# FIXED AND COMMON FIXED POINT THEOREMS FOR WARDOWSKI TYPE MAPPINGS IN UNIFORM SPACES

by Muhammad Usman Ali[1], Tayyab Kamran[2], Fahimuddin[3] and Muhammad Anwar[4]

*In this paper we generalize Wardowski type mappings in the setting of uniform spaces. By using these new notions we prove some fixed and common fixed point theorems in uniform spaces. We also provide some examples to show the applicability of our results.*

**Keywords:** $F$-contraction, Uniform Space, $A$-distance, $E$-distance.

**MSC2010:** 47H10, 54H25.

## 1. Introduction

Samet *et al.* [1] introduced the notions of $\alpha$-admissible and $\alpha$-$\psi$-contractive type mappings and proved fixed point theorems on complete metric space by using these notions. The results of Samet *et al.* [1] were generalized by many authors in different settings see for example [1–16]. Wardowski [17] introduced the notion of $F$-contraction which is a nice generalization of the classical contraction condition. He also proved a fixed point theorem for a mapping satisfying the $F$-contraction. This result has been extended in different ways as mentioned in [18–23]. Aamri and Moutawakil [26] introduced the notions of $A$-distance and $E$-distance on uniform spaces and proved some common fixed point theorems on uniform spaces.

The purpose of this paper is to prove some fixed and common fixed point theorems on uniform spaces for mappings satisfying the contraction conditions obtained by refining the ideas of [1] and [17].

Now, we recollect some basic definitions, notions and results which we require subsequently.

Let $X$ be a nonempty set. A nonempty family "$\vartheta$" of subsets of $X \times X$ is called a uniform structure of $X$, if it satisfies the following properties:

(i) if $G$ is in $\vartheta$, then $G$ contains the diagonal $\{(x,x)|x \in X\}$;
(ii) if $G$ is in $\vartheta$ and $H$ is a subset of $X \times X$ which contains $G$, then $H$ is in $\vartheta$;
(iii) if $G$ and $H$ are in $\vartheta$, then $G \cap H$ is in $\vartheta$;
(iv) if $G$ is in $\vartheta$, then there exists $H$ in $\vartheta$, such that, whenever $(x,y)$ and $(y,z)$ are in $H$, then $(x,z)$ is in $G$;
(v) if $G$ is in $\vartheta$, then $\{(y,x)|(x,y) \in G\}$ is also in $\vartheta$.

[1] Department of Mathematics, COMSATS Institute of Information Technology, Attock Pakistan, e-mail: muh_usman_ali@yahoo.com (M. U. Ali)

[2] Department of Mathematics, Quaid-i-Azam University, Islamabad Pakistan, & Department of Mathematics, School of Natural Sciences, National University of Sciences and Technology, Islamabad Pakistan, e-mail: tayyabkamran@gmail.com (T. Kamran)

[3] Center for Advanced Studies in Engineering (CASE), Islamabad Pakistan, e-mail: fahamiiu@gmail.com (Fahimuddin)

[4] Capital University of Science and Technology, Islamabad Pakistan, e-mail: muhammadanwar.imcb@gmail.com(M. Anwar)

(v) if $G$ is in $\vartheta$, then $\{(y,x)|(x,y)\in G\}$ is also in $\vartheta$.

The pair $(X,\vartheta)$ is called a uniform space and the element of $\vartheta$ is called entourage or neighborhood or surrounding. The pair $(X,\vartheta)$ is called a quasiuniform space (see e.g. [24,25] ) if property (v) is omitted.

For a subset $V\in\vartheta$, a pair of points $x$ and $y$ are said to be $V$-close if $(x,y)\in V$ and $(y,x)\in V$. Moreover, a sequence $\{x_n\}$ in $X$ is called a Cauchy sequence for $\vartheta$, if for any $V\in\vartheta$, there exists $N\geq 1$ such that $x_n$ and $x_m$ are $V$-close for $n,m\geq N$. For $(X,\vartheta)$, there is a unique topology $\tau(\vartheta)$ on $X$ generated by $V(x)=\{y\in X|(x,y)\in V\}$ where $V\in\vartheta$.

A sequence $\{x_n\}$ in $X$ is convergent to $x$ for $\vartheta$, denoted by $\lim\limits_{n\to\infty}x_n=x$, if for any $V\in\vartheta$, there exists $n_0\in\mathbb{N}$ such that $x_n\in V(x)$ for every $n\geq n_0$. Let $\Delta=\{(x,x)|x\in X\}$ be the diagonal of $X$. For $V\subseteq X\times X$, we define

$$V^{-1}=\{(x,y)|(y,x)\in V\}.$$

A uniform space $(X,\vartheta)$ is called Hausdorff if the intersection of all the $V\in\vartheta$ is equal to $\Delta$ of $X$, that is, if $(x,y)\in V$ for all $V\in\vartheta$ implies $x=y$. If $V=V^{-1}$ then we say that a subset $V\in\vartheta$ is symmetric. Throughout the paper, we consider that each $V\in\vartheta$ is symmetric. For more details, see e.g. [26–29].

Now, we recall the notions of $A$-distance and $E$-distance.

**Definition 1.1.** *[26,27] Let $(X,\vartheta)$ be a uniform space. A function $p:X\times X\longrightarrow[0,\infty)$ is said to be an $A$-distance if for any $V\in\vartheta$ there exists $\delta>0$ such that if $p(z,x)\leq\delta$ and $p(z,y)\leq\delta$ for some $z\in X$, then $(x,y)\in V$.*

**Definition 1.2.** *[26,27] Let $(X,\vartheta)$ be a uniform space. A function $p:X\times X\longrightarrow[0,\infty)$ is said to be an $E$-distance if*

(i) *$p$ is an $A$-distance,*
(ii) *$p(x,y)\leq p(x,z)+p(z,y),\ \forall x,y,z\in X$.*

**Example 1.1.** *[26,27] Let $(X,\vartheta)$ be a uniform space and let $d$ be a metric on $X$. It is evident that $(X,\vartheta_d)$ is a uniform space where $\vartheta_d$ is a set of all subsets of $X\times X$ containing a "band" $U_\epsilon=\{(x,y)\in X^2|d(x,y)<\epsilon\}$ for some $\epsilon>0$. Moreover, if $\vartheta\subseteq\vartheta_d$, then $d$ is an $E$-distance on $(X,\vartheta)$.*

**Lemma 1.1.** *[26,27] Let $(X,\vartheta)$ be a Hausdorff uniform space and $p$ is an $A$-distance on $X$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$ and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0. Then, for $x,y,z\in X$, the following results hold:*

(a) *If $p(x_n,y)\leq\alpha_n$ and $p(x_n,z)\leq\beta_n$ for all $n\in\mathbb{N}$, then $y=z$. In particular, if $p(x,y)=0$ and $p(x,z)=0$, then $y=z$.*
(b) *If $p(x_n,y_n)\leq\alpha_n$ and $p(x_n,z)\leq\beta_n$ for all $n\in\mathbb{N}$, then $\{y_n\}$ converges to $z$.*
(c) *If $p(x_n,x_m)\leq\alpha_n$ for all $n,m\in\mathbb{N}$ with $m>n$, then $\{x_n\}$ is a Cauchy sequence in $(X,\vartheta)$.*

Let $p$ be an $A$-distance. A sequence in a uniform space $(X,\vartheta)$ with an $A$-distance is said to be a $p$-Cauchy if for every $\epsilon>0$ there exists $n_0\in\mathbb{N}$ such that $p(x_n,x_m)<\epsilon$ for all $n,m\geq n_0$.

**Definition 1.3.** *[26,27] Let $(X,\vartheta)$ be a uniform space and $p$ is an $A$-distance on $X$. Then:*

(i) *$X$ is called $S$-complete if for each $p$-Cauchy sequence $\{x_n\}$, there exists $x\in X$, $\lim_{n\to\infty}p(x_n,x)=0$.*
(ii) *$X$ is called $p$-Cauchy complete if for each $p$-Cauchy sequence $\{x_n\}$, there is $x$ in $X$ with $\lim_{n\to\infty}x_n=x$ with respect to $\tau(\vartheta)$ .*

(iii) $T : X \to X$ *is* $p$*-continuous if* $\lim_{n\to\infty} p(x_n, x) = 0$*, then we have* $\lim_{n\to\infty} p(T(x_n), T(x)) = 0$.

**Remark 1.1.** *[26] Note that if $(X, \vartheta)$ is a Hausdorff uniform space which is $S$-complete, then it is also $p$-Cauchy complete.*

Wardowski [17] introduced the following functions and proved the following fixed point theorem:

**Definition 1.4.** *[17] A function $F: R^+ = (0, \infty) \to R$ is called $F$-function if it satisfies the following conditions:*

- : $(F_1)$ *$F$ is strictly increasing function i.e., $x_1 < x_2$ implies $F(x_1) < F(x_2)$.*
- : $(F_2)$ *For each sequence $\{x_n\}$ of positive real numbers, we have $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} F(x_n) = -\infty$.*
- : $(F_3)$ *For each sequence $\{x_n\}$ of positive real numbers with $\lim_{n\to\infty} x_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} x_n^k F(x_n) = 0$.*

*The family of $F$-functions is denoted by $\mathfrak{F}$.*

**Theorem 1.1.** *[17] Let $(X, d)$ be a complete metric space and let $T: X \to X$ be a $F$-contraction, that is, there exists a function $F \in \mathfrak{F}$ and constant $\tau > 0$ such that*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

*for each $x, y \in X$, whenever $d(Tx, Ty) > 0$. Then $T$ has a unique fixed point.*

Samet *et al*. [1] stated the notion of $\alpha$-admissible mappings in the following way:

**Definition 1.5.** *[1] A mapping $T: X \to X$ is an $\alpha$-admissible if for any $x, y \in X$ with $\alpha(x, y) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$.*

Abdeljawad [7] extended this notion in the following way:

**Definition 1.6.** *[7] A pair of two self mappings $T, S: X \to X$ is an $\alpha$-admissible if for any $x, y \in X$ with $\alpha(x, y) \geq 1$, we have $\alpha(Tx, Sy) \geq 1$ and $\alpha(Sx, Ty) \geq 1$.*

## 2. **Main Results**

We begin this section with the following definition.

**Definition 2.1.** *Let $(X, \vartheta)$ be a uniform space such that $p$ is an $E$-distance on $X$. A mapping $T: X \to X$ is said to be an $(\alpha, F)$-contractive mapping if there exist the functions $\alpha: X \times X \to [0, \infty)$, $F \in \mathfrak{F}$ and constant $\tau > 0$ such that*

$$\tau + F(\alpha(x, y)p(Tx, Ty)) \leq F(p(x, y)), \ \forall x, y \in X \tag{1}$$

*whenever* $\min\{\alpha(x, y)p(Tx, Ty), p(x, y)\} > 0$.

**Theorem 2.1.** *Let $(X, \vartheta)$ be a $S$-complete Hausdorff uniform space such that $p$ is an $E$-distance on $X$. Let $T: X \to X$ be an $(\alpha, F)$-contractive mapping which satisfies the following conditions:*

- : *(i) $T$ is $\alpha$-admissible;*
- : *(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*
- : *(iii) $T$ is $p$-continuous.*

*Then $T$ has a fixed point.*

*Proof.* By hypothesis (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0$, then $x_{n_0}$ is a fixed point of $T$. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n$. As $T$ is $\alpha$-admissible then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$ implies $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$. Inductively, we have

$$\alpha(x_n, x_{n+1}) \geq 1, \ \forall n \in \mathbb{N} \cup \{0\}.$$

Thus from (1), we have

$$\begin{aligned} \tau + F(p(x_n, x_{n+1})) &= \tau + F(p(Tx_{n-1}, Tx_n)) \\ &\leq \tau + F(\alpha(x_{n-1}, x_n)p(Tx_{n-1}, Tx_n)) \\ &\leq F(p(x_{n-1}, x_n)) \ \forall n \in \mathbb{N}. \end{aligned}$$

This inequality yields to the following

$$F(p(x_n, x_{n+1})) \leq F(p(x_0, x_1)) - n\tau \ \forall n \in \mathbb{N}. \tag{2}$$

Letting $n \to \infty$ in the above inequality, we get $\lim_{n\to\infty} F(p(x_n, x_{n+1})) = -\infty$. Thus by using the property $(F_2)$, we get $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. For convenience we denote $p_n = p(x_n, x_{n+1})$ for each $n$. Property $(F_3)$ implies that there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} p_n^k F(p_n) = 0$. From (2), we have

$$p_n^k F(p_n) - p_n^k F(p_0) \leq -np_n^k \tau.$$

Letting $n \to \infty$ in the above inequality, we have $\lim_{n\to\infty} np_n^k = 0$. Then there exists $n_0 \in \mathbb{N}$ such that $np_n^k \leq 1$ for all $n \geq n_0$. Thus, we have $p_n \leq \frac{1}{n^{1/k}}$. As $p$ is $E$-distance, then by using triangular inequality for $m > n > n_0$ we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots + p(x_{m-1}, x_m) \\ &= \sum_{i=n}^{\infty} p_i - \sum_{i=m}^{\infty} p_i \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}} - \sum_{i=m}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Letting $n, m \to \infty$ in the above inequality we get $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Since $p$ is not symmetric, by using the assumption $\alpha(Tx_0, x_0) \geq 1$ and hypothesis of the theorem in the similar manner as mentioned above we get $\lim_{n\to\infty} p(x_m, x_n) = 0$. Therefore, $\{x_n\}$ is a $p$-Cauchy sequence in $X$. By $S$-completeness of $X$, we have $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = 0$. Further by hypothesis (iii), we have $\lim_{n\to\infty} p(Tx_n, Tu) = 0$, that is, $\lim_{n\to\infty} p(x_{n+1}, Tu) = 0$. Hence, we have $\lim_{n\to\infty} p(x_n, u) = 0$ and $\lim_{n\to\infty}(x_n, Tu) = 0$. Thus by Lemma 1.1-(a), we have $Tu = u$. □

In next theorem, $p$-continuity of the mapping is replaced by another condition which is imposed on the space.

**Theorem 2.2.** *Let $(X, \vartheta)$ be a $S$-complete Hausdorff uniform space such that $p$ is an $E$-distance on $X$. Let $T \colon X \to X$ be an $(\alpha, F)$-contractive mapping which satisfies the following conditions:*

- : *(i) $T$ is $\alpha$-admissible;*
- : *(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*
- : *(iii) for any sequence $\{x_n\}$ in $X$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

*Then $T$ has a fixed point.*

*Proof.* From the proof of Theorem 2.1, we know that $\{x_n\}$ is a $p$-Cauchy sequence in $X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Further, there exists $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = 0$. By hypothesis (iii), we have $\alpha(x_n, u) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Thus by using (1) and triangular inequality of $p$, we have

$$\begin{aligned} p(x_n, Tu) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, Tu) \\ &\leq p(x_n, x_{n+1}) + \alpha(x_n, u)p(Tx_n, Tu) \\ &< p(x_n, x_{n+1}) + p(x_n, u). \end{aligned}$$

Letting $n \to \infty$ in the above inequality we get $p(x_n, Tu) = 0$. Hence, we have $\lim_{n\to\infty} p(x_n, u) = 0$ and $\lim_{n\to\infty} p(x_n, Tu) = 0$. Thus by Lemma 1.1-(a), we have $Tu = u$. □

**Example 2.1.** *Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ be endowed with the usual metric $p$. Define $\vartheta = \{U_\epsilon : \epsilon > 0\}$. It can be seen that $(X, \vartheta)$ is a uniform space. Define $T : X \to X$ as*

$$Tx = \begin{cases} 0 \text{ if } x = 0 \\ \frac{1}{3n+2} \text{ if } x = \frac{1}{n} : n > 1 \\ 1 \text{ if } x = 1, \end{cases}$$

*and $\alpha : X \times X \to [0, \infty)$ as*

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x, y \in X - \{1\} \\ 0 \text{ otherwise.} \end{cases} .$$

*It is easy to see that $T$ is $(\alpha, F)$-contractive mapping with $F(x) = \ln x$ for each $x > 0$ and $\tau = 1$. For $x_0 = \frac{1}{2}$, we have $\alpha(x_0, Tx_0) = \alpha(Tx_0, x_0) = 1$. Further, for any sequence $\{x_n\}$ in $X$ with $x_n \to x$ and $\alpha(x_{n-1}, x_n) = 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) = 1$ for all $n \in \mathbb{N}$. Thus, by Theorem 2.2, we say that $T$ has a fixed point.*

To find out the uniqueness of fixed point, we consider the following condition:

**:** (J) For all $x, y \in Fix(T)$ we have $z \in X$ such that $\alpha(z, x) \geq 1$ and $\alpha(z, y) \geq 1$,

where, $Fix(T)$ represents the set of all fixed points of $T$. The following theorem guarantees the uniqueness of fixed point.

**Theorem 2.3.** *If we add the condition $(J)$ in the hypothesis of Theorem 2.1 (and Theorem 2.2), we get the uniqueness of fixed point of $T$.*

*Proof.* On contrary suppose that $u$ and $v$ are two distinct fixed points of $T$. From the condition $(J)$, we have $z \in X$ such that

$$\alpha(z, u) \geq 1 \text{ and } \alpha(z, v) \geq 1. \tag{3}$$

Since $T$ is $\alpha$-admissible, thus we get

$$\alpha(T^n z, u) \geq 1 \text{ and } \alpha(T^n z, v) \geq 1, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{4}$$

We define the sequence $\{z_n\}$ in $X$ by $z_{n+1} = Tz_n = T^n z_0$ for all $n \in \mathbb{N} \cup \{0\}$ and $z_0 = z$. Thus from (1), we get

$$\tau + F(p(z_{n+1}, u)) \leq \tau + F(\alpha(z_n, u)p(Tz_n, Tu)) \leq F(p(z_n, u)) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Iteratively we get the following

$$F(p(z_n, u)) \leq F(p(z_0, u)) - n\tau \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality, we get $\lim_{n\to\infty} F(p(z_n, u)) = -\infty$. By using Property $(F_2)$ we reach $\lim_{n\to\infty} p(z_n, u) = 0$. Similarly, we have $\lim_{n\to\infty} p(z_n, v) = 0$. Thus, by Lemma 1.1-(a), we get $u = v$. □

In the following definition we introduce the notion of $(\alpha, F)$-contractive pair for self mappings:

**Definition 2.2.** *Let $(X, \vartheta)$ be a uniform space such that $p$ is an $E$-distance on $X$. A pair of two self mappings $T, S\colon X \to X$ is an $(\alpha, F)$-contractive pair if there exist the functions $\alpha\colon X \times X \to [0, \infty)$, $F \in \mathfrak{F}$ and constant $\tau > 0$ such that*

$$\tau + F(\alpha(x, y) \max\{p(Tx, Sy), p(Sx, Ty)\}) \leq F(p(x, y)) \tag{5}$$

*for all $x, y \in X$ whenever* $\max\{\alpha(x, y) \max\{p(Tx, Sy), p(Sx, Ty)\}, p(x, y)\} > 0$.

With the help of above notion we prove the following common fixed point theorem.

**Theorem 2.4.** *Let $(X, \vartheta)$ be a $S$-complete Hausdorff uniform space such that $p$ is an $E$-distance on $X$. Let a pair of two self mappings $T, S\colon X \to X$ be an $(\alpha, F)$-contractive pair which satisfies the following conditions:*

**:** *(i) $(T, S)$ is $\alpha$-admissible;*
**:** *(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*
**:** *(iii) for any sequence $\{x_n\}$ in $X$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, then $\alpha(x_n, x) \geq 1$ for each $n \in N \cup \{0\}$.*

*Then $T$ and $S$ have a common fixed point.*

*Proof.* By hypothesis $(ii)$ we have $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$. As $(T, S)$ is an $\alpha$-admissible pair, we construct a sequence $\{x_n\}$ in $X$ such that $Tx_{2n} = x_{2n+1}, Sx_{2n+1} = x_{2n+2}$ and $\alpha(x_n, x_{n+1}) \geq 1$, $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. From (5), we get

$$\begin{aligned} \tau + F(p(x_{2n+1}, x_{2n+2})) &= \tau + F(p(Tx_{2n}, Sx_{2n+1})) \\ &\leq \tau + F(\alpha(x_{2n}, x_{2n+1}) \times \\ &\quad \max\{p(Tx_{2n}, Sx_{2n+1}), p(Sx_{2n}, Tx_{2n+1})\}) \\ &\leq F(p(x_{2n}, x_{2n+1})) \ \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{6}$$

Likewise, we get the following

$$\begin{aligned} \tau + F(p(x_{2n+2}, x_{2n+3})) &= \tau + F(p(Sx_{2n+1}, Tx_{2n+2})) \\ &\leq \tau + F(\alpha(x_{2n+1}, x_{2n+2}) \times \\ &\quad \max\{p(Tx_{2n+1}, Sx_{2n+2}), p(Sx_{2n+1}, Tx_{2n+2})\}) \\ &\leq F(p(x_{2n+1}, x_{2n+2})) \ \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{7}$$

Thus from (6) and (7), we get

$$\tau + F(p(x_{n+1}, x_{n+2})) \leq F(p(x_n, x_{n+1})) \ \forall n \in \mathbb{N} \cup \{0\}.$$

This inequality yields the following

$$F(p(x_n, x_{n+1})) \leq F(p(x_0, x_1)) - n\tau \text{ for all } n \in \mathbb{N}. \tag{8}$$

Following the details given in proof of Theorem 2.1, we conclude that $\{x_n\}$ is a $p$-Cauchy sequence in $X$. By $S$-completeness of $X$, we have $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = 0$, which implies $\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1} = u$. By hypothesis (iii), we get $\alpha(x_n, u) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Thus, by using (5) and triangular inequality of $p$, we have

$$\begin{aligned} p(x_n, Tu) &\leq p(x_n, x_{2n+2}) + p(x_{2n+2}, Tu) \\ &= p(x_n, x_{2n+2}) + p(Sx_{2n+1}, Tu) \\ &\leq p(x_n, x_{2n+2}) + \alpha(x_{2n+1}, u) \max\{p(Tx_{2n+1}, Su), p(Sx_{2n+1}, Tu)\}) \\ &< p(x_n, x_{2n+2}) + p(x_{2n+1}, u). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get $\lim_{n\to\infty} p(x_n, Tu) = 0$. Further, we already have $\lim_{n\to\infty} p(x_n, u) = 0$. Thus by Lemma 1.1-(a), we have $Tu = u$. Analogously, we prove that $Su = u$. Hence, $u$ is a common fixed point of $T$ and $S$. □

**Remark 2.1.** *Note that Theorem 2.4 is valid if we replace condition (ii) with the condition stated below:*
*There exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$ and $\alpha(Sx_0, x_0) \geq 1$.*

We can proof the following theorem on the same lines as the proofs of previous theorems are done.

**Theorem 2.5.** *Let $(X, \vartheta)$ be a $S$-complete Hausdorff uniform space such that $p$ is an $E$-distance on $X$. Let $T\colon X \to X$ be a mapping for which there exist the functions $\alpha : X \times X \to [0, \infty)$, $F \in \mathfrak{F}$ and constant $\tau > 0$ satisfying the following condition:*

$$\tau + F(\alpha(x, y) \max\{p(Tx, y), p(x, Ty)\}) \leq F(p(x, y))$$

*for each $x, y \in X$, whenever $\max\{\alpha(x, y) \max\{p(Tx, y), p(x, Ty)\}, p(x, y)\} > 0$. Further assume that the following conditions hold:*

- **:** *(i) $T$ is $\alpha$-admissible;*
- **:** *(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*
- **:** *(iii) for any sequence $\{x_n\}$ in $X$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

*Then $T$ has a fixed point.*

**Example 2.2.** *Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ be endowed with the usual metric $p$. Define $\vartheta = \{U_\epsilon : \epsilon > 0\}$. It can be seen that $(X, \vartheta)$ is a uniform space. Define $T : X \to X$ as*

$$Tx = \begin{cases} 0 \ \textit{if } x = 0 \\ \frac{1}{4n+1} \ \textit{if } x = \frac{1}{n} : n > 1 \\ 1 \ \textit{if } x = 1, \end{cases}$$

*and $S : X \to X$ as*

$$Sx = \begin{cases} 0 \ \textit{if } x = 0 \\ \frac{1}{4n+1} \ \textit{if } x = \frac{1}{n} : n > 1 \\ 0 \ \textit{if } x = 1, \end{cases}$$

*and $\alpha : X \times X \to [0, \infty)$ as*

$$\alpha(x, y) = \begin{cases} 1 \ \textit{if } x, y \in X - \{1\} \\ 0 \ \textit{otherwise.} \end{cases}$$

*It is easy to see that $(T, S)$ is $(\alpha, F)$-contractive pair with $F(x) = \ln x$ for each $x > 0$ and $\tau = 1$. For $x_0 = \frac{1}{2}$, we have $\alpha(x_0, Tx_0) = \alpha(Tx_0, x_0) = 1$. Further, for any sequence $\{x_n\}$ in $X$ with $x_n \to x$ and $\alpha(x_{n-1}, x_n) = 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) = 1$ for all $n \in \mathbb{N}$. Thus, by Theorem 2.4, we say that $T$ and $S$ have a common fixed point.*

To prove the uniqueness of common fixed point of mappings, we use the following condition:

- **:** (I) For each $x, y \in CFix(T, S)$ we have $\alpha(x, y) \geq 1$;

where $CFix(T, S)$ is the set of all common fixed points of $T$ and $S$.

**Theorem 2.6.** *By including condition (I) in the hypothesis of Theorem 2.4, we obtain the uniqueness of common fixed point of $T$ and $S$.*

*Proof.* Suppose on contrary that $u, v \in X$ are two different common fixed points of $T$ and $S$. By condition (I), we have $\alpha(u, v) \geq 1$. From (5), we have

$$\begin{aligned} \tau + F(p(u,v)) &\leq \tau + F(\alpha(u,v)\max\{p(Tu,Sv), p(Su,Tv)\}) \\ &\leq F(p(u,v)) \end{aligned}$$

which is not possible for $p(u, v) > 0$. Thus, we have $p(u, v) = 0$. Further, we get $p(u, u) = 0$. Therefore, Lemma 1.1-(a) implies that $u = v$. This is contradiction to our supposition. Hence $T$ and $S$ have a unique common fixed point. □

The following results are immediately follow from our results by taking $\alpha(x, y) = 1$ for each $x, y \in X$.

**Corollary 2.1.** *Let $(X, \vartheta)$ be a $S$-complete Hausdorff uniform space such that $p$ is an E-distance on $X$. Let $T : X \to X$ be a mapping for which there exists a function $F \in \mathfrak{F}$ and constant $\tau > 0$ satisfying the following condition:*

$$\tau + F(p(Tx, Ty)) \leq F(p(x, y))$$

*for each $x, y \in X$, whenever $p(Tx, Ty) > 0$. Then $T$ has a unique fixed point.*

**Corollary 2.2.** *Let $(X, \vartheta)$ be a $S$-complete Hausdorff uniform space such that $p$ is an E-distance on $X$. Let $T, S : X \to X$ be two mappings for which there exists a function $F \in \mathfrak{F}$ and constant $\tau > 0$ satisfying the following condition:*

$$\tau + F(\max\{p(Tx, Sy), p(Sx, Ty)\}) \leq F(p(x, y))$$

*for each $x, y \in X$, whenever $\max\{\max\{p(Tx, Sy), p(Sx, Ty)\}, p(x, y)\} > 0$. Then $T$ and $S$ have a unique common fixed point.*

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