \preceq x_{n+1}$ and $x_n \to_b x$, then $x_n \preceq x$ for each $n \in \mathbb{N}$.
Then f has a fixed point.

If we consider $G = (X, X \times X)$ then Theorem 2.1 reduces to following result:

Corollary 2.3. Let (X, m, s) be a complete b-multiplicative metric space and let $f : X \to X$ be a mapping such that for each $x, y \in X$, we have

$$m(fx, fy) \le \max\{m(x,y), m(x,fx), m(y,fy), m(x,fy)^{\frac{1}{2s}} \cdot m(y,fx)\}^{\kappa}$$

where $\kappa \in [0, \frac{1}{s})$. Then f has a fixed point.

Letting $\omega = 0$ in Theorem 2.2, we have the following result:

Corollary 2.4. Let (X, m, s) be a complete b-multiplicative metric space endowed the graph G and let $F : X \to CL(X)$ be a mapping such that for each $(x,y) \in E$ and $u \in Fx$, there exists $v \in Fy$ satisfying

$$m(u,v) \le m(x,y)^{\kappa}$$

where $\kappa \in [0, \frac{1}{s})$. Assume that the following conditions hold:

(i) there exists $x_0 \in X$ and $x_1 \in Fx_0$ such that $(x_0, x_1) \in E$;

(ii) for each $u \in Fx$ and $v \in Fy$ such that $m(u,v) < m(x,y)$ we have $(u,v) \in E$, whenever $(x,y) \in E$;

(iii) for each sequence $\{x_n\} \subseteq X$ such that $(x_n, x_{n+1}) \in E$ and $x_n \to_b x$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then F has a fixed point.

By taking $G = (X, X \times X)$ in Theorem 2.2, we get the following result:

Corollary 2.5. Let (X, m, s) be a complete b-multiplicative metric space and let $F\colon X \to CL(X)$ be a mapping such that for each $x, y \in X$ and $u \in Fx$, there exists $v \in Fy$ satisfying

$$m(u,v) \leq m(x,y)^{\kappa} \cdot m(y,u)^{\omega}$$

where $\kappa \in [0, \frac{1}{s})$ and $\omega \geq 0$. Then F has a fixed point.

Remark 2.1. Note that if $d(x,y)$ is a b metric on X then the function $m\colon X \times X \to [1, \infty)$ defined by $m(x,y) = e^{d(x,y)}$ is a b-multiplicative metric on X. Moreover, if f is a self map on a b-metric space (X, d) satisfying following inequality

$$d(fx, fy) \leq \kappa \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)}{2s} + d(y,fx)\right\},$$

then by taking $m(x,y) = e^{d(x,y)}$ it follows that $m(x,y)$ satisfies the inequality in Theorem 2.1. Thus one may obtain following result as direct consequence of our Theorem 2.1.

Theorem 2.3. Let (X, d, s) be a complete b- metric space endowed the graph G and let $f\colon X \to X$ be an edge preserving mapping such that for each $(x,y) \in E$, we have

$$d(fx, fy) \leq \kappa \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)}{2s} + d(y,fx)\right\},$$

where $\kappa \in [0, \frac{1}{s})$. Assume that the following conditions hold:

(i) there exists $x_0 \in X$ such that $(x_0, fx_0) \in E$;

(ii) a. f is G-continuous, with respect to d;

or

b. for each sequence $\{x_n\} \subseteq X$ such that $(x_n, x_{n+1}) \in E$ and $x_n \to x$, with respect to d then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then f has a fixed point.

Similarly, one may deduce following result from Theorem 2.2

Theorem 2.4. Let (X, d, s) be a complete b-metric space endowed the graph G and let $F\colon X \to CL(X)$ be a mapping such that for each $(x,y) \in E$ and $u \in Fx$, there exists $v \in Fy$ satisfying

$$d(u,v) \leq \kappa d(x,y) + \omega d(y,u),$$

where $\kappa \in [0, \frac{1}{s})$ and $\omega \geq 0$. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Fx_0$ such that $(x_0, x_1) \in E$;

(ii) for each $u \in Fx$ and $v \in Fy$ such that $d(u,v) < d(x,y)$ we have $(u,v) \in E$, whenever $(x,y) \in E$;

(iii) for each sequence $\{x_n\} \subseteq X$ such that $(x_n, x_{n+1}) \in E$ and $x_n \to x$, with respect to d, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then F has a fixed point.

Remark 2.2. By the best of our knowledge, we note that these fixed point theorems are new in the setting of b-metric spaces, endowed with a graph.

3. Application

Let $X = C([a,b],\mathbb{R}_+)$, $a > 0$ and $\mathbb{R}_+ = (0,\infty)$, be the space of all positive, continuous real valued functions, endowed with the b-multiplicative metric

$$m(x,y) = \begin{cases} \sup_{t\in[a,b]} \left|\frac{x(t)}{y(t)}\right|^2 & \text{if } \frac{x(t)}{y(t)} > 1 \\ \sup_{t\in[a,b]} \left|\frac{y(t)}{x(t)}\right|^2 & \text{if } \frac{x(t)}{y(t)} < 1 \end{cases}$$

and graph $G = (V,E)$ such that $V = X$ and $E = \{(x,y) : x(t) \geq y(t), \forall t \in [a,b]\}$.

As an application, we give an existence theorem for the Fredholm multiplicative integral equation of the following type.

$$x(t) = \int_a^b K(t,s,x(s))^{ds},\ t,s \in [a,b] \tag{4}$$

where $K\colon [a,b]\times[a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing function.

Theorem 3.1. Let $X = C([a,b],\mathbb{R}_+)$, $a > 0$, endowed with the graph G and let the operator

$$F\colon X \to X,\quad Fx(t) = \int_a^b K(t,s,x(s))^{ds}$$

where $K\colon [a,b]\times[a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing function. Assume that the following conditions hold:

(i) for each $t,s \in [a,b]$ and $x,y \in X$ with $(x,y) \in E$, there exists a constant $\eta > 0$ such that

$$\left|\frac{K(t,s,x(s))}{K(t,s,y(s))}\right| \leq \left(\left|\frac{x(s)}{y(s)}\right|\right)^{\eta};$$

(ii) the constant η is such that $\eta < \frac{1}{2(b-a)}$;

(iii) there exists $x_0 \in X$ such that $(x_0, Fx_0) \in E$.

Then the integral equation (4) has at least one solution.

Proof. First we show that for each $(x,y) \in E$, the the inequalities (i) of Corollary 2.1 holds. For each $(x,y) \in E$, we have

$$\begin{aligned} \left|\frac{Fx(t)}{Fy(t)}\right|^2 &\leq \left(\int_a^b \left|\frac{K(t,s,x(s))}{K(t,s,y(s))}\right|^{ds}\right)^2 \\ &\leq \left(\int_a^b \left(\left|\frac{x(s)}{y(s)}\right|^{\eta}\right)^{ds}\right)^2 \\ &\leq \left(\int_a^b \left(m(x,y)^{\frac{\eta}{2}}\right)^{ds}\right)^2 \\ &= \left(\left(m(x,y)^{b-a}\right)^{\frac{\eta}{2}}\right)^2 \\ &= m(x,y)^{\eta(b-a)} \text{ for each } t \in [a,b]. \end{aligned}$$

Thus, we get $m(Fx,Fy) \leq m(x,y)^{\kappa}$, $\kappa = \eta(b-a) \in [0,\frac{1}{2})$, for each $(x,y) \in X$. Since K is nondecreasing, for each $(x,y) \in E$ we have $(Fx,Fy) \in E$. Moreover, by hypothesis (iii), $(x_0, Fx_0) \in E$. Also, by continuity of K, F is a G-continuous. Therefore by Corollary 2.1, there exists at least one fixed point of the operator F, that is, integral equation (4) has at least one solution. □

4. **Conclusion**

Maginniss [16] used non-Newtonian calculus to create a theory of probability that is adopted to human behavior and decision making. Many authors have contributed in this field. The collective book [25] presents how a non-Newtonian calculus could be applied to re-postulate and analyze the neoclassical (Solow-Swan) exogenous growth made in economics. Non-Newtonian way to look at differential equations has been a great surprise to us, it opens the question if there are major fields of economic analysis which can be profoundly re-thought in the light of this discovery. Non-Newtonian calculus has been used in the study of biomedical image analysis by Florak *et al.* [9, 10], while Mora used non-Newtonian calculus in the study of contour detection in images with multiplicative noise [18]. Several applications regarding these subjects can be seen in differential equations, calculus of variations, finite differential method, complex analysis, actuarial science, finance, economics, biology and demographics.

A semimetric space (X, d) is called a b-metric space if for each $x, y, z \in X$, we have

$$d(x,y) \leq s[d(x,z) + d(z,y)]. \tag{5}$$

This condition was put together by Czerwik [5] in order to generalize the Banach contraction principle. The same inequality given in (5) was also discussed by Fagin *et al.* [8], who called this new distance a measure of nonlinear elastic matching. In this paper we discuss the analogue version of b-metric space as b-multiplicative metric space. One application has been discussed here, and possible application regarding this discovery are discussed above.

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