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THE NON-HOMOGENEOUS MULTIVARIATE GREY MODEL NMGM (1, N) AND ITS APPLICATION

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There is higher bias on fitting approximation non-homogenous series for building the traditional MGM (1, n) model by fitting data with homogenous exponential function. In fact, there is a lot of approximation non-homogenous series. Based on the modeling principle of the traditional MGM (1, n) model, a non-homogenous multivariate grey model NMGM (1, n) was put forward. The parameters were estimated of the proposed model by least square method and the time respond function was given. Two kinds of optimization models were established: one is taking the coefficient of the background value as design variable and the minimum average relative error as the objective function, the other is taking the coefficient of the background value and the initial value of the response function as the design variables and the minimum average relative error as the objective function. The solution program based on Matlab was written. Finally, the example validates new optimization model has better fitting and prediction accuracy than the traditional MGM (1,n) model.

Keywords: Grey system theory; non- homogenous; multivariate; optimization; NMGM (1, n).

1. Introduction

Grey system model is new method researching the uncertainty problem about a small amount of data and poor information. GM (1,1) is the most commonly grey system model, which reveals the inherent development law by first-order differential equation model with single variable [1]. GM (1,1) with single variable was extended to the multivariate grey model MGM (1, n) [2]. MGM (1,n) is neither the simple combination of GM (1, n), nor GM (1, n) that establishing only a first-order differential equation with n variables. In MGM (1,n), n differential equations with n variables are established and solved, in order that the parameters in the model can reflect the relationships of mutual influence and restriction among multiple variables. Many scholars and practitioners have extensively and profoundly researched from theory to application, and successfully resolved a large number of practical problems in the production, life and scientific research [3-9].

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Model accuracy is the key and the difficulty of modeling. Many scholars put forward many methods to improve the accuracy, such as the reconstruction of background value [6], the coefficient optimization of background value [7], the optimization on initial value [7-8] and new information optimization model [9]. To some degree, the above methods improve the prediction accuracy of the model, but they do not fundamentally eliminate errors due to the defects caused by modeling method. The analysis found that most of the optimizations are for the parameters of the grey model, that is, that constantly correcting the original model better fits the data sequence with the pure exponential characteristics. But in real life, the data complete with pure exponential characteristic is minimal, and more of the original sequence data meet approximation non-homogeneous exponential law. Xie N.M. put forward the discrete grey model based on approximate non-homogeneous exponential sequence and researched the model features to achieve certain results [10]. Cui J. constructed a kind of grey model based on approximate non-homogeneous exponential discrete function according to the classical modeling mechanism of grey model [11]. But this model is very strict with the data, and when the data does not meet the requirement the error between the fitting and the prediction is too large. Wang Y.N. put forward the direct modeling method on approximate non-homogeneous exponential sequence [12]. It has a good effect when the original data is monotonous rise-fall or concavo-convex in this model, but in practice the raw data is not necessarily monotone. The non-homogeneous exponential grey model based on equidistant sequence was established and the desired effect was achieved [13,14]. But this is for GM (1,1) with a single variable, and the non-homogeneous multivariate MGM (1, n) model has not seen in published papers. In this paper, according to the modeling mechanism of the traditional grey model, firstly the raw data was accumulated to increase exponential law and reduce the randomness, and then a non-homogeneous multivariate grey model NMGM (1, n) was built to fit the raw data with non-homogeneous exponential law. New model can fit and forecast for any multiple sets of data with non-homogeneous exponential law, and also for approximated homogeneous exponential data. The parameters were estimated of the proposed model by least square method and the time respond function was given. Two kinds of optimization models were established: one is taking the coefficient of the background value as design variable and the minimum average relative error as the objective function, the other is taking the coefficient of the background value and the initial value of the response function as the design variables and the minimum average relative error as the objective function. The solution program based on Matlab was written. The example validates new optimization model has better fitting and prediction accuracy than the traditional MGM (1,n) model.

2. Modeling mechanism of multivariate grey model MGM (1, n)

Supposed the non-negative sequence $X_i^{(0)} = [x_i^{(0)}(1), \cdots, x_i^{(0)}(j), \cdots, x_i^{(0)}(m)]$, where $i = 1,2,\cdots,n$, $j = 2,\cdots,m$, n is the number of variables and m is the sequence number of each variable, $X_i^{(1)} = [x_i^{(1)}(1), \cdots, x_i^{(1)}(j), \cdots, x_i^{(1)}(m)]$ is first-order accumulated generation of $\mathbf{X}_i^{(0)}$, and it is denoted by 1-AGO, where

$$x_i^{(1)}(k) = \sum_{j=1}^{k} x^{(0)}(j)(j = 1,2,\cdots,m] \tag{1}$$

Let multivariate raw data matrix is:

$$X^{(0)} = \left\{X_1^{(0)} \quad X_2^{(0)} \quad \cdots \quad X_n^{(0)}\right\}^{\mathbf{T}} = \begin{bmatrix} x_1^0(1) & x_1^0(2) & x_1^0(m) \\ x_2^0(1) & x_2^0(2) & x_2^0(m) \\ \cdots & & \\ x_n^0(1) & x_n^0(2) & x_n^0(m) \end{bmatrix} \tag{2}$$

The equations of MGM (1, n) are first-order albino differential equations with n variables.

$$\begin{cases} \dfrac{dx_1^{(1)}}{dt} = a_{11}x_1^{(1)} + a_{12}x_2^{(1)} + \cdots + a_{1n}x_n^{(1)} + b_1 \\ \dfrac{dx_2^{(1)}}{dt} = a_{21}x_1^{(1)} + a_{22}x_2^{(1)} + \cdots + a_{2n}x_n^{(1)} + b_2 \\ \cdots \\ \dfrac{dx_n^{(1)}}{dt} = a_{n1}x_1^{(1)} + a_{n2}x_2^{(1)} + \cdots + a_{nn}x_n^{(1)} + b_n \end{cases} \tag{3}$$

Supposed

$X^{(0)}(k) = (x_1^{(0)}(k), x_2^{(0)}(k), \cdots, x_n^{(0)}(k))^{\mathbf{T}}$ and $X^{(1)}(k) = (x_1^{(1)}(k), x_2^{(1)}(k), \cdots, x_n^{(1)}(k))^{\mathbf{T}}$,

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$, Eq.(3) can be expressed as:

$$\frac{dX^{(1)}}{dt} = AX^{(1)} + B \tag{4}$$

Taken the first component $x_i^{(1)}(1)$ of the sequence $x_i^{(1)}(j)(j = 1,2,\cdots,m)$ as an initial condition of the grey differential equation, the continuous time response of Eq.(4) is as:

$$X^{(1)}(t) = e^{At}X^{(1)}(1) + A^{-1}(e^{At} - I)B \tag{5}$$

Where， $e^{At}=I+\sum_{k=1}^{\infty}\frac{A^k}{k!}t^k, I$ is an unit matrix.

With known matrix A and time t, we can easily calculate exponent matrix e^{At} without appearing singular matrix with function expm.m in Matlab software.

In order to identify $\mathbf{A}$ and B, Eq. (3) is discretized and $X^{(0)}=AZ^{(1)}+B$ can be obtained through the difference grey derivative in $[k-1,k]$. Taken $z_i^{(1)}(k)=0.5*(x_i^{(1)}(k)+x_i^{(1)}(k-1))$, the following equation can be obtained:

$$x_i^{(0)}(k)=\sum_{j=1}^{n}\frac{a_{ij}}{2}(x_j^{(1)}(k)+x_j^{(1)}(k-1))+b_i(i=1,2,\cdots,n;k=2,3,\cdots,m) \quad (6)$$

Supposed $a_i=(a_{i1},a_{i2},\cdots,a_{in},b_i)^{\mathrm{T}}(i=1,2,\cdots,n)$， the identified value $\hat{\mathbf{a}}_i$ of a_i can be obtained through the least square method:

$$\hat{a}_i=(\hat{a}_{i1},\hat{a}_{i2},\cdots,\hat{a}_{in},\hat{b}_i)^{\mathrm{T}}=(Z^{\mathrm{T}}Z)^{-1}Z^{\mathrm{T}}Y_i, i=1,2,\cdots,n \quad (7)$$

Where,

$$Z=\begin{bmatrix} \frac{1}{2}(x_1^{(1)}(2)+x_1^{(1)}(1)) & \frac{1}{2}(x_2^{(1)}(2)+x_2^{(1)}(1)) & \cdots & \frac{1}{2}(x_n^{(1)}(2)+x_n^{(1)}(1)) & 1 \\ \frac{1}{2}(x_1^{(1)}(3)+x_1^{(1)}(2)) & \frac{1}{2}(x_2^{(1)}(3)+x_2^{(1)}(2)) & \cdots & \frac{1}{2}(x_n^{(1)}(3)+x_n^{(1)}(2)) & 1 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ \frac{1}{2}(x_1^{(1)}(m)+x_1^{(1)}(m-1)) & \frac{1}{2}(x_2^{(1)}(m)+x_2^{(1)}(m-1)) & \cdots & \frac{1}{2}(x_n^{(1)}(m)+x_n^{(1)}(m-1)) & 1 \end{bmatrix} \quad (8)$$

$$Y_i=[x_i^{(0)}(2),x_i^{(0)}(3),\cdots,x_i^{(0)}(m)]^{\mathrm{T}} \quad (9)$$

Then the identified values of A and B can be get:

$$\hat{A}=\begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{a}_{n1} & \hat{a}_{n2} & \cdots & \hat{a}_{nn} \end{bmatrix}, \quad B=\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \cdots \\ \hat{b}_n \end{bmatrix} \quad (10)$$

The calculated value in MGM (1, n) is:

$$\hat{X}_i^{(1)}(k)=e^{\hat{\mathbf{A}}(k-1)}X_i^{(1)}(1)+\hat{A}^{-1}(e^{\hat{A}(k-1)}-I)\hat{B}, k=1,2,\cdots,m \quad (11)$$

$\hat{X}_i^{(1)}$ is restored to the original sequence $\hat{X}_i^{(0)}$.

$$\hat{X}_i^{(0)}(k)=\begin{cases} X_i^{(1)}(1) & k=1 \\ (X_i^{(1)}(1)+\hat{A}^{-1}\hat{B})(e^{\hat{A}k}-e^{\hat{A}(k-1)}) & k=1,2,\cdots,m \end{cases} \quad (12)$$

The absolute error of the ith variable:

$$q_i(k)=\hat{x}_i^{(0)}(k)-x_i^{(0)}(k) \quad (13)$$

The relative error of the ith variable (%):

$$e_i(k)=\frac{\hat{x}_i^{(0)}(k)-x_i^{(0)}(k)}{x_i^{(0)}(k)} \tag{14}$$

The mean of the relative error of the ith variable:

$$\frac{1}{m}\sum_{k=1}^{m}\left|e_i(k)\right| \tag{15}$$

The average error of all the data:

$$f=\frac{1}{nm}\sum_{i=1}^{n}\left(\sum_{k=1}^{m}\left|e_i(k)\right|\right) \tag{16}$$

The background value of the above model is generated by the mean, denoted as MGM-1. After amended and taken advantage of known information, the background value can be get:

$$Z=\begin{bmatrix} \lambda x_1^{(1)}(2)+(1-\lambda)x_1^{(1)}(1) & \lambda x_2^{(1)}(2)+(1-\lambda)x_2^{(1)}(1) & \cdots & \lambda x_n^{(1)}(2)+(1-\lambda)x_n^{(1)}(1) & 1 \\ \lambda x_1^{(1)}(3)+(1-\lambda)x_1^{(1)}(2) & \lambda x_2^{(1)}(3)+(1-\lambda)x_2^{(1)}(2) & \cdots & \lambda x_n^{(1)}(3)+(1-\lambda)x_n^{(1)}(2) & 1 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ \lambda x_1^{(1)}(m)+(1-\lambda)x_1^{(1)}(m-1) & \lambda x_2^{(1)}(m)+(1-\lambda)x_2^{(1)}(m-1) & \cdots & \lambda x_n^{(1)}(m)+(1-\lambda)x_n^{(1)}(m-1) & 1 \end{bmatrix} \tag{17}$$

where $\lambda\in[0,1]$ [7,8].

That Eq.(17) substituting Eq.(8) can obtain λ through the optimization method. The MGM model is referred to as MGM-2.

In MGM-1, the first column of data is taken as the initial value of the solution $x_i^{(1)}(1)=x_i^{(0)}(1)$. After it is amended, that $x_i^{(1)}(1)+\beta$ takes the place of $x_i^{(1)}(1)$, where β is a vector whose dimension is equal to $x_i^{(1)}(1)$, that is, $\beta=\left[\beta_1,\beta_2,\cdots,\beta_n\right]^T$. Eq.(11) is changed as:

$$\hat{X}_i^{(1)}(k)=e^{\hat{\mathbf{A}}(k-1)}(X_i^{(1)}(1)+\beta)+\hat{A}^{-1}(e^{\hat{\mathbf{A}}(k-1)}-I)\hat{B},k=1,2,\cdots,m \tag{18}$$

It is restored to obtain the fitting value of the original sequence.

$$\hat{X}_i^{(0)}(k)=\begin{cases} X_i^{(1)}(1)+\beta & k=1 \\ (X_i^{(1)}(1)+\beta+\hat{A}^{-1}\hat{B})(e^{\hat{A}k}-e^{\hat{A}(k-1)}) & k=1,2,\cdots,m \end{cases} \tag{19}$$

That Eq.(17) substituting Eq.(8) and Eq.(18) substituting Eq.(11) can obtain λ and β by using the optimization method. The MGM model is referred to as MGM-3.

After analyzing Eq.(12), it is found $\hat{X}_i^{(0)}$ has homogeneous exponent characteristic. Because the collected data in the practical application are often approximate non-homogeneous, the non-homogeneous exponent sequence is used to fit the original data in this paper.

3. Modeling mechanism of the non-homogeneous multivariate grey model NMGM (1, n)

Supposed the non-negative sequence $X_i^{(0)} = [x_i^{(0)}(1),\cdots,x_i^{(0)}(j),\cdots,x_i^{(0)}(m)]$, where $i = 1,2,\cdots,n$, $j = 2,\cdots,m$, n is the number of variables and m is the sequence number of each variable, $X_i^{(1)} = [x_i^{(1)}(1),\cdots,x_i^{(1)}(j),\cdots,x_i^{(1)}(m)]$ is called as is one-time accumulated generation of $X_i^{(0)}$, denoted as 1-AGO. Supposed that $Z^{(1)}(t)$ is the background value of $X^{(1)}(t)$, $X^{(0)}(t) = AZ^{(1)}(t) + B_1 + B_2 t$ is defined as the differential equation of NMGM (1, n) under the optimization on the grey action.

$$\begin{cases} \dfrac{dx_1^{(1)}}{dt} = a_{11}x_1^{(1)} + a_{12}x_2^{(1)} + \cdots + a_{1n}x_n^{(1)} + b_1 + b_{21}t \\ \dfrac{dx_2^{(1)}}{dt} = a_{21}x_1^{(1)} + a_{22}x_2^{(1)} + \cdots + a_{2n}x_n^{(1)} + b_2 + b_{22}t \\ \cdots \\ \dfrac{dx_n^{(1)}}{dt} = a_{n1}x_1^{(1)} + a_{n2}x_2^{(1)} + \cdots + a_{nn}x_n^{(1)} + b_n + b_{2n}t \end{cases} \tag{20}$$

Noting $X^{(0)}(k) = (x_1^{(0)}(k), x_2^{(0)}(k),\cdots,x_n^{(0)}(k))^{\mathrm{T}}$,

$X^{(1)}(k) = (x_1^{(1)}(k), x_2^{(1)}(k),\cdots,x_n^{(1)}(k))^{\mathrm{T}}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $B_1 = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$,

$B_2 = \begin{bmatrix} b_{11} \\ b_{22} \\ \cdots \\ b_{nn} \end{bmatrix}$ Eq.(20) can be expressed as:

$$\frac{dX^{(1)}}{dt} = AX^{(1)} + B_1 + B_2 t \tag{21}$$

Taken the first component $x_i^{(1)}(1)$ of the sequence $x_i^{(1)}(j)(j = 1,2,\cdots,m)$ as an initial condition of the grey differential equation, that is $x_i^{(1)}(1) = x_i^{(0)}(1)$, the continuous time response of Eq.(5) is as:

$$X^{(1)}(t) = (X^{(1)}(1) + A^{-1}B_1 + A^{-1}B_2 + A^{-2}B_2)e^{A(t-1)} - A^{-1}B_1 - A^{-1}B_2 t - A^{-2}B_2 \tag{22}$$

where, $e^{At} = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} t^k$, where I is an unit matrix.

In order to identify A, B_1 and B_2, Eq.(20) is discreted and the following equation can be obtained:

$X^{(0)}(t) = AZ^{(1)}(t) + B_1 + B_2 t$,taken $z_i^{(1)}(k) = 0.5*(x_i^{(1)}(k) + x_i^{(1)}(k-1))$,

$$x_i^{(0)}(k) = \sum_{j=1}^{n} \frac{a_{ij}}{2}(x_j^{(1)}(k) + x_j^{(1)}(k-1)) + b_{1i} + b_{2i}t(i = 1,2,\cdots,n; k = 2,3,\cdots,m) \quad (23)$$

The definite integral is taken on both sides of the equation in [k-1,k]:

$$x_i^{(0)}(k)(k-(k-1)) = (k-(k-1))\sum_{j=1}^{n} \frac{a_{ij}}{2}(x_j^{(1)}(k) + x_j^{(1)}(k-1)) + (k-(k-1))b_{1i} + b_{2i}(k\char`^2 - (k-1)\char`^2)/2, (i = 1,2,\cdots,n; k = 2,3,\cdots,m)$$

that is:

$$x_i^{(0)}(k) = \sum_{j=1}^{n} \frac{a_{ij}}{2}(x_j^{(1)}(k) + x_j^{(1)}(k-1)) + b_{1i} + b_{2i}(2k-1)/2, (i = 1,2,\cdots,n; k = 2,3,\cdots,m)$$

Noting $a_i = (a_{i1}, a_{i2}, \cdots, a_{in}, b_{1i}, b_{2i})^{\mathrm{T}} (i = 1,2,\cdots,n)$, the identified value $\hat{\mathbf{a}}_i$ of a_i can be obtained through the least square method:

$$\hat{a}_i = (\hat{a}_{i1}, \hat{a}_{i2}, \cdots, \hat{a}_{in}, \hat{b}_{1i}, b_{2i})^{\mathrm{T}} = (Z^{\mathrm{T}}Z)^{-1}Z^{\mathrm{T}}Y_i, i = 1,2,\cdots,n \quad (24)$$

where:

$$Z = \begin{bmatrix} \frac{1}{2}(x_1^{(1)}(2) + x_1^{(1)}(1)) & \frac{1}{2}(x_2^{(1)}(2) + x_2^{(1)}(1)) & \cdots & \frac{1}{2}(x_n^{(1)}(2) + x_n^{(1)}(1)) & 1 & 3/2 \\ \frac{1}{2}(x_1^{(1)}(3) + x_1^{(1)}(2)) & \frac{1}{2}(x_2^{(1)}(3) + x_2^{(1)}(2)) & \cdots & \frac{1}{2}(x_n^{(1)}(3) + x_n^{(1)}(2)) & 1 & 5/2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2}(x_1^{(1)}(m) + x_1^{(1)}(m-1)) & \frac{1}{2}(x_2^{(1)}(m) + x_2^{(1)}(m-1)) & \cdots & \frac{1}{2}(x_n^{(1)}(m) + x_n^{(1)}(m-1)) & 1 & (2m-1)/2 \end{bmatrix} \quad (25)$$

$$Y_i = [x_i^{(0)}(2), x_i^{(0)}(3), \cdots, x_i^{(0)}(m)]^{\mathrm{T}} \quad (26)$$

Then the identified values of A, B_1 and B_2 can be get:

$$\hat{A} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{a}_{n1} & \hat{a}_{n2} & \cdots & \hat{a}_{nn} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \hat{b}_{11} \\ \hat{b}_{12} \\ \cdots \\ \hat{b}_{1n} \end{bmatrix}, \quad B_2 = \begin{bmatrix} \hat{b}_{21} \\ \hat{b}_{22} \\ \cdots \\ \hat{b}_{2n} \end{bmatrix} \quad (27)$$

The calculated value in NMGM(1,n) is:

$$\hat{X}^{(1)}(k) = (X^{(1)}(1) + A^{-1}B_1 + A^{-1}B_2 + A^{-2}B_2)e^{\mathrm{A}(k-1)} - A^{-1}B_1 - A^{-1}B_2 k - A^{-2}B_2 \quad (28)$$

$\hat{X}_i^{(1)}$ is restored to the original sequence $\hat{X}_i^{(0)}$.

$$\hat{X}^{(0)}(k) = \begin{cases} X^{(1)}(1) & k = 1 \\ (X^{(1)}(1) + \hat{A}^{-1}\hat{B}_1 + \hat{A}^{-1}B_2 + \hat{A}^{-2}\hat{B}_2)e^{\hat{A}(k-1)} - \hat{A}^{-1}\hat{B}_1 - \hat{A}^{-1}\hat{B}_2 k - \hat{A}^{-2}\hat{B}_2, & k = 2,3,\cdots,m \end{cases} \quad (29)$$

After analyzing the above equation, it is found $\hat{X}_i^{(0)}$ has non-homogeneous exponent characteristic. There, the above MGM (1, n) is referred to NMGM(1,n), where N represents non-homogeneous exponent.

The absolute error of the ith variable:

$$q_i(k)=\hat{x}_i^{(0)}(k)-x_i^{(0)}(k) \tag{30}$$

The relative error of the ith variable (%):

$$e_i(k)=\frac{\hat{x}_i^{(0)}(k)-x_i^{(0)}(k)}{x_i^{(0)}(k)} \tag{31}$$

The mean of the relative error of the ith variable:

$$\text{MAPLE(i)}=\frac{1}{m}\sum_{k=1}^{m}\left|e_i(k)\right|\times 100 \tag{32}$$

The average error of all the data:

$$f=\frac{1}{nm}\sum_{i=1}^{n}\left(\sum_{k=1}^{m}\left|e_i(k)\right|\right)\times 100 \tag{33}$$

The background value of the above model is generated by the mean, denoted as NMGM-1. After amended and taken advantage of known information, the background value can be get:

$$\mathbf{Z}=\begin{bmatrix} \lambda x_1^{(1)}(2)+(1-\lambda)x_1^{(1)}(1) & \lambda x_2^{(1)}(2)+(1-\lambda)x_2^{(1)}(1) & \cdots & \lambda x_n^{(1)}(2)+(1-\lambda)x_n^{(1)}(1) & 1 & 3/2 \\ \lambda x_1^{(1)}(3)+(1-\lambda)x_1^{(1)}(2) & \lambda x_2^{(1)}(3)+(1-\lambda)x_2^{(1)}(2) & \cdots & \lambda x_n^{(1)}(3)+(1-\lambda)x_n^{(1)}(2) & 1 & 5/2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda x_1^{(1)}(m)+(1-\lambda)x_1^{(1)}(m-1) & \lambda x_2^{(1)}(m)+(1-\lambda)x_2^{(1)}(m-1) & \cdots & \lambda x_n^{(1)}(m)+(1-\lambda)x_n^{(1)}(m-1) & 1 & (2m-1)/2 \end{bmatrix} \tag{34}$$

where, $\lambda\in[0,1]$ [7,8].

That Eq.(34) substituting Eq.(25) can obtain λ through the optimization method. The NMGM model is referred to as NMGM-2.

In NMGM-1, the first column of data is taken as the initial value of the solution $x_i^{(1)}(1)=x_i^{(0)}(1)$. After it is amended, that $x_i^{(1)}(1)+\beta$ takes the place of $x_i^{(1)}(1)$, where β is a vector whose dimension is equal to $x_i^{(1)}(1)$, that is, $\beta=\left[\beta_1,\beta_2,\cdots,\beta_n\right]^T$. Eq.(29) is changed as:

$$\hat{X}^{(1)}(k)=(X^{(1)}(1)+\beta+A^{-1}B_1+A^{-1}B_2+A^{-2}B_2)e^{A(k-1)}-A^{-1}B_1-A^{-1}B_2k-A^{-2}B_2 \tag{35}$$

It is restored to the fitting value of the raw sequence:

$$\hat{X}^{(0)}(k)=\begin{cases} X^{(1)}(1)+\beta & k=1 \\ (X^{(1)}(1)+\beta+\hat{A}^{-1}\hat{B}_1+\hat{A}^{-1}\hat{B}_2+\hat{A}^{-2}\hat{B}_2)e^{\ddot{A}(k-1)}-\hat{A}^{-1}\hat{B}_1-\hat{A}^{-1}\hat{B}_2k-\hat{A}^{-2}\hat{B}_2, & k=2,3,\cdots,m \end{cases} \tag{36}$$

That Eq.(34) substituting Eq.(25) and Eq.(35) substituting Eq.(28) can obtain λ and β through the optimization method such as genetic optimization function ga.m in MATLAB, quantum chaos particle swarm optimization [15] and other modern intelligent optimization method. The NMGM model is referred to as NMGM-3. After solving the model, the model should be tested to determine whether the model is appropriate. There are commonly three methods for testing MGM (1, n) [1,3]: residual test, relational coefficient test and post-error examine. NMGM (1, n) also is test by the method in MGM (1, n) to test.

In this paper, NMGM (1, n) becomes a grey MGM (1, n) model when $B_2 = 0$. NMGM (1, n) is the promotion of MGM (1, n), and MGM (1, n) is a special case of NMGM (1, n) when $B_2 = 0$. This model with important theoretical and practical value widens application of grey prediction theory.

4. Examples

In the project of Xiong'ao metro section in Beijing Metro Line 10, the open-cut method is used for construction. The cast-in-place pile with φ800mm@1400 is supporting structure of foundation pit. The deepest depth of foundation pit is 20m. Due to the geological conditions in the project is complex, the depth of foundation digging is deeper. In order to ensure the safety of the structure and surrounding buildings, the deformation of foundation pit need be predicted. By comparison and screening to the raw data, three groups of the raw data sequence with representative and truly reflecting the deformation of foundation pit were obtained. In this paper three groups of nine data behind the raw data sequence reflecting the deformation of foundation pit were selected as shown in Table 1 [16]. The proposed grey MGM (1, n) model in this paper and the traditional multi-variable model were established, and the simulated predictions were respectively conducted to the deformation of supporting structure of deep foundation pit.

Table 1.

Three groups of the raw data sequence reflecting the deformation of foundation pit

k	1	2	3	4	5	6	7	8	9
$X_1^{(0)}$	8.48	12.77	15.10	17.87	19.66	22.30	24.32	26.10	28.90
$\hat{X}_2^{(0)}$	9.29	13.67	16.23	19.00	20.84	23.33	25.39	27.22	29.35
$X_3^{(0)}$	10.07	14.52	17.28	20.05	21.84	24.28	26.34	28.15	30.4

In this paper, the previous seven data in Table 1 were used to model and the following two data were used to predict to test the predicted effect. The parameters of each model are as follows:

$$\text{MGM-1: } A = \begin{vmatrix} 6.0461 & -14.7184 & 8.6881 \\ 9.1062 & -21.2487 & 12.1549 \\ 12.1097 & -27.5473 & 15.4514 \end{vmatrix}, B = \begin{vmatrix} 9.6664 \\ 10.3281 \\ 11.0004 \end{vmatrix}, \lambda = \mathbf{0}.5$$

MGM-2: $A=\begin{vmatrix} 6.1741 & -14.9721 & 8.8148 \\ 9.1985 & -21.4326 & 12.2474 \\ 12.2006 & -27.7300 & 15.5439 \end{vmatrix}, B=\begin{vmatrix} 9.6499 \\ 10.3113 \\ 10.9822 \end{vmatrix}, \lambda = \mathbf{0}.50509$

MGM-3:

$$A=\begin{vmatrix} 5.7678 & -14.1661 & 8.4130 \\ 8.9001 & -20.8369 & 11.9487 \\ 11.9033 & -27.1316 & 15.2417 \end{vmatrix}, B=\begin{vmatrix} 9.7014 \\ 10.3640 \\ 11.0395 \end{vmatrix}, \beta=\begin{vmatrix} 1.2088985e-05 \\ -1515.7966 \\ -3.1087836 \end{vmatrix}, \lambda = 0.48920486$$

NMGM-1: $A=\begin{vmatrix} 4.7979 & -11.1491 & 6.3435 \\ 6.4762 & -13.7178 & 7.2155 \\ 7.7115 & -14.9466 & 7.1914 \end{vmatrix}, B_1=\begin{vmatrix} 9.2742 \\ 9.5018 \\ 9.6185 \end{vmatrix}, B_2=\begin{vmatrix} 1.2217 \\ 2.5740 \\ 4.3047 \end{vmatrix}, \lambda = 0.5000$

NMGM-2: $A=\begin{vmatrix} -0.41710 & 0.7349 & -0.3512 \\ 1.9275 & -3.3713 & 1.4004 \\ 2.9522 & -4.2921 & 1.2818 \end{vmatrix}, B_1=\begin{vmatrix} 8.8331 \\ 9.0803 \\ 9.1536 \end{vmatrix}, B_2=\begin{vmatrix} 2.8854 \\ 3.9361 \\ 5.4734 \end{vmatrix}, \lambda = 0.38447$

NMGM-3: $A=\begin{vmatrix} 6.1418 & -14.3186 & 8.1909 \\ 7.5613 & -16.2772 & 8.7095 \\ 8.8158 & -17.5002 & 8.6598 \end{vmatrix}, B_1=\begin{vmatrix} 9.3729 \\ 9.5934 \\ 9.7208 \end{vmatrix}, B_2=\begin{vmatrix} 0.6575 \\ 2.1332 \\ 3.9335 \end{vmatrix},$

$$\beta=\begin{vmatrix} 4.1228258e-06 \\ 4989.4922 \\ -1392.2162 \end{vmatrix}, \lambda = 0.53024657$$

Table 2

Comparison of the fitting values of the traditional model

k	MGM-1			MGM-2			MGM-3		
	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$
1	8.48	9.29	10.07	8.48	9.29	10.07	8.48001	9.29001	10.07
2	12.8217	13.766	14.6617	12.8129	13.7564	14.6512	12.8624	13.8096	14.708
3	15.1825	16.2634	17.2628	15.1812	16.2621	17.2619	15.2306	16.3134	17.3136
4	17.5811	18.7244	19.759	17.591	18.7367	19.7735	17.6354	18.7774	19.8102
5	19.8958	21.0289	22.0381	19.9171	21.0536	22.0656	19.9635	21.0946	22.1029
6	22.0854	23.1606	24.1131	22.114	23.1914	24.1448	22.181	23.2593	24.2164
7	24.2201	25.2362	26.1461	24.2533	25.2682	26.1764	24.3552	25.3828	26.3043

Table 3

Comparison of the accuracy of the traditional model

Average relative error (%) of MGM-1			Average relative error (%) of MGM-2			Average relative error (%) of MGM-3		
$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$
0.73433	0.65662	0.69397	0.69293	0.61635	0.65689	0.7319	0.60854	0.61241
	0.69497		`	0.65539			0.65095	

Table 4

Predictive value of the traditional model

k	Predicted value of MGM-1			Predicted value of MGM-2			Predicted value of MGM-3		
	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$
8	26.4887	27.5014	28.4352	26.5351	27.5451	28.4775	26.6521	27.681	28.6272
9	29.1738	30.294	31.3641	29.2663	30.3899	31.4674	29.3096	30.436	31.5028

Table 5

Accuracy of the traditional model

k	Average relative error (%) of MGM-1			Average relative error (%) of MGM-2			Average relative error (%) of MGM-3		
	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$
8	-1.4893	-1.0338	-1.0131	-1.6670	-1.1943	-1.1634	-2.1153	-1.6936	-1.6952
9	-0.9474	-3.2164	-3.1714	-1.2675	-3.5431	-3.5112	-1.4173	-3.7002	-3.6276

Table 6

Fitting values of new mode

k	NMGM-1			NMGM-2			NMGM-3		
	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$
1	8.48	9.29	10.07	8.48	9.29	10.07	8.48	9.29	10.07
2	12.5646	13.4813	14.3526	12.735	13.5732	14.3542	12.7311	13.6781	14.5798
3	14.9213	16.0354	17.0746	15.2055	16.2731	17.2716	15.2948	16.4245	17.4758
4	17.1635	18.3107	19.3447	17.6114	18.7394	19.7629	17.7398	18.8939	19.9298
5	19.2224	20.3432	21.3384	19.9469	21.0611	22.0592	19.9708	21.0948	22.0854
6	21.2007	22.2843	23.2486	22.2066	23.2907	24.2588	22.128	23.2102	24.1662
7	23.1782	24.2276	25.172	24.3906	25.4443	26.3854	24.3499	25.3914	26.3275

Table 7

Comparison of the accuracy of new model

Average relative error (%) of NMGM-1			Average relative error (%) of NMGM-2			Average relative error (%) of NMGM-3		
$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$
2.6567	2.5216	2.4054	0.6554	0.54124	0.55514	0.68559	0.50825	0.54065
2.5279			0.58393			0.57816		

Table 8

Comparison of the predicted values of new model

k	Predicted value of NMGM-1			Predicted value of NMGM-2			Predicted value of NMGM-3		
	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$
8	25.1900	26.2088	27.1389	26.5009	27.5262	28.4425	26.7253	27.7282	28.6537
9	27.2458	28.2353	29.1529	28.5400	29.5385	30.431	29.3038	30.2682	31.1872

Table 9

Comparison of the accuracy of predictive value in new model

k	Average relative error (%) of NMGM-1			Average relative error (%) of NMGM-2			Average relative error (%) of NMGM-3		
	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$	$\hat{X}_1^{(0)}$	$\hat{X}_2^{(0)}$	$\hat{X}_3^{(0)}$
8	3.4866	3.7149	3.5918	-1.5360	-1.1249	-1.0391	-2.3958	-1.8670	-1.7893
9	5.7239	3.7980	4.1023	1.2457	-0.6422	-0.1020	-1.3972	-3.1284	-2.5895

The calculation results of six models are shown in Table 2- Table 9. Thus, the optimized model is better than the original model. The more the optimization parameters, the better the optimization results. The accuracy of MGM-3 is better than of MGM-2, and one of MGM-2 is better than of MGM-1. The accuracy of NMGM-3 is better than of NMGM-2, and one of NMGM-2 is better than of NMGM-1. In these models, the precision of NMGM-3 is the best.

5. Conclusions

(1) In the traditional MGM (1, n) model, the homogeneous exponential data is used to fit the raw data. Based on the modeling principle of the traditional MGM (1, n) model, a non-homogeneous exponential multivariate grey model NMGM (1, n) was put forward, in which the homogeneous exponential data is used to fit the raw data. The parameters were estimated of the proposed model by least square method and the time respond function was given.

(2) Two kinds of optimization models were established: one is taking the coefficient of the background value as design variable and the minimum average relative error as the objective function, the other is taking the coefficient of the background value and the initial value of the response function as the design variables and the minimum average relative error as the objective function. The solution program based on Matlab was written.

(3) NMGM (1, n) becomes a grey MGM (1, n) model when $B_2 = 0$. NMGM (1, n) is the promotion of MGM (1, n), and MGM (1, n) is a special case of NMGM (1, n) when $B_2 = 0$. This model with important theoretical and practical value widens application of grey prediction theory.

(4) The example validates new optimization model has better fitting and prediction accuracy than the traditional MGM (1, n) model.

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