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BINARY RELATIONS – ADDENDA 1 (KERNEL, RESTRICTIONS AND INDUCING, RELATIONAL MORPHISMS)

Mihai REBENCIUC[1]

Această lucrare (în două părţi) conţine unele completări la teoria relaţiilor binare într-un cadru extins prin operaţii categoriale generalizate – relativ la categoria neregulată a relaţiilor binare $\mathcal{Rel}$.asociată categoriei regulate $\mathcal{Set}$. Primele două completări din această parte a lucrării se referă la nucleul, respectiv la restricţiile şi indusa în mulţimi arbitrare a unei relaţii binare – în legătură cu operaţii de algebră Booleană şi categoriale generalizate Ultima completare constă într-o ierarhie de morfisme relaţionale în paralel în cazurile omogen şi neomogen la care se raportează şi noţiunea de (bi)simulare (generalizată pentru cazul neomogen) – esenţială în programarea concurentă .

This paper(in two parts) contains some addenda to binary relations theory in a background witch is extended by generalized categorical operations – relative to the unregulated category of binary relations $\mathcal{Rel}$ associated with the regulate category $\mathcal{Set}$. The first two addenda from this part of the paper refer to the kernel, respectively to the restrictions and the induced relation in arbitrary sets of a binary relation - in connection with Boolean algebra operations and generalized categorical operations. The last addendum consists in a hierarchy of relational morphisms in parallel in homogeneous and inhomogeneous cases to witch the notion of (bi)simulation (generalized for inhomogeneous case) is reported - important in concurrency programming.

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1. Introduction

The concept of multivocity is illustrated in a naturally way by the notion of binary relation in the regulate category of sets $\mathcal{Set}$ – in witch the binary relations form the unregulated category $\mathcal{Rel}$ in regard to the categorical operation of composition and the binary relations between the same sets have a structure of complete Boolean algebra – with the known properties [1], [2], [3], [4]; in a topoi $\mathcal{E}$ it is maintained the corresponding category $\mathcal{Rel}_{\mathcal{E}}$ – but the structure of a

[1] Lecturer, Department of Mathematics III, University "Politehnica" of Bucharest, ROMANIA , e-mail: m_rebenciuc@yahoo.com

complete Boolean algebra it is replaced by the structure of Heyting algebra [5] and in a category that in the finite case is more generally then a topoi (wellpowered and wellcopowered category with products, coproducts and finite intersections) remains just the structure of complete lattice - with a conditionally distributivity of the composition in regard to union [6], [7].

The first addendum from this part of the paper is relates to the functional type – by defining the kernel of a relation and the determination of some properties as a special homogeneous relation (of tolerance and equivalence) respectively of connection with generalized categorical operations.

In the second addendum, we associate the concept of multivocity with the one of partiality - with a unification of functional and order approach by defining in the same generalized way of the notions of restrictions and induced binary relation in arbitrary sets, followed by the study of Boolean algebra and generalized categorical operations.

These generalisations - relative to categorical operations and restrictions and inducing in arbitrary sets are categorical validated in categories with intersections and unions of “objects” [8], [9].

The last addendum refers to a hierarchy of relational morphisms with a parallel between homogeneous and inhomogeneous cases - to witch the notion of (bi)simulation (generalized for inhomogeneous case) is reported ; the notion of (bi)simulation – that induces the notion of (bi)similarity is important in concurrency programming [10], [11].

We close up with an example relative to ones of the above generalizations.

Example 1.1 (operations with subtotal and subdiagonal relations) Relative to the set $\mathcal{Rel}(A, B)$ of binary relations between A, B we consider the set $\mathcal{Rel}_{st}(A, B)$ $= \{\omega_{A', B'} = A' \times B' \,/\, A' \in \mathcal{P}(A),\ B' \in \mathcal{P}(B)\} \subset \mathcal{Rel}(A, B)$ of the subtotal relations with $\omega_{A, B}$ the total relation and for $A' \in \mathcal{P}^*(A)$, $B' \in \mathcal{P}^*(B)$ with $\omega_{A', \varnothing} = \omega_{\varnothing, B'} = \omega_{\varnothing, \varnothing} = \varnothing$ the vide relation in $\mathcal{Rel}(A, B)$; τ_1: $\mathcal{P}(A) \times \{\bullet\} \rightarrow \mathcal{Rel}_{st}(A, \bullet)$, $\tau_1(A', \bullet) = \omega_{A', \bullet}$ (and analogously τ_2) is complete isomorphism of Boolean algebras , but τ: $\mathcal{P}(A) \times \mathcal{P}(B) \rightarrow \mathcal{Rel}_{st}(A, B)$, $\tau(A', B') = \omega_{A', B'}$ remains only morphism of inferior semilattices (and partial order semiembedding) with $\mathcal{P}(A) \times \mathcal{P}(B)$ complete Boolean algebra and with $\mathcal{Rel}_{st}(A, B)$ bounded inferior semilattice by Ø, $\omega_{A, B}$. Consequently $\mathcal{Rel}_{st}(A, \bullet)$, $\mathcal{Rel}_{st}(\bullet, B)$ are complete Boolean subalgebras of complete Boolean algebra $\mathcal{Rel}(A, B)$ - through the medium of inferior semilattice $\mathcal{Rel}_{st}(A, B)$; analogously in homogeneous case $\mathcal{Rel}(A) = \mathcal{Rel}(A, A)$ - in addition with $\mathcal{Rel}_{sd}(A) = \{\Delta_X = \{(x, x)/x \in X\} / X \in \mathcal{P}(A)\} \subset \mathcal{Rel}(A)$ (the set of subdiagonal relation in A with Δ_A the diagonal relation in A) complete sublattice of $\mathcal{Rel}(A)$ according to complete lattice isomorphism δ: $\mathcal{P}(A) \to \mathcal{Rel}_{sd}(A)$, $\delta(A')=\Delta_{A'}$. $\mathcal{Rel}_{tot}$ corresponding to the sets $\mathcal{Rel}_{tot}(A, B) = \{\omega_{A,B}\}$, $\mathcal{Rel}_{tot}(A) = \{\omega_A\}$ is preordered subcategory of $\mathcal{Rel}$ with $(\omega_{A,B})^{-1} = \omega_{B,A}$; categorical operation of composition " $\circ$" (graphically omitted) is generalized by $\omega_{C,D}\omega_{A,B} = \varnothing$ for $B \cap C = \varnothing$, respectively $\omega_{C,D}\omega_{A,B} = \omega_{A,D}$ for $B \cap C \neq \varnothing$. If $\mathcal{Rel}(A, B)$ is non-strictly, i.e. $O = A \cap B \neq \varnothing$ [12], then $\mathcal{Rel}(A, B)$ is semigroup relative to composition (Δ_A, Δ_B the neutral elements to the right , respectively to the left) which $\mathcal{Rel}_{st}(A, B)$ subsemigroup . In homogeneous case $\mathcal{Rel}(A)$ is monoid with $\mathcal{Rel}_{sd}(A)$ submonoid because $\Delta_Y\Delta_X = \Delta_X\Delta_Y = \Delta_{X\cap Y}$; in addition we have $\Delta_{X\times Y} = \Delta_X \times \Delta_Y$.

2. Kernel

Definition 2.1 ((co)kernel of a relation). The kernel of the relation $R \in \mathcal{Rel}(A, B)$ is the homogeneous relation of $\mathcal{Rel}(A)$, noted ker R and defined by $\ker R = R^{-1}R$; dually $co\ker R = RR^{-1} \in \mathcal{Rel}(B)$ is the cokernel of R.

Observation 2.1.i (duality) We have $co\ker R = \ker R^{-1}$, $\ker R = co\ker R^{-1}$ because $co\ker R = RR^{-1} = (R^{-1}R)^{-1} = \ker R^{-1}$ and analogously for the other equality.

ii (symmetry) Ker R is symmetric relation because $(\ker R)^{-1} = (R^{-1}R)^{-1} = R^{-1}R = \ker R$ and analogously for cokernel.

iii(tolerance) Ker R is D-tolerance relation, D = dom(R) = subfield(ker R) (= dom(ker R)$\cap$codom(ker R), where dom(ker R) = codom(ker R) = dom(R)) because ker R is reflexive in any $a \in D$ - there is $b \in B$ such that $(a,b) \in R$ and hence $(a,a) \in \ker R$ and according to ii; analogously(but dually) coker R is C-tolerance relation, C = codom(R) = subfield(coker R). Particularly, if R is left-total (dom(R) = A), respectively right-total (codom(R) = B), then ker R, respectively coker R are tolerance relations.

iv (equivalence) If in addition $R \in \mathcal{Rel}(A, B)$ is dysfunctional [13], i. e. $R = RR^{-1}R$ ($RR^{-1}R \subseteq R$ is a sufficient condition) – which is named also i-regulate or preunivocal(for unification of terminology in the category $\mathcal{Rel}$ [14]), then ker R is D-equivalence and coker R is C-equivalence; in the case R left-total, respectively right-total these became equivalences. Indeed we have $(\ker R)^2 = (R^{-1}R)(R^{-1}R) = R^{-1}(RR^{-1}R) = R^{-1}R = \ker R$ and analogously for cokernel.

Theorem 2.1(inclusion, categorical operations) Let be $R \in \mathcal{Rel}(A, B)$; for $S \in \mathcal{Rel}(A, B)$, $S \subseteq R$ implies (co)ker $S \subseteq$ (co)ker R. For $S \in \mathcal{Rel}(C, D)$ we have the equalities ker (R, S) = ker $R \cap$ ker S, coker (R, S) = coker $(R \times S)$, (co)ker $(R \times S)$ = (co)ker $R \times$ (co)ker S.

Proof . Relative to inclusion we have $S^{-1} \subseteq R^{-1}$, $\ker S = S^{-1}S \subseteq S^{-1}R \subseteq R^{-1}R = \ker R$ and analogously for cokernel. Relative to categorical operations we have $\ker(R \times S) = (R \times S)^{-1}(R \times S) = (R^{-1} \times S^{-1})(R \times S) = (R^{-1}R) \times (S^{-1}S) = \ker R \times \ker S$ and analogously for cokernel. In addition, we have $(a, a') \in \ker(R, S)$ iff there exists $(b, d) \in B \times D$ such that $(a, (b, d) \in (R, S)$, $((b, d), a') \in (R, S)^{-1}$ iff $(a, b), (a', b) \in R, (a, d), (a', d) \in S$ iff $(a, a') \in$ ker $R \cap$ ker S, respectively $((b, d), (b', d')) \in co\ker(R, S)$ iff there exists $a \in A \cap C$ such that $((b, d), a) \in (R, S)^{-1}, (a, (b', d')) \in (R, S)$ iff $(a, b), (a, b') \in R$, $(a, d), (a, d') \in S$ iff $(b, b') \in co\ker R, (d, d') \in$ coker S iff $((b, d), (b', d')) \in co\ker R \times co\ker S$.

Definition 2.2(w-composability) The relations $R \in \mathcal{Rel}(A, B)$, $S \in \mathcal{Rel}(C, D)$ are weak composable – for short w-composable if there exist composable pairs of R, S, i. e. $.SR \neq \varnothing$. Particularly, R is weak self-composable – for short w-self-composable if $R^2 = RR \neq \varnothing$; more generally, for $n \in IN^* \setminus \{1\}$ (implicitly n = 2) R is n-weak self-composable – for short nw-self-composable if $R^n \neq \varnothing$.

Observation 2.2.i (sufficient condition) A relation is w-self-composable if it is non-banal transitive.

ii (monotony) For $m, n \in IN^* \setminus \{1\}$ if $m < n$, then R n-w-self-composable implies R m-w-self-composable.

iii (the w-self-composability of the (co)kernel) Ker R(and analogously coker R) is w-self-composable because whichever of the inclusion $R \subseteq RR^{-1}R$ or

$\Delta_D \subseteq \ker R$, $D = \mathrm{dom}(R)$ (see the points iii and iv of observation 2.1) imply the inclusion $\ker R \subseteq (\ker R)^2$ with $\ker R \neq \varnothing$.

3. Restrictions and inducing

Definition 3.1 ((co)restriction, induced relation) Let $R \in \mathcal{Rel}(A, B)$ be a relation and let X, Y be arbitrary sets; the restriction of R to X and dually, the corestriction of R to Y and the relation which is induced by R in X, Y are respectively the relations $R|_X = R \cap \omega_{X,B}$, ${}_Y|R = R \cap \omega_{A,Y}$, $R_{X,Y} = R \cap \omega_{X,Y} \in \mathcal{Rel}(A, B)$.

Observation 3.1 (nuances, terminology) More exactly we have $R|_X \in \mathcal{Rel}(A \cap X, B)$, ${}_Y|R \in \mathcal{Rel}(A, B \cap Y)$ (which are named also the left restriction of R to X, respectively the right restriction of R to Y) and $R_{X,Y} \in \mathcal{Rel}(A \cap X, B \cap Y)$; R is the extension of $R|_X$, ${}_Y|R$ and $R_{X,Y}$ respectively to A, B and A, B. For $R \in \mathcal{Rel}(A)$ $R_X = R_{X,X}$ is the relation which is induced by R in X.

Theorem 3.1 (connections) Let $R \in \mathcal{Rel}(A, B)$ be a relation and let X, Y be arbitrary sets. We have the equalities $R|_X = R\Delta_X$, ${}_Y|R = \Delta_Y R$, ${}_Y|(R|_X) = ({}_Y|R)|_X$ $(= {}_Y|R|_X$ - "associativity"), $R_{X,Y} = {}_Y|R|_X = R|_X \cap {}_Y|R = \Delta_Y R \Delta_X$.

Proof. The first two equalities, the last equality and the equalities of the "associativity" follow at once (by definition or according to the last equality for the equalities of the "associativity") – and imply the equality $R_{X,Y} = {}_Y|R|_X$. Finally, we have $R|_X \cap {}_Y|R = (R \cap \omega_{X,B}) \cap (R \cap \omega_{A,Y}) = R \cap \omega_{X,B} \cap \omega_{A,Y} = {}_Y|R|_X = R_{X,Y}$.

Example 3.1 (restrictions and induced relations of the subtotal and subdiagonal relations) Relative to the arbitrary sets X, Y we have the following restrictions and induced relations of the relations $\omega_{U,V} \in \mathcal{Rel}_{\mathrm{st}}(A, B)$, $\Delta_U \in \mathcal{Rel}_{\mathrm{sd}}(A)$:

$\omega_{U,V}|_X = \omega_{U,V} \cap \omega_{X,B} = \omega_{U \cap X, V} = \omega_{U,V}\Delta_X$, ${}_Y|\omega_{U,V} = \omega_{U,V} \cap \omega_{A,Y} = \omega_{U, V \cap Y} = \Delta_Y \omega_{U,V}$, $(\omega_{U,V})_{X,Y} = \omega_{U,V}|_X \cap {}_Y|\omega_{U,V} = \omega_{U \cap X, V} \cap \omega_{U, V \cap Y} = \omega_{U \cap X, V \cap Y} = \Delta_Y \omega_{U,V} \Delta_X$ - particularly for $X \in \mathcal{P}^*(A)$, $Y \in \mathcal{P}^*(B)$ $\omega_{A,B}|_X = \omega_{X,B}$, ${}_Y|\omega_{A,B} = \omega_{A,Y}$, $(\omega_{A,B})_{X,Y} = \omega_{X,Y}$, respectively $(\Delta_U)_X = \Delta_X \Delta_U \Delta_X = \Delta_{U \cap X}$ - particularly for $X \in \mathcal{P}^*(A)$ $(\Delta_A)_X = \Delta_X$. In addition, for $R \in \mathcal{Rel}(A, B)$ we have $R \setminus R_{X,Y} = R \setminus \omega_{X,Y}$

Theorem 3.2 (relations and operation of Boolean algebra) Let R, S$\in$ $\mathcal{Rel}(A, B)$ be relations and let X, X', Y, Y' be arbitrary sets. i (inclusion preserving) $X' \subseteq X$ imply $R|_{X'} \subseteq R|_X$, $S \subseteq R$ imply $S|_X \subseteq R|_X$ - and analogously it is the inclusion preserving relative to corestriction, $X' \subseteq X$, $Y' \subseteq Y$ imply $R_{X',Y'} \subseteq R_{X,Y}$, $S \subseteq R$ imply $S_{X,Y} \subseteq R_{X,Y}$;

ii (union preserving) $R|_{U\cup X} = R|_U \cup R|_X$, $(R \cup S)|_X = R|_X \cup S_X$ - and analogously it is the union preserving relative to corestriction, $R_{U\cup X, V\cup Y} = R_{U,V} \cup R_{U,Y} \cup R_{X,V} \cup R_{X,Y}$, $(R\cup S)_{X,Y} = R_{X,Y} \cup S_{X,Y}$;

iii (behaviour towards intersection) $R|_{U\cap X} = (R|_U)_X = (R|_X)_U$, $(R \cap S)|_X = R|_X \cap S|_X$ - and analogously for corestriction, $R_{U\cap X, V\cap Y} = (R_{U,V})_{X,Y} = (R_{X,Y})_{U,V} = (R_{U,Y})_{X,V} = (R_{X,V})_{U,Y}$, $(R\cap S)_{X,Y} = R_{X,Y} \cap S_{X,Y}$.

Proof. i. It is easy – for example by making use of the expressions of the restrictions and of the induced relation with subdiagonal relations (as operands of the composition – see theorem 3.1) and by inclusion preserving by composition.

ii. We have (see theorem 3.1) $R|_{U\cup X} = R\Delta_{U\cup X} = R(\Delta_U \cup \Delta_X) = R\Delta_U \cup R\Delta_X = R|_U \cup R|_X$, $(R\cup S)|_X = (R\cup S)\Delta_X = R\Delta_X \cup S\Delta_X = R|_X \cup S|_X$ (or by definition and according to the example 1.1) - and analogously for the preservation of the union relative to corestriction, $R_{U\cup X, V\cup Y} = {}_{V\cup Y}|(R|_{U\cup X}) = {}_{V\cup Y}|(R|_U \cup R|_X) = {}_{V\cup Y}|(R|_U) \cup {}_{V\cup Y}|(R|_X) = {}_V|R|_U \cup {}_Y|R|_U \cup {}_V|R|_X \cup {}_Y|R|_X = R_{U,V} \cup R_{U,Y} \cup R_{X,V} \cup R_{X,Y}$, $(R\cup S)_{X,Y} = {}_Y|((R\cup S)|_X) = {}_Y|(R|_X \cup S|_X) = {}_Y|R|_X \cup {}_Y|S|_X = R_{X,Y} \cup S_{X,Y}$.

iii. We have (by definition) $R|_{U\cap X} = R \cap \omega_{U\cap X, B} = R \cap (\omega_{U,B} \cap \omega_{X,B}) = (R \cap \omega_{U,B}) \cap \omega_{X,B} = (R|_U)|_X$ - and analogously for the other expression and for corestriction, $R_{U\cap X, V\cap Y} = R \cap \omega_{U\cap X, V\cap Y} = (R \cap \omega_{U,V}) \cap \omega_{X,Y} = (R_{U,V})_{X,Y}$ - and analogously for the other expressions, $(R\cap S)_{X,Y} = (R\cap \omega_{X,Y}) \cap (S \cap \omega_{X,Y}) = R_{X,Y} \cap S_{X,Y}$.

Observation 3.2 (the taking of the restrictions and of the induced relation as morphisms) The taking of the restriction ρ :$\mathcal{P}$(A) × $\mathcal{Rel}$(A, B) → $\mathcal{Rel}$(A, B), χ : $\mathcal{Rel}$(A, B)×$\mathcal{P}$(B)→ $\mathcal{Rel}$(A, B), respectively of the induced relation ι : $\mathcal{P}$(A) × $\mathcal{Rel}$(A, B) ×$\mathcal{P}$(B)→ $\mathcal{Rel}$(A, B) are order morphisms and lattice morphisms in the relation argument; they only are the superior semilattice morphisms in the set argument – respectively in an set argument.

Theorem 3.3 (generalized categorical operations) Let $R\in$ $\mathcal{Rel}$(A, B), $S\in\mathcal{Rel}$(C, D) be relations and let U, X, Y, Z be arbitrary sets. i (inversion) $(R|_X)^{-1}={}_X|R^{-1}$ - and analogously for corestriction, $(R_{X,Y})^{-1}=(R^{-1})_{Y,X}$;

ii (composition) $(SR)_X=S(R|_X)$ - and analogously for corestriction, $(SR)_{X,Y}=({}_Y|S)(R|_X)$;

iii (other operations) $(R,S)|_X=(R|_X,S|_X)$, ${}_{Y\times Z}|(R,S)=({}_Y|R,\ {}_Z|S)$, $(R,S)_{X,Y\times Z}=(R_{X,Y},S_{X,Z})$, $(R\times S)|_{U\times X}=(R|_U)\times(S|_X)$ - and analogously for corestriction, $(R\times S)_{U\times X,Y\times Z}=R_{U,Y}\times S_{X,Z}$.

Proof. i. We have (see theorem 3.1) $(R|_X)^{-1}=(R\Delta_X)^{-1}=\Delta_X R^{-1}={}_X|R^{-1}$ - and analogously for corestriction, $(R_{X,Y})^{-1}=({}_Y|(R|_X))^{-1}=(R|_X)^{-1}|_Y=$ $({}_X|R^{-1})|_Y=(R^{-1})_{Y,X}$.

ii. We have successively (see theorem 3.1) $(SR)|_X=(SR)\Delta_X=S(R\Delta_X)=$ $S(R|_X)$ - and analogously for corestriction, $(SR)_{X,Y}={}_Y|((SR)|_X)={}_Y|(S(R|_X))=$ $({}_Y|S)(R|_X$.

iii. We have successively (see theorem 3.1, example 1.1 and [4]) $(R,S)|_X=(R,S)\Delta_X=(R\Delta_X,S\Delta_X)=(R|_X,S|_X)$, ${}_{Y\times Z}|(R,S)=\Delta_{Y\times Z}(R,S)=$ $(\Delta_Y\times\Delta_Z)(R,S)=(\Delta_Y R,\Delta_Z S)=({}_Y|R,\ {}_Z|S)$, $(R,S)_{X,Y\times Z}={}_{Y\times Z}|((R,S)|_X)=$ ${}_{Y\times Z}|(R|_X,S|_X)=({}_Y|R|_X,\ {}_Z|S|_X)=(R_{X,Y},S_{X,Z})$, respectively $(R\times S)|_{U\times X}=$ $(R\times S)\Delta_{U\times X}=(R\times S)(\Delta_U\times\Delta_X)=(R\Delta_U)\times(S\Delta_X)=(R|_U)\times(S|_X)$ - and analogously for corestriction, $(R\times S)_{U\times X,Y\times Z}={}_{Y\times Z}|((R\times S)|_{U\times X})=$ ${}_{Y\times Z}|((R|_U)\times(S|_X))=({}_Y|R|_U)\times({}_Z|S|_X)=R_{U,Y}\times S_{X,Z}$.

4. Relational morphisms

Definition 4.1 (r-(bi)morphism) The inhomogeneous relation $F \in \mathcal{Rel}(A, B)$ is relational morphism – for short r-morphism between the homogeneous relational structures $(A, R_A), (B, R_B)$ if it is compatible with R_A, R_B, i. e. for each $a, a' \in A, b, b' \in B$ with $(a, b), (a', b') \in F$ (or equivalently with $((a, a'), (b, b')) \in F^2 = F \times F$), $(a, a') \in R_A$ implies $(b, b') \in R_B$; F is relational bimorphism(relational semiembedding) for short r-bimorphism(r-semiembedding) if F, F^{-1} are r-morphisms.

Observation 4.1 (partial – but non-banal compatibility) The compatibility with R_B from the r-morphism condition can be partially – but it is totally non-banal because $a, a' \in \mathrm{dom}\ (F)$; other two distinct conditions with $(a', b') \in F$, respectively $(a, b), (a', b') \in F$ after implication lead to two partially non-banal variants.

Definition 4.2 (variants) The inhomogeneous relation $F \in \mathcal{Rel}(A, B)$ is r′-morphism between the homogeneous relational structures $(A, R_A), (B, R_B)$ if for each $a, a' \in A, b \in B$ with $(a, b) \in F, (a, a') \in R_A$ implies for each $b' \in B$, $(a', b') \in F$, $(b, b') \in R_B$; F is r″-morphism if for each $a, a' \in A, (a, a') \in R_A$ imply for each $b, b' \in B, (a, b), (a', b') \in F, (b, b') \in R_B$. F is r′-bimorphism (r′-semiembedding) if F, F^{-1} are r′-morphisms – and analogously for r″-bimorphism (r″-semiembedding).

Theorem 4.1 Let be $(A, R_A), (B, R_B)$, $F \in \mathcal{Rel}(A, B)$. i (connections) The condition of r-bimorphism is equivalent with the condition of r-morphism with equivalence (instead of implication). The condition of F r′-morphism with equivalence imply F^{-1} r′-morphism between $(A, (R_A)^{-1}), (B, (R_B)^{-1})$; the condition of F r″-morphism with equivalence imply F^{-1} r-morphism.

ii (hierarchy) We have the implications F r-morphism imply F r′-morphism imply F r″-morphism – with equivalences if dom(F) $\supseteq$ field (R_A); analogously for the relational bimorphisms – with the equivalences condition in hierarchy completed with codom(F) $\supseteq$ field (R_B).

Proof. i. The statements follow at once (by definition).

ii. We have $r(a, a') \to r'(a, a') \to r''(a, a')$, where F r-morphism iff $\forall a, a' \in A, r(a, a')$, F r′-morphism iff $\forall a, a' \in A, r'(a, a')$, F r″-morphism iff $\forall a, a' \in A, r''(a, a')$, $r(a, a') = t(a) \wedge t'(a') \wedge (p(a, a') \to q)$, $r'(a, a') = t(a) \wedge$

$(p(a, a') \to (t'(a') \wedge q)) \approx t(a) \wedge (p(a, a') \to t'(a')) \wedge (p(a, a') \to q)$, $r''(a, a') =$ $p(a, a'') \to (t(a) \wedge t'(a, a') \wedge q) \approx (p(a, a') \to t(a)) \wedge (p(a, a') \to t'(a')) \wedge$ $(p(a, a') \to q)$, $p(a, a') = (a, a') \in R_A$, $q = \forall b, b' \in B, (b, b') \in R_B$, $t(a) = \forall b \in B, (a, b) \in F$, $t'(a') = \forall b' \in B, (a', b') \in F$.
The conditioned equivalences by dom (*F*) $\supseteq$ field (R_A) are consequences of the above implications and of the total non-banality of the implication from the condition of r"-morphism because for $q(b, b') = (b, b') \in R_B$, $r(a, a') = \forall b, b' \in$ codom(*F*), $p(a, a') \to q(b, b')$, $r''(a, a') = p(a, a') \to \forall b, b' \in$ codom(F), $q(b, b')$ we have $r(a, a') \approx r''(a, a')$.
In the case of the relational bimorphisms the above results are valid for the inverse relation $F^{-1} \in \mathcal{R}el(B, A)$ – with codom(*F*) = dom(F^{-1}) $\supseteq$ field (R_B).

Observation 4.2.i (categorical composition) The categorical composite of two (w-composable) r-morphisms is r-morphism; analogously for the other relational morphisms and for the relational bimorphisms – w-composed under the conditions of the equivalences in relational bimorphisms hierarchy.

ii (the case of the left-total and right-total relations) The left-total relations satisfy the equivalences condition in relational morphisms hierarchy.; the left-total and right-total relations satisfy the equivalences condition in relational bimorphisms hierarchy.

iii (duality) The condition $(b, b') \in (R_B)^{-1}$ (instead of $(b, b') \in R_B$) lead to dual r-(bi)morphism – with invariant, respectively partial variant composite towards dualizing.

iv (the inhomogeneous case – vs. the homogeneous case) In the inhomogeneous case relative to the inhomogeneous relational structures $(A, A', R_{A, A'})$, $(B, B', R_{B, B'})$ a (inhomogeneous) r-morphism is a ordered pair $(F, F') \in \mathcal{R}el(A, B) \times \mathcal{R}el(A', B')$ (with $(F, F') \approx F \times F' \in \mathcal{R}el(A \times A', B \times B')$) which satisfies the compatibility condition with $R_{A, A'}$, $R_{B, B'}$, i. e. for each $a \in A, a' \in A', b \in B, b' \in B'$ with $(a, b) \in F$, $(a', b') \in F'$ (or equivalently with $((a, a'), (b, b')) \in F \times F'$), $(a, a') \in R_{A, A'}$ implies $(b, b') \in R_{B, B'}$; in fact, in homogeneous case a r-morphism is a singlet $\{F\} \approx (F, F)$, noted F which satisfies the compatibility condition. The other relational morphisms and the relational bimorphisms can be defined similarly. In addition in the inhomogeneous case are valid theorem 4.1 and points i-iii, where $(F, F')^{-1} = (F^{-1}, (F')^{-1}), (G, G')(F, F') = (GF, G'F')$, (co)dom (*F*, *F'*) = ((co)dom(*F*), (co)dom(*F'*)).

v (bi)simulation) A simulation $S \in \mathcal{Rel}(A, B)$ between the homogeneous relational structures (A, R_A), (B, R_B) is defined by a non-banal existential variant of the compatibility condition with R_A, R_B of a r′-morphism, i. e. for each $a, a' \in A, b \in B$ with $(a, b) \in S, (a, a') \in R_A$ implies there exists $b' \in B$, $(a', b') \in S$, $(b, b') \in R_B$ - which is equivalently with the usual condition(see [10], [11]) "for each $a \in A, b \in B$ with $(a, b) \in S$ and for each $a' \in A, (a, a') \in R_A$ implies there exists $b' \in B$, $(a', b') \in S$, $(b, b') \in R_B$"; but it is weaker than the r-morphism condition which is conditioned equivalently with the r′-morphism condition(see theorem 4.1 and point i) – with $(a', b') \in S$. There is analogously for bisimulation vs. r′-bimorphism, respectively r-bimorphism – with the mention of the non-equivalence between the bisimulation condition and the simulation condition with equivalence instead of implication.

Definition 4.3 ((bi)simulation in the inhomogeneous case) A simulation between the inhomogeneous relational structures $(A, A', R_{A, A'})$, $(B, B', R_{B, B'})$ is a ordered pair $(S, S') \in \mathcal{Rel}(A, B) \times \mathcal{Rel}(A', B')$ (with $(S, S') \approx S \times S' \in \mathcal{Rel}(A \times A', B \times B')$) which satisfies the non-banal existential variant of the compatibility condition with $R_{A, A'}$, $R_{B, B'}$, i. e. for each $a \in A, a' \in A', b \in B$ with $(a, b) \in S, (a, a') \in R_{A, A'}$ implies there exists $b' \in B$, $(a', b') \in S'$, $(b, b') \in R_{B, B'}$; (S, S') is bisimulation if it is simulation – along with $(S, S')^{-1}$.

Observation 4.3.i (equivalence) The compatibility condition is equivalently with the condition "for each $a \in A, b \in B$ with $(a, b) \in S$ and for each $a' \in A', (a, a') \in R_{A, A'}$ implies there exists $b' \in B'$, $(a', b') \in S'$, $(b, b') \in R_{B, B'}$" – a inhomogeneous analogue of the usual condition in homogeneous case (see observation 4.2.v).

ii (two-way similarity) In addition can be defined similarity, bisimilarity, two-way similarity respectively $\underset{\sim}{<}$, $\underset{\sim}{\leftrightarrow}$, $\underset{\sim}{-} \in \mathcal{Rel}(A, B)$ by $a \underset{\sim}{<} b$ iff there exists $(S, S') \in \mathcal{Rel}(A, B) \times \mathcal{Rel}(A', B')$ simulation with $(a, b) \in S$, $a \underset{\sim}{\leftrightarrow} b$ iff there exists $(S, S') \in \mathcal{Rel}(A, B) \times \mathcal{Rel}(A', B')$ bisimulation with $(a, b) \in S$, $a \underset{\sim}{-} b$ iff $a \underset{\sim}{<} b$, $b \underset{\sim}{<} a$, i. e. there exists $(S, S') \in \mathcal{Rel}(A, B) \times \mathcal{Rel}(A', B')$, $(T, T') \in \mathcal{Rel}(B, A) \times \mathcal{Rel}(B', A')$ simulations with $(a, b) \in S$, $(b, a) \in T$ - with the inclusion $\underset{\sim}{\leftrightarrow} \subseteq \underset{\sim}{-}$.

iii (strict inclusion) Generally, the inclusion is strictly (see next example), hence we have not equality – possibly we have only conditioned equality.

Example 4.1 (counterexample) For $A = \{a\}$, $A' = \{a', a''\}$, $\dot{A}' = \{a''\}$, $B = \{b\}$, $B' = \{b'\}$, $R_{A, A'} = \omega_{A, A'}$, $R_{B, B'} = \omega_{B, B'}$ and the simulations $(S, S') \in \mathcal{R}el(A, B) \times \mathcal{R}el(A', B')$, $S = \omega_{A, B}$, $S' = \omega_{A', B'}$, $(T, T') \in \mathcal{R}el(B, A) \times \mathcal{R}el(B', A')$, $T = \omega_{B, A}$, $T' = \omega_{B', \dot{A}'}$ we have $a \underset{\sim}{-} b$ and non $a \underset{\sim}{\leftrightarrow} b$.

5. Conclusions

In the first two addenda of this part of the paper we define the notions of kernel and restriction (which are dualized), respectively the notion of induced relation, where the last two are in arbitrary sets - in connection with Boolean algebra operations and generalized categorical operations (see the theorems 2.1, 3.1, 3.2, 3.3 and the observation 3.2); so that we have done a unification of functional and order approach and more generally an association of the multivocity and partiality concept – existent in some domains of the theoretical computer science. These generalizations (relative to categorical operations, restrictions and inducing in sets) are categorical valid [8], [9].

The last addendum refers to a hierarchy of relational morphisms in parallel in homogeneous and inhomogeneous cases - with equivalence conditions (see the theorem 4.1 and the observation 4.2.iv); the notion of (bi)simulation (generalized for the inhomogeneous case) is reported to this hierarchy and it is important in concurrency programming [10], [11] along with the notion of (bi)similarity which it induces.

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