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ANALYSIS OF RING SOURCED DIFFRACTION WITH RIGID AND IMPEDANCE BOUNDARY CONDITION

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The diffraction of sound waves emanating from a ring source is investigated rigorously by using the Wiener-Hopf technique. Two different geometry is considered which are semi-lined and full-lined with different linings. An exact solution is obtained based on the boundary condition. At the end of the analysis, the influence of the problem parameters and comparison of geometries are presented graphically.

Keywords: Fourier transform, Wiener-Hopf, Diffraction, Duct
MSC2010: 78A45, 47A68, 42B10, 34B30.

1. Introduction

The diffraction of acoustic waves along duct systems is an important topic in diffraction theory and relevant to many applications including reduction of noise in exhaust systems, in modern aircraft jet and turbofan engines, etc. For this reason, a rigorous analysis of such engineering problems is required.

The reduction of noise in duct systems is generally achieved by silencers. The most well-known of such silencers is acoustically absorbent linings, which have been widely analyzed in literature [6], [11], [12], [14]. Rawlins proved the effectiveness of absorbing lining who considered the radiation of sound from an unfanged rigid cylindrical duct with an acoustically absorbing internal surface [12].

When analysing sound diffraction in a duct system with an absorbing lining, two possible linings are commonly used which are classified as locally reacting lining or bulk reacting lining. The more completely investigated case is that the liner may be treated as locally reacting and this case results in a simplification of the analysis. In this case, the liner is treated as though it may be characterised by a local impedance that is independent of whatever occurs at any other part of the liner and the assumption is implicit that sound propagation does not occur in the material in any other direction than normal to the surface [2]. Bulk reacting liner is one where sound can propagate in all directions, and therefore sound can propagate in the liner parallel to the axis of the duct. It is not easy to perform acoustic analyses of this type of liner [5], [8]. This paper focuses on the local reacting lining.

The aim of this work is to consider the diffraction of acoustic waves emanating from a ring source by an infinite semi-lined and full-lined duct. Duct walls are assumed to be infinitely thin and rigid from inside. This geometry can be considered as a model of an acoustic waveguide for use in noise reduction. The ring source provides the total field to have angular symmetry which makes the problem simpler than the asymmetric case [3], [15]. In this study, an analytical solution is obtained based on the Wiener-Hopf technique [7]. By applying direct Fourier transform, the problem is reduced into the solution of a Wiener-Hopf equation. Then, numerical solution is obtained for various values of the

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problem parameters such as frequency, impedance etc. The effect of these parameters on the diffraction phenomenon is presented graphically by using the MATLAB programming.

2. Semi-Lined Duct

We consider the diffraction of sound waves by circular cylindrical duct. Duct walls are assumed to be infinitely thin and they occupy the region $\{r = a, z \in (-\infty, \infty)\}$ illuminated by a ring source located at $\{r = b > a, z = -c, c > 0\}$ (see Fig. 1). The inner surface of cylinder $(z \in (-\infty, \infty))$ and the outer surface of cylinder $(z < l)$ are assumed to be rigid, while the outer surface for $z > l$ is assumed to be lined with acoustically absorbent material. The liner impedance is characterized by Z. From the symmetry of the geometry of the problem and of the ring source, the total field will be independent of azimuth θ everywhere in circular cylindrical coordinate system (r, θ, z). The velocity potential ψ will be used to obtain acoustic pressure p and velocity v via $p = -\rho_0(\partial/\partial t)\psi$ and $\overrightarrow{v} = \operatorname{grad}\psi$, where ρ_0 is the density of the undisturbed medium.

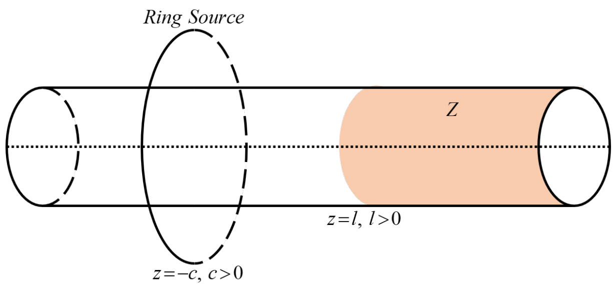

Fig. 1. Semi-lined geometry.

For analysis purposes, it is convenient to express the total field as follows:

$$\psi^T(r, z, t) = \begin{cases} \psi_1(r, z)\exp(i\omega t) & ; \quad r > b \\ \psi_2(r, z)\exp(i\omega t) & ; \quad a < r < b \end{cases} \tag{1}$$

where $\omega = 2\pi f$ is the angular frequency. Time dependence is assumed to be $e^{i\omega t}$ and suppressed throughout this work.

2.1. Derivation of the Wiener-Hopf Equation

The unknown fields $\psi_1(r, z)$ and $\psi_2(r, z)$ satisfy the wave equation for $z \in (-\infty, \infty)$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} + k^2\right]\psi_j(r, z) = 0 \quad , \quad j = 1, 2 \tag{2}$$

with wave number $k = \omega/c_0$ and speed of the sound c_0. By taking Fourier transform of these two equations we obtain the following integral representations

$$\psi_1(r, z) = \frac{k}{2\pi}\int_{\Upsilon} A(u)\, H_0^{(2)}(\lambda k r)\, e^{-iukz} du \tag{3}$$

$$\psi_2(r, z) = \frac{k}{2\pi}\int_{\Upsilon} [B(u)\, J_0(\lambda k r) + C(u)\, Y_0(\lambda k r)]\, e^{-iukz} du \tag{4}$$

where Υ is a suitable inverse Fourier transform integration contour along or near the real axis in the complex u-plane (see Fig. 2). J_0 and Y_0 are the Bessel and Neumann functions of order zero, $H_0^{(2)} = J_0 - iY_0$ is the Hankel function of the second type. λ is square root function which is defined as $\lambda(u) = \sqrt{1 - u^2}$, $\operatorname{Im}(\lambda) \leq 0$

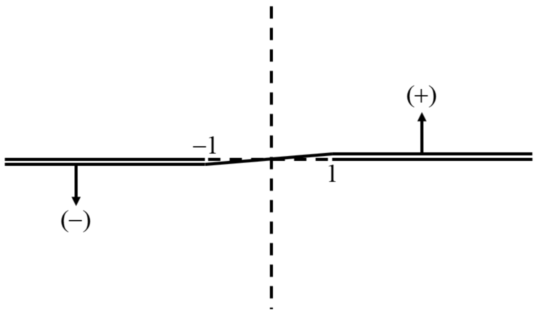

Fig. 2. Complex u plane.

Branch cuts for λ is taken on the line from 1 to ∞ and from $-\infty$ to -1. As usual in this kind of Wiener-Hopf problem, we will assume that the surrounding medium is slightly lossy and k has a small negative imaginary part. The lossless case can be obtained by letting $\operatorname{Im} k \to 0$ at the end of the analysis. The spectral coefficients $A(u)$, $B(u)$ and $C(u)$, which are introduced in the solution of velocity potential function, are to be determined with the aid of the following boundary and continuity relations valid along $r = a$ and $r = b$.

$$\frac{\partial}{\partial r}\psi_2(a,z) = 0 \qquad , \qquad z < l \tag{5}$$

$$\frac{\partial}{\partial r}\psi_2(a,z) = \frac{ik}{Z}\psi_2(a,z) \qquad , \qquad l < z \tag{6}$$

$$\frac{\partial}{\partial r}\psi_1(b,z) - \frac{\partial}{\partial r}\psi_2(b,z) = \delta(z+c) \qquad , \qquad -\infty < z < \infty \tag{7}$$

$$\psi_1(b,z) = \psi_2(b,z) \qquad , \qquad -\infty < z < \infty \tag{8}$$

$A(u)$, $B(u)$ and $C(u)$ are related to each other by the definition of the ring source given in (7,8), application of the boundary conditions on $r = b$ yields

$$\lambda k A(u) H_1^{(2)}(\lambda kb) = \lambda k B(u) J_1(\lambda kb) + \lambda k C(u) Y_1(\lambda kb) - e^{-iukc} \tag{9}$$

$$A(u) H_0^{(2)}(\lambda kb) = B(u) J_0(\lambda kb) + C(u) Y_0(\lambda kb) \tag{10}$$

From the relations (9) and (10), we obtain

$$B(u) = A(u) + e^{-iukc}\frac{\pi b}{2}Y_0(\lambda kb) \tag{11}$$

$$C(u) = -iA(u) - e^{-iukc}\frac{\pi b}{2}J_0(\lambda kb) \tag{12}$$

Applying the boundary conditions on $r = a$ and taking Fourier transforms gives

$$-\lambda k B(u) J_1(\lambda ka) - \lambda k C(u) Y_1(\lambda ka) = \frac{ik}{Z}e^{iukl}\Phi_1^+(u) \tag{13}$$

$$k\left[B(u) J(Z,u) + C(u) Y(Z,u)\right] = \frac{ik}{Z}e^{iukl}\Phi_1^-(u) \tag{14}$$

where

$$J(Z,u) = iJ_0(\lambda ka)/Z + \lambda J_1(\lambda ka) \tag{15}$$

$$Y(Z,u) = iY_0(\lambda ka)/Z + \lambda Y_1(\lambda ka) \tag{16}$$

$\Phi_1^{\pm}$ are a function analytic at the upper and lower half plane respectively, and defined as

$$\Phi_1^+(u) = \int_l^\infty \psi_2(a,z)\, e^{iuk(z-l)} dz \quad (17)$$

$$\Phi_1^-(u) = \int_{-\infty}^l \psi_2(a,z)\, e^{iuk(z-l)} dz \quad (18)$$

The substitution of $B(u)$ and $C(u)$ into (13), (14) yields

$$A(u) = -\frac{i}{Z}\frac{e^{iukl}}{\lambda H_1^{(2)}(\lambda ka)}\Phi_1^+(u) - \frac{e^{-iukc}\pi b}{2H_1^{(2)}(\lambda ka)}\left[Y_0(\lambda kb)\, J_1(\lambda ka) - J_0(\lambda kb)\, Y_1(\lambda ka)\right] \quad (19)$$

$$A(u) = \frac{i}{Z}\frac{e^{iukl}}{H(Z,u)}\Phi_1^-(u) - \frac{e^{-iukc}\pi b}{2H(Z,u)}\left[Y_0(\lambda kb)\, J(Z,u) - J_0(\lambda kb)\, Y(Z,u)\right] \quad (20)$$

where

$$H(Z,u) = iH_0^{(2)}(\lambda ka)/Z + \lambda H_1^{(2)}(\lambda ka)(\lambda ka) \quad (21)$$

$A(u)$ can be eliminated from equations (19) and (20), we get the following Wiener-Hopf equation:

$$L(u)\,\Phi_1^+(u) = -\Phi_1^-(u) - \frac{b}{a}e^{-iuk(c+l)}\frac{H_0^{(2)}(\lambda kb)}{\lambda k H_1^{(2)}(\lambda ka)} \quad (22)$$

where

$$L(u) = \frac{H(Z,u)}{\lambda H_1^{(2)}(\lambda ka)} \quad (23)$$

2.2. Solution of the Wiener-Hopf Equation

We consider equation (22). By using the classical factorization and decomposition procedure, we get

$$L^+(u)\,\Phi_1^+(u) - Q_1^+(u) = -L^-(u)\,\Phi_1^-(u) + Q_1^-(u) \quad (24)$$

where the split functions $L^+(u)$ and $L^-(u)$, result from the factorization of $L(u)$ as,

$$L(u) = \frac{L^+(u)}{L^-(u)} \quad (25)$$

they are regular and free of zeros in the upper and lower half planes, respectively [13]. Decomposing $Q_1(u)$ we obtain split functions $Q_1^+(u)$ and $Q_1^-(u)$ which are regular in the upper and lower half planes, respectively.

$$Q_1(u) = -\frac{b}{a}\frac{H_0^{(2)}(\lambda kb)}{\lambda k H_1^{(2)}(\lambda ka)}L^-(u)\, e^{-iuk(c+l)} = Q_1^+(u) + Q_1^-(u) \quad (26)$$

and $Q_1^+(u)$ is defined by

$$Q_1^+(u) = \frac{1}{2\pi i}\int_{\Upsilon_+}\frac{Q_1(\tau)}{\tau - u}d\tau \quad (27)$$

The explicit expression of the integral Q_1^+ is given in the appendix. Now both sides of (24) are analytical functions on upper and lower regions, but they are equal to each other on the strip $\operatorname{Im} k < \operatorname{Im} u < \operatorname{Im}(-k)$. From analytical continuation principle and Liouville's theorem, we get the Wiener Hopf solution

$$L^+(u)\,\Phi_1^+(u) = Q_1^+(u) \quad (28)$$

2.3. Far Field

The total field in the region $r > b$ can be evaluated from (3)

$$\psi_1(r,z) = \frac{k}{2\pi}\int_{\Upsilon} A(u)\, H_0^{(2)}(\lambda kr)\, e^{-iukz} du \tag{29}$$

Using (19) we may write the total field as follows

$$\psi_1(r,z) = -\frac{ik}{Z}\frac{1}{2\pi}\int_{\Upsilon} \frac{\Phi_1^+(u)}{\lambda H_1^{(2)}(\lambda ka)} H_0^{(2)}(\lambda kr)\, e^{-iuk(z-l)} du$$
$$-\frac{kb}{4}\int_{\Upsilon} \frac{Y_0(\lambda kb)\, J_1(\lambda ka) - J_0(\lambda kb)\, Y_1(\lambda ka)}{H_1^{(2)}(\lambda ka)} H_0^{(2)}(\lambda kr)\, e^{-iuk(z+c)} du \tag{30}$$

Taking into account the asymptotic expression of the Hankel function $H_0^{(2)}(\lambda kr)$ for large arguments ($kr \gg 1$) and applying the saddle point technique [9], we get,

$$\psi_1(r,z) \sim -\frac{ik}{Z}\frac{i}{\pi}\frac{\Phi_+(\cos\theta_1)}{\sin\theta_1 H_1^{(2)}(\sin\theta_1 ka)}\frac{e^{-ikR_1}}{kR_1}$$
$$-\frac{ikb}{2}\frac{Y_0(\sin\theta_2 kb)\, J_1(\sin\theta_2 ka) - J_0(\sin\theta_2 kb)\, Y_1(\sin\theta_2 ka)}{H_1^{(2)}(\sin\theta_2 ka)}\frac{e^{-ikR_2}}{kR_2} \tag{31}$$

where R_1, θ_1 and R_2, θ_2 are spherical coordinates.

$$r = R_1 \sin\theta_1 \quad , \quad z - l = R_1 \cos\theta_1 \tag{32}$$

and

$$r = R_2 \sin\theta_2 \quad , \quad z + c = R_2 \cos\theta_2 \tag{33}$$

3. Full-Lined Duct

We now consider the same geometry with different lining. The inner surface of cylinder is assumed to be rigid, while the outer surface is assumed to be lined with acoustically absorbent material. The liner impedances are characterized by Z_1 ($z < l$) and Z_2 ($z > l$) (see Fig.3).

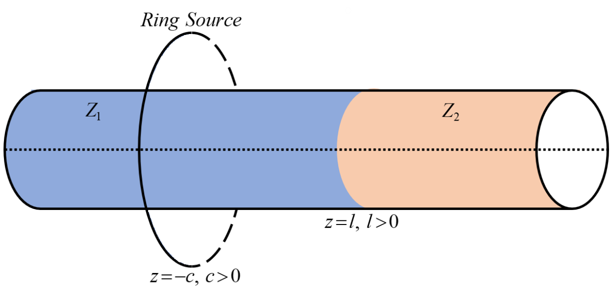

Fig. 3. Different lined geometry.

Due to different linig, equation (5,6) have to be modified as

$$\frac{\partial}{\partial r}\psi_2(a,z) = \frac{ik}{Z_1}\psi_2(a,z) \quad , \quad z < l \tag{34}$$

$$\frac{\partial}{\partial r}\psi_2(a,z) = \frac{ik}{Z_2}\psi_2(a,z) \quad , \quad l < z \tag{35}$$

Applying the boundary conditions on $r = a$ and taking Fourier transforms gives

$$k\left[B\left(u\right)J(Z_1,u)+C\left(u\right)Y(Z_1,u)\right]=e^{iukl}\Phi_2^+\left(u\right) \tag{36}$$

$$k\left[B\left(u\right)J(Z_2,u)+C\left(u\right)Y(Z_2,u)\right]=e^{iukl}\Phi_2^-\left(u\right) \tag{37}$$

where

$$J(Z_j,u)=iJ_0\left(\lambda ka\right)/Z_j+\lambda J_1\left(\lambda ka\right) \quad , \quad j=1,2 \tag{38}$$

$$Y(Z_j,u)=iY_0\left(\lambda ka\right)/Z_j+\lambda Y_1\left(\lambda ka\right) \quad , \quad j=1,2 \tag{39}$$

$\Phi_2^\pm$ are a function analytic at the upper and lower half plane and defined as

$$\Phi_2^+\left(u\right)=\int_l^\infty\left[\frac{ik}{Z_1}\psi_2\left(a,z\right)-\frac{\partial}{\partial r}\psi_2\left(a,z\right)\right]e^{iuk(z-l)}dz \tag{40}$$

$$\Phi_2^-\left(u\right)=\int_{-\infty}^l\left[\frac{ik}{Z_2}\psi_2\left(a,z\right)-\frac{\partial}{\partial r}\psi_2\left(a,z\right)\right]e^{iuk(z-l)}dz \tag{41}$$

Similarly, one can obtain the following spectral coefficients

$$A\left(u\right)=\frac{e^{iukl}}{kH(Z_1,u)}\Phi_2^+\left(u\right)-\frac{\pi b}{2}\frac{M(Z_1,u)}{H(Z_1,u)}e^{-iukc} \tag{42}$$

$$A\left(u\right)=\frac{e^{iukl}}{kH(Z_2,u)}\Phi_2^-\left(u\right)-\frac{\pi b}{2}\frac{M(Z_2,u)}{H(Z_2,u)}e^{-iukc} \tag{43}$$

where

$$H(Z_j,u)=iH_0^{(2)}\left(\lambda ka\right)/Z_j+\lambda H_1^{(2)}\left(\lambda ka\right) \ ,j=1,2 \tag{44}$$

$$M(Z_j,u)=Y_0\left(\lambda kb\right)J(Z_j,u)-J_0\left(\lambda kb\right)Y(Z_j,u) \ ,j=1,2 \tag{45}$$

$A\left(u\right)$ can be eliminated from equations (42) and (43), we get the following Wiener-Hopf equation:

$$M\left(u\right)\Phi_2^+\left(u\right)=\Phi_2^-\left(u\right)-\frac{b}{a}\left(\frac{i}{Z_1}-\frac{i}{Z_2}\right)e^{-iuk(c+l)}\frac{H_0^{(2)}\left(\lambda kb\right)}{H(Z_1,u)} \tag{46}$$

where

$$M\left(u\right)=\frac{H(Z_2,u)}{H(Z_1,u)}=\frac{M^+\left(u\right)}{M^-\left(u\right)} \tag{47}$$

The total field in the region $r > b$ can be evaluated similarly

$$\psi_1\left(r,z\right)\sim\frac{i}{\pi}\frac{\Phi_2^+(\cos\theta_1)}{H(Z_1,\cos\theta_1)}\frac{e^{-ikR_1}}{kR_1}-\frac{ikb}{2}\frac{M(Z_1,\cos\theta_2)}{H(Z_1,\cos\theta_2)}\frac{e^{-ikR_2}}{kR_2} \tag{48}$$

where

$$\Phi_2^+(u)=Q_2^+(u)/M^+\left(u\right) \tag{49}$$

$$Q_2(u)=-\frac{b}{a}\left(\frac{i}{Z_1}-\frac{i}{Z_2}\right)\frac{H_0^{(2)}\left(\lambda kb\right)}{H(Z_1,u)}M^-\left(u\right)e^{-iuk(c+l)}=Q_2^+(u)+Q_2^-(u) \tag{50}$$

4. Results

In this section some graphics displaying the effects of the parameters of the problem on the diffracted field are presented. Numerical results are produced for the total diffracted field as

$$20log|\psi_1\left(R_1,\theta_1\right)|$$

with the observation angle θ_1 changing from 0 to π. Some parameter values remain unchanged in all examples are given below [4]

Speed of Sound	(c_0)	$= 340.17$ m/s
Far Radius	(R_1)	$= 46$ m
Duct Radius	(a)	$= 0.1191$ m
Ring Source Radius	(b)	$= 0.1985$ m
Ring Source Axis	(c)	$= 0.2000$ m
Lining Length	(l)	$= 0.1191$ m

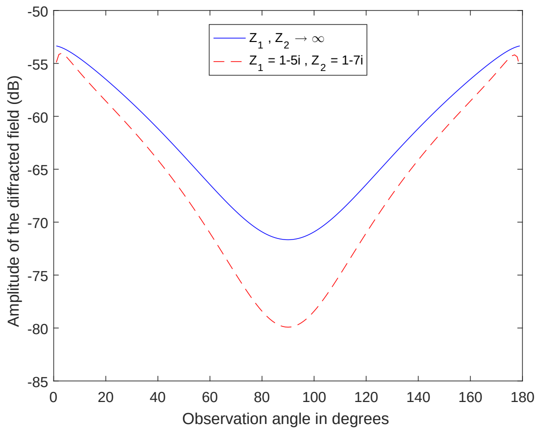

Fig. 4. Diffracted field for the frequency $f_1 = 1000$ Hz.

Fig. 4 shows the variation of the amplitude of the diffracted field as a function of the observation angle θ_1 for the frequency and surface impedance. The surface impedance is taken differently for hard wall $(Z_1, Z_2 \to \infty)$ and soft wall $(Z_1, Z_2 \sim finite)$ cases. It is observed that the diffracted field amplitude decreases with the lining impedance Z_1 and Z_2.

In figure 5, it can be seen that, especially for the main diffracted region, the diffracted field decreasing with lining for higher frequency.

From figure 6, one can see the effect of the acoustic impedance (Z_1) of the outer surface on the diffracted field amplitute. Diffracted field amplitude exhibits an oscillatory behaviour with increasing value of $Im(Z_1)$. The other result is that the diffracted field amplitude is insensitive to the variations of the outer surface impedance Z_2. So it can be deduced that the diffracted field is affected merely by the variations in the outer surface impedance Z_1.

Fig. 7 and Fig. 8 depict an excellent agreement both semi-lined and full-lined condition between the Fig. 1 and Fig. 3.

5. Conclusions

In this work, a rigorous Wiener-Hopf solution is presented for the diffraction of sound waves emanating from a ring source by a circular cylindrical duct whose exterior surface

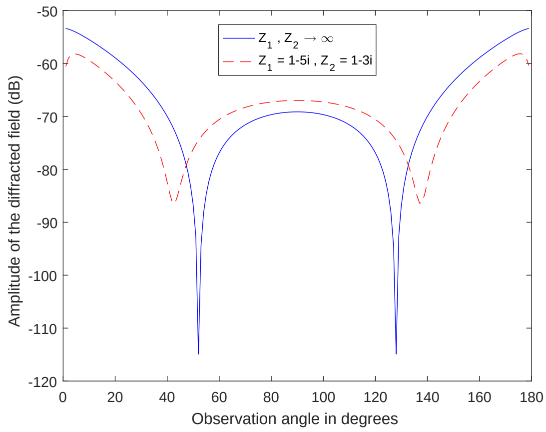

Fig. 5. Diffracted field for the frequency $f_1 = 1500$ Hz.

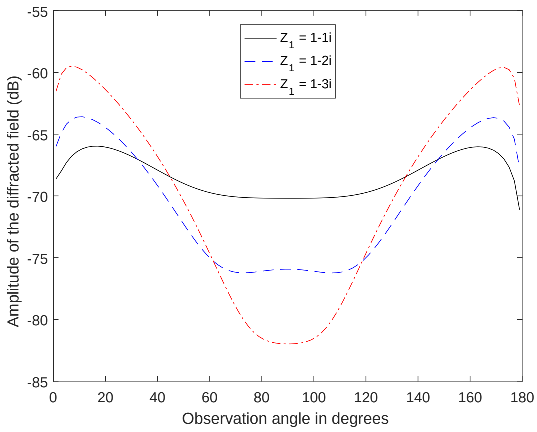

Fig. 6. Diffracted field with different values of Z_1 for $f = 1000$ Hz, $Z_2 = 1 - 7.27i$.

is treated by an acoustically absorbing lining. An analytical solution is derived for this problem by solving the Wiener-Hopf equation. Numerical solution is obtained for various values of the problem parameters. In Fig.3, when the exterior surface is rigid for $z < l$, the geometry same as Fig. 1 (see Fig. 7-8). This can be considered a good check for the analysis made in this paper.

6. Appendix

In this section we give explicit expressions for the integrals. Consider the asymptotic evaluation of $Q_1^+(u)$ for $k(c+l) \gg 1$.

$$Q_1^+(u) = -\frac{b}{a}\frac{1}{2\pi i}\int_{\Upsilon_+} \frac{H_0^{(2)}(\lambda kb)\, L^-(u)\, e^{-i\tau k(c+l)}}{\lambda k H_1^{(2)}(\lambda ka)(\tau - u)} d\tau \tag{A.1}$$

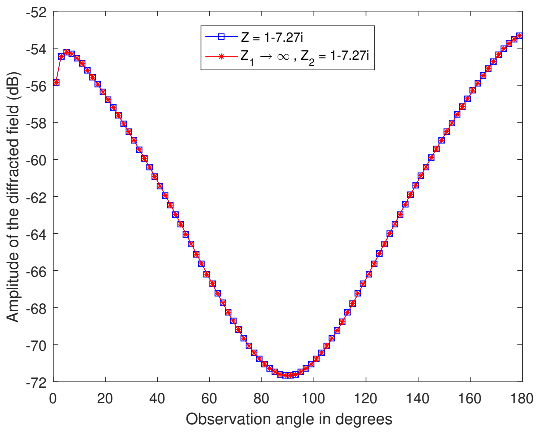

Fig. 7. Comparison of the Fig.1 and Fig.3 for $f = 1000$ Hz.

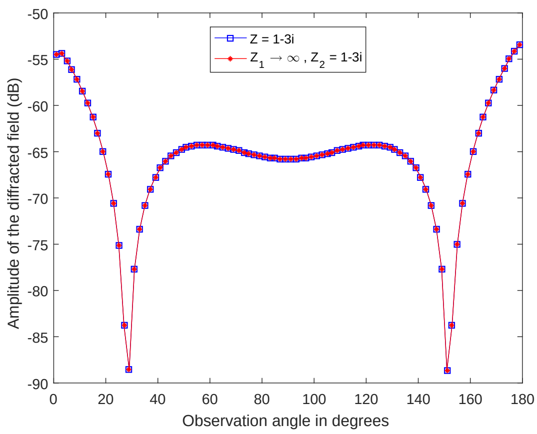

Fig. 8. Comparison of the Fig.1 and Fig.3 for $f = 2500$ Hz.

according to Jordan's Lemma, the integration line Υ_+ can be deformed onto the branch cut $\Upsilon_1 + \Upsilon_2$ through the branch point $\tau = 1$.

$$Q_1^+(u) = \frac{b}{a}\frac{1}{2\pi i}\left[\int\limits_{\Upsilon_1} \frac{H_0^{(2)}(\lambda kb)\, L^-(u)\, e^{-i\tau k(c+l)}}{\lambda k H_1^{(2)}(\lambda ka)(\tau - u)} d\tau + \int\limits_{\Upsilon_2} \frac{H_0^{(2)}(\lambda kb)\, L^-(u)\, e^{-i\tau k(c+l)}}{\lambda k H_1^{(2)}(\lambda ka)(\tau - u)} d\tau\right] \tag{A.2}$$

Using the properties [1]

$$H_0^{(2)}\left(e^{i\pi} z\right) = H_0^{(2)}(z) + 2J_0(z)\,, \quad (-\lambda)\, H_1^{(2)}\left(e^{i\pi} z\right) = \lambda H_1^{(2)}(z) + 2\lambda J_1(z) \tag{A.3}$$

and making the following substitution

$$-1 + \tau = t e^{-i\pi/2} \qquad , \qquad t > 0 \tag{A.4}$$

the integral in (A.2) can be reduced to the following equation

$$Q_1^+(u) = \frac{b}{\pi ka} e^{-ik(c+l)} \int\limits_0^\infty \frac{L^-(1-it)}{1-it-u} P_1(t)\, e^{-tk(c+l)} dt \tag{A.5}$$

$$P_1(t) = \frac{J_1(\lambda ka) H_0^{(2)}(\lambda kb) - J_0(\lambda kb) H_1^{(2)}(\lambda ka)}{\lambda H_1^{(2)}(\lambda ka)\left(H_1^{(2)}(\lambda ka) + 2J_1(\lambda ka)\right)} \tag{A.6}$$

If $k(c+l)$ is large, the main contribution to the integral in (A.5) comes from the end point $t = 0$ [10].

$$Q_1^+(u) = \frac{b}{\pi ka} e^{-ik(c+l)} L^-(1)\, \xi_1(u) \tag{A.7}$$

where

$$\xi_1(u) = \int_0^\infty \frac{P_1(t)\, e^{-tk(c+l)}}{1 - it - u} dt \tag{A.8}$$

By proceeding similarly, we get the following approximate expressions for $Q_2^+(u)$

$$Q_2^+(u) = \frac{b}{\pi a}\left(\frac{i}{Z_1} - \frac{i}{Z_2}\right) e^{-ik(c+l)} M^-(1)\, \xi_2(u) \tag{A.9}$$

where

$$\xi_2(u) = \int_0^\infty \frac{P_2(t)\, e^{-tk(c+l)}}{1 - it - u} dt \tag{A.10}$$

and

$$P_2(t) = \frac{H_0^{(2)}(\lambda kb)\, J(Z_1, 1-it) - J_0(\lambda kb)\, H(Z_1, 1-it)}{H(Z_1, 1-it)\left[H(Z_1, 1-it) + 2J(Z_1, 1-it)\right]} \tag{A.11}$$

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